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DISTRIBUTION OF ZEROS AND CRITICAL POINTS OF A POLYNOMIAL, AND SENDOV'S CONJECTURE

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Abstract. According to the Gauss Lucas theorem, the critical points of a Complex polynomial $p(z) := \sum_{j=0}^{n} a_j z^j$ where $a_j \in \mathbb{C}$ always lie in the convex hull of its zeros. In this paper, we prove certain relations between the distribution of zeros of a polynomial and its critical points. Using these relations, we prove the well-known Sendov's conjecture for certain special cases.

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1. Introduction and statement of results

Polynomials are the mathematical expressions of the form $p(z) := \sum_{j=0}^{n} a_j z^j$ where $a_j \in \mathbb{C}$ and have been studied since ancient times with regard to their zeros, the values of the variable z that make $p(z)$ vanish. If we plot the zeros of a polynomial $p(z)$ and the zeros of the derived polynomial $p'(z)$ (the critical points of $p(z)$) in the complex plane, there are interesting geometric relations between the two sets of points. It was shown by Gauss that the zeros of $p'(z)$ are the positions of equilibrium in the force field due to particles of equal mass/charge situated at each zero of $p(z)$, if each particle attracts/repels the other particle with a force equal to the inverse of the distance between them. Concerning the location of critical points of a polynomial , the Gauss-lucas theorem states that the critical points of a polynomial p lie in the convex hull of its zeros. Regarding the distribution of critical points of p within the convex hull of its zeros the well known Sendov's Conjecture asserts:

"If all the zeros of a polynomial p lie in $|z| \leq 1$ and if z_0 is any zero of $p(z)$, then there is a critical point of p in the disk $|z - z_0| \leq 1$."

The conjecture was posed by Bulgarian mathematician Blagovest Sendov in 1958, but is often attributed to Ilieff because of a reference in Hayman's *Research Problems in Function Theory* [5] in 1967. A good number of papers have been published on this conjecture (for details see [6]) but the general conjecture remains open. Rubenstein [11] in 1968 proved the conjecture for all polynomials of degree 3 and 4. In 1969 Schmeisser [12] showed that, if the convex hull containing all zeros of p has its vertices on $|z| = 1$, then p satisfies the conjecture (for the proof see [10, Theorem 7.3.4]). Later Schmeisser [13] also proved the conjecture for the Cauchy class of polynomials. In 1996 Borcea [3] showed that the conjecture holds true for polynomials with atmost six distinct zeros and in 1999 Brown and Xiang [4] proved the conjecture for polynomials of degree upto eight. Dégot [6] proved that for every zero (say) z_0 of a polynomial p there exists lower bound N_0 depending upon the modulus of z_0 such that $|z - z_0| \leq 1$ contains a critical point of p if $deg(p) > N_0$. Chalebgwa [5] gave an explicit formula for such a N_0. More recent important work in this area includes that of Kumar [8] and Sofi, Ahanger and Gardner [14]. As for the latest, Terence Tao [15] following on the work of Degot [6], proved that the Sendov's conjecture holds for polynomials with sufficiently high degree.

In this paper, we prove certain relations between the distribution of zeros of a polynomial p in the complex plane and the distribution of its critical points. Using these relations we prove the Sendov's Conjecture for the case when all the zeros of a polynomial lie on a circle or a line within the closed unit disk.

Theorem 1.1. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial with zeros $z_1, z_2, \cdots, z_n$. Suppose z_1 is a zero of p such that $|z_j - z_1 + re^{i\theta}| \leq r$ for all $2 \leq j \leq n$, $0 < \theta \leq 2\pi$ and $r > 0$, then $|z - z_1| \leq r$ always contains a critical point of p.*

In case z_1 is the largest zero of p in modulus, then $|z_j| \leq |z_1|$ for all $2 \leq j \leq n$, and therefore we have $|z_j - z_1 + |z_1|e^{i\theta}| \leq |z_1|$ where $\theta = arg(z_1)$ for all $2 \leq j \leq n$. Hence by Theorem 1.1, $|z - z_1| \leq |z_1|$ contains a critical point of p. So we have proved the following:

Corollary 1.1. *If $p(z) := \sum_{j=0}^{n} a_j z^j$ is a polynomial with all its zeros $z_1, z_2, \cdots, z_n$ and suppose z_1 is the zero of p with largest modulus, that is, $|z_j| \leq |z_1|$ for all $2 \leq j \leq n$, then $|z - z_1| \leq |z_1|$ always contains a critical point of p.*

Theorem 1.2. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a non-constant polynomial with all its zeros $z_1, z_2, \cdots, z_n$ lying inside the closed disk $|z| \leq r$. Suppose all $z_j, 1 \leq j \leq n$ lie on a circle or on a line within the closed disk $|z| \leq r$, then for every $z_i, 1 \leq i \leq n$, there exists a critical point of p in $|z - z_i| \leq r$.*

Taking $r = 1$, Theorem 1.2 shows that the Sendov's Conjecture holds when all the zeros of p lie on a circle or a line within the disk containing all the zeros of p.

Sendov's conjecture is about how far away are critical points of a polynomial p from a given zero of p. In the converse direction we prove the following :

Theorem 1.3. *Let $p(z) := \sum_{j=0}^{n} a_j z^j$ be a non-constant polynomial with its zeros $z_1, z_2, \cdots, z_n$ and critical points $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$. Then for every critical point $\zeta_j, 1 \leq j \leq n-1$ there exists a zero $z_i, 1 \leq i \leq n$ such that*

$$|z_i - \zeta_j|^2 \leq |z_i|^2 - |\zeta_j|^2. \tag{1.1}$$

and

$$\left|\frac{z_i}{2} - \zeta_j\right| \leq \left|\frac{z_i}{2}\right|. \tag{1.2}$$

Remark 1: In the above result if we assume that all the zeros of p lie in the closed unit disk, we get from Theorem 1.3 that for every critical point $\zeta_j, 1 \le j \le n-1$ of p there exists a zero $z_i, 1 \le i \le n$ of p such that

$$|z_i - \zeta_j| \le 1. \tag{1.3}$$

and $\left|\frac{z_i}{2} - \zeta_j\right| \le \frac{1}{2}$, which is the main result by A.Aziz in [1]. Our proof of the result from which it follows is very short and simple.

Remark 2: If a unique zero of p satisfies (1.3) for all $\zeta_j, 1 \le j \le n-1$ then by Biernacki's theorem [10, Theorem 4.5.2] Sendov's conjecture is true for p. Therefore, if Sendov's conjecture is false, (1.3) must be true for at least two different zeros of p and hence for any polynomial p with all its zeros in the closed unit disk there will always be at least two different zeros say z_1, z_2 such that $|z - z_i| \le 1$ for $i = 1, 2$ contains some critical point of p.

2. Proofs of the theorems

Proof of Theorem 1.1 . Let $\zeta_1, \zeta_2 \ldots, \zeta_{n-1}$ be the critical points of p and assume to the contrary $|z_1 - \zeta_j| > r$ for all $1 \le j \le n-1$.Then

$$\left|\frac{p''(z_1)}{p'(z_1)}\right| = \left|\sum_{j=1}^{n-1} \frac{1}{z_1 - \zeta_j}\right| < \frac{n-1}{r}.$$

Let $p(z) = a_n(z - z_1)q(z)$, where $q(z) = \prod_{j=2}^{n}(z - z_j)$. Then $p'(z_1) = q(z_1)$ and $p''(z_1) = 2q'(z_1)$, so that

$$\left|\frac{2q'(z_1)}{q(z_1)}\right| = \left|\frac{p''(z_1)}{p'(z_1)}\right| < \frac{n-1}{r} \quad \text{and hence} \quad \left|\frac{q'(z_1)}{q(z_1)}\right| = \left|\sum_{j=2}^{n} \frac{1}{z_1 - z_j}\right| < \frac{n-1}{2r}.$$

Now by our assumption, for all $2 \le j \le n$

$$\left|\frac{z_j - z_1 + re^{i\theta}}{re^{i\theta}}\right| \le 1,$$

we have

$$\mathfrak{Re}\left(\frac{1}{1 - \dfrac{z_j - z_1 + re^{i\theta}}{re^{i\theta}}}\right) \ge \frac{1}{2}.$$

Equivalently $\mathfrak{Re}\left(\dfrac{re^{i\theta}}{z_1 - z_j}\right) \ge \dfrac{1}{2}$. This gives

$$\mathfrak{Re}\left(\sum_{j=2}^{n} \frac{re^{i\theta}}{z_1 - z_j}\right) \ge \frac{n-1}{2}.$$

Therefore

$$\left|\sum_{j=2}^{n} \frac{re^{i\theta}}{z_1 - z_j}\right| \ge \frac{n-1}{2}.$$

Hence

$$\left|\frac{q'(z_1)}{q(z_1)}\right| = \left|\sum_{j=2}^{n}\frac{1}{z_1 - z_j}\right| \geq \frac{n-1}{2r},$$

a contradiction to (2.1). □

Proof of Theorem 1.3. Case I : All the zeros of p lie on the circle. Since the relative distances between zeros and critical points will not change under a translation we may assume without loss of generality that all the zeros of p lie on a circle with centre at the origin and in that case $|z_i| = |z_j|$ for all $1 \leq i, j \leq n$. Therefore by Corollary 1.1 for every $i, 1 \leq i \leq n$, there exists a zero of p' inside $|z - z_i| \leq |z_j| \leq r$.

Case II : All the zeros of p lie on a line.

Again since a rigid motion will not alter the relative distances between zeros and critical points of p, it is sufficient to prove the result in case all the zeors of p lie on the real line. Let $\zeta_1, \zeta_2 \ldots, \zeta_{n-1}$ be the critical points of p and assume to the contrary that there exists a zero of p say z_1 such that $|z_1 - \zeta_j| > r$ for all $1 \leq j \leq n-1$. By the same argument as in the proof of Theorem 1.1 we have

$$\left|\sum_{j=2}^{n}\frac{1}{z_1 - z_j}\right| < \frac{n-1}{2r}. \tag{2.1}$$

We may also assume that $z_1 > 0$. If there exists any zero of p, say z_j such that

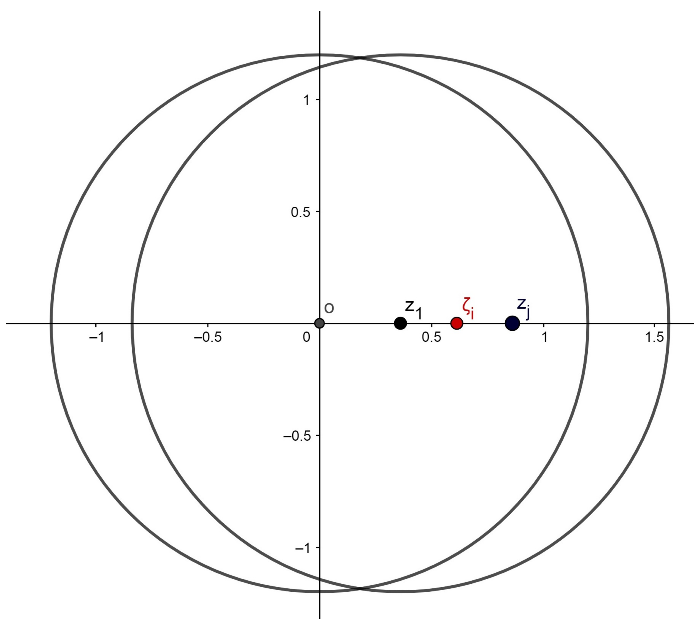

Рис. 1. If $z_j > z_1$ for some $2 \leq j \leq n$ then $|z - z_1| \leq r$ contains a critical point of p.

$z_j > z_1$, then by Rolle's theorem p will have a critical point between z_1 and z_j

which will be within a distance r of z_1 and there is nothing to prove (see Figure 1). So we may assume $z_j \leq z_1$ for all $2 \leq j \leq n$ and hence $0 \leq z_1 - z_j \leq 2r$ for all $2 \leq j \leq n$. This implies that for all $2 \leq j \leq n$ $\frac{1}{z_1 - z_j} \geq \frac{1}{2r}$, which gives

$$\left| \sum_{j=2}^{n} \frac{1}{z_1 - z_j} \right| = \sum_{j=2}^{n} \frac{1}{z_1 - z_j} \geq \frac{n-1}{2r}$$

a contradiction to (2.2) and hence the result. $\square$

Proof of Theorem 1.3. To prove (1.1) assume the contrary. Therefore there exist a critical point say ζ_1 of p such that

$$|z_i - \zeta_1|^2 > |z_i|^2 - |\zeta_1|^2, 1 \leq i \leq n. \tag{2.2}$$

Without loss of generality we may assume that $\zeta_1 \geq 0$. From (2.3) we get

$$|z_i|^2 + \zeta_1^2 - 2\zeta_1 Re(z_i) > |z_i|^2 - \zeta_1^2, 1 \leq i \leq n.$$

This gives $Re(z_i) < \zeta_1$ for all $1 \leq i \leq n$. But this implies that ζ_1, a critical point of p, lies outside the convex hull of $z_1, z_2, \cdots, z_n$, the zeros of p violating the Gauss-Lucas theorem and this contradiction proves (1.1). A similar argument proves (1.2). $\square$

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