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FURTHER RESULTS ON SHARED-VALUE PROPERTIES OF $f'(z) = f(z+c)$

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Abstract. In this paper, we will continue to consider "under what sharing value conditions, does $f'(z) = f(z+c)$ hold?" For example, we prove the following result: Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z+c)$ share ∞ CM and a, b IM, and if $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$, then $f'(z) = f(z+c)$. The research also includes some improvements of earlier results of such studies.

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1. Introduction

In the study of complex differential equations, Nevanlinna theory has a wide range of applications. In addition, with the difference correspondence of the logarithmic derivative lemma obtained by Chiang-Feng [4], and Halburd-Korhonen [7] respectively, the complex domain differences and the complex difference equations have also been rapidly developed. The related results, readers can refer to [3].

The study of complex differential-difference equations can be traced back to Naftalevich's work in [6, 16, 17], but the results of using Nevanlinna theory to study differential-difference equations are relatively limited, the reader is invited to see [5, 9, 11, 13, 14].

The delay equation $f'(x) = f(x-k)$, $(k > 0)$ have been studied extensively in real analysis. The related results can be found in [1]. Inspired by such results, Liu and Dong [12] discussed the properties of the solutions of complex differential-difference equation $f'(z) = f(z+c)$, $(c(\neq 0) \in \mathbb{C})$ by using Nevanlinna theory.

We have tried to clarify the form of the solutions to the equation $f'(z) = f(z+c)$, but unfortunately, this attempt has not been successful. Then, we investigated this equation from another point of view, namely, "under what sharing value conditions, does $f'(z) = f(z+c)$ hold?" And in [18, Theorem 1.4], we obtained:

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Theorem A. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0, a CM, then $f'(z) = f(z+c)$.*

Afterwards, for entire functions, Qi et al. improved Theorem A to "share 0 CM and a IM " in [19, Theorem 1.2] and "share two distinct constants a, b CM" in [20, Theorem 2.1]. Further, Huang and Fang [10, Theorem 1] improved the value sharing assumption to "share two distinct constants a, b IM". In addition, some authors tried to extend Theorem A to meromorphic functions:

Theorem B [19, Theorem 1.1]**.** *Let $f(z)$ be a non-constant meromorphic function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM, and satisfy $f(z+c) = 0 \to f'(z) = 0$, $f(z+c) = \infty \leftarrow f'(z) = \infty$, then $f'(z) = f(z+c)$. Further, $f(z)$ is a transcendental entire function.*

Remark. Let $z_n (n = 1, 2, \ldots)$ be zeros of $f - \alpha$ with multiplicity $\nu(n)$. If z_n are also $\nu(n)$ multiple zeros of $g - \alpha$ at least, then we write $f = \alpha \to g = \alpha$, where $\alpha \in \mathbb{C} \cup \{\infty\}$.

From Theorem 2.1 in [2], we know that:

Theorem C. *Let $f(z)$ be a non-constant meromorphic function of hyper order $\rho_2(f) < 1$. If $f'(z)$ and $f(z+c)$ share $0, \infty$ CM and 1 IM, then $f'(z) = f(z+c)$.*

In this paper, we will continue to consider the above question as $f(z)$ is a meromorphic function. We, for instance, get "Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z+c)$ share ∞ CM and a, b IM, and if $N(r, f) = O(\overline{N}(r, f)), (r \to \infty)$, then $f'(z) = f(z+c)$." The reminder of this paper is organized as follows: In Sections 3 and 4, we will improve Theorem B and Theorem C, respectively. In Section 5, we will give some partially shared values results for $f'(z)$ and $f(z+c)$, which can be seen as the improvements of Theorems B and C as well.

2. Lemmas

Lemma 2.1. [8, Theorem 5.1] *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. Then,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Throughout the paper, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

Lemma 2.2. [22, Lemma 1.2] *Let $f_1(z), f_2(z)$ be two meromorphic functions, then*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

Lemma 2.3. [15],[21, Theorem 1.13] *Let $f(z)$ be a non-constant meromorphic function, and $R(f)=\frac{P(f)}{Q(f)}$, where $P(f)=\sum\limits_{i=0}^{p}\alpha_i f^i$ and $Q(f)=\sum\limits_{j=0}^{q}\beta_j f^j$ are two mutually prime polynomials in $f(z)$. If the coefficients $\{\alpha_i(z)\}$, $\{\beta_j(z)\}$ are small functions of $f(z)$ and $\alpha_p(z)\not\equiv 0$, $\beta_q(z)\not\equiv 0$, then*

$$T(r,R(f))=max\{p,q\}\cdot T(r,f)+S(r,f).$$

Lemma 2.4. *Let $f(z)$ be a non-constant meromorphic function of hyper-order strictly less than 1, and let $a_1,\ldots,a_p\in\mathbb{C}$, $p\geq 2$, be distinct points. Then,*

$$\begin{aligned}(p-1)T(r,f(z+c))\leq{}&\sum_{k=1}^{p}N\left(r,\frac{1}{f(z+c)-a_k}\right)-N(r,f(z+c))\\&+N(r,f')-N\left(r,\frac{1}{f'}\right)+S(r,f).\end{aligned}$$

Proof. Let

$$P(f)=\prod_{k=1}^{p}(f(z+c)-a_k),$$

then we have

$$\frac{1}{P(f)}=\sum_{k=1}^{p}\frac{b_k}{f(z+c)-a_k}, \tag{2.1}$$

for some constants b_k. From Lemma 2.1 and the lemma of logarithmic derivative, we have

$$m\left(r,\frac{f'}{f(z+c)-a_k}\right)=m\left(r,\frac{f'}{f(z)-a_k}\frac{f(z)-a_k}{f(z+c)-a_k}\right)=S(r,f). \tag{2.2}$$

Hence, by (2.1) and (2.2), it follows that

$$m\left(r,\frac{f'}{P(f)}\right)\leq\sum_{k=1}^{p}m\left(r,\frac{f'}{f(z+c)-a_k}\right)+S(r,f)=S(r,f).$$

From the above equation, we get

$$m\left(r,\frac{1}{P(f)}\right)=m\left(r,\frac{f'}{P(f)}\frac{1}{f'}\right)\leq m\left(r,\frac{1}{f'}\right)+S(r,f). \tag{2.3}$$

From (2.3), we have

$$\begin{aligned}T(r,f')&=m\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{f'}\right)+O(1)\\&\geq m\left(r,\frac{1}{P(f)}\right)+N\left(r,\frac{1}{f'}\right)+S(r,f)\\&=\sum_{k=1}^{p}m\left(r,\frac{1}{f(z+c)-a_k}\right)+N\left(r,\frac{1}{f'}\right)+S(r,f),\end{aligned}$$

which means

$$
\begin{aligned}
& m(r, f(z+c)) + \sum_{k=1}^{p} m\left(r, \frac{1}{f(z+c)-a_k}\right) \\
& \leq m(r, f') + N(r, f') + T(r, f(z+c)) - N(r, f(z+c)) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\
& \leq m\left(r, \frac{f'}{f(z+c)}\right) + m(r, f(z+c)) + N(r, f') + T(r, f(z+c)) \\
& - N(r, f(z+c)) - N\left(r, \frac{1}{f'}\right) + S(r, f) \\
& = 2T(r, f(z+c)) - 2N(r, f(z+c)) + N(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(p-1)T(r, f(z+c)) \leq & \sum_{k=1}^{p} N\left(r, \frac{1}{f(z+c)-a_k}\right) - N(r, f(z+c)) \\
& + N(r, f') - N\left(r, \frac{1}{f'}\right) + S(r, f).
\end{aligned}
$$

From Lemma 8.3 in [8] and Lemma 2.1, we have the following lemma:

Lemma 2.5. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, then we have*

$$
N(r, f(z+c)) = N(r, f) + S(r, f), \quad \overline{N}(r, f(z+c)) = \overline{N}(r, f) + S(r, f),
$$
$$
N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r, f),
$$

and

$$
T(r, f(z+c)) = T(r, f) + S(r, f).
$$

Lemma 2.6. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = \infty \leftarrow f' = \infty$, then $\overline{N}(r, f(z+c)) = \overline{N}(r, f') = \overline{N}(r, f) = S(r, f)$.*

Proof. By the assumption and Lemma 2.5, we have

$$
\begin{aligned}
N(r, f) + \overline{N}(r, f) = N(r, f') & \leq N(r, f(z+c)) + S(r, f) \\
& = N(r, f) + S(r, f),
\end{aligned}
$$

which means that

$$
\overline{N}(r, f) = S(r, f),
$$

and

$$
\overline{N}(r, f(z+c)) = \overline{N}(r, f') = \overline{N}(r, f) = S(r, f).
$$

Lemma 2.7. [21, Lemma 4.3] *Suppose that $f(z)$ is a non-constant meromorphic function and $P(f) = a_p f^p + a_{p-1} f^{p-1} + \cdots + a_0 \, (a_p \neq 0)$ is a polynomial in $f(z)$ with degree p and coefficients $a_i \, (i = 0, 1, \ldots, p)$ are constants, suppose furthermore that $b_j \, (j = 1, \ldots, q) \, (q > p)$ are distinct finite values. Then,*

$$m\left(r, \frac{P(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = S(r,f).$$

Lemma 2.8. *Suppose that $f(z)$ and $g(z)$ are meromorphic functions such that $N(r,f) = N(r,g) = S(r,f)$ and a, b are two distinct finite values. Let*

$$V(z) = \left(\frac{f'}{f-a} - \frac{f'}{f-b}\right) - \left(\frac{g'}{g-a} - \frac{g'}{g-b}\right).$$

If $V(z) \equiv 0$, then either

$$2T(r,f) \leq \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r,f),$$

or

$$f(z) = g(z).$$

Proof. From $V(z) \equiv 0$, we have

$$\frac{f-a}{f-b} = A\frac{g-a}{g-b}, \tag{2.4}$$

where A is a non-zero constant. If $A = 1$, then we obtain $f(z) = g(z)$. If $A \neq 1$, then it follows from (2.4) that

$$\frac{A-1}{A}\frac{f - \frac{Ab-a}{A-1}}{f-b} = \frac{a-b}{g-b}.$$

Since $N(r,f) = N(r,g) = S(r,f)$, we get $N\left(r, \frac{1}{f-\frac{Ab-a}{A-1}}\right) = S(r,f)$. Clearly, $\frac{Ab-a}{A-1} \neq a$ and $\frac{Ab-a}{A-1} \neq b$, and then from the second main theorem, we obtain

$$2T(r,f) \leq \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f-b}\right) + S(r,f).$$

3. The improvement of Theorem B

Proposition 3.1. Let $f(z)$ be a non-constant meromorphic function. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = 0 \to f' = 0$ and $f(z+c) = \infty \leftarrow f' = \infty$, then $f(z)$ must be transcendental.

Proof. Suppose $f(z)$ is a non-constant rational function. Then, set

$$f(z) = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are two mutually prime polynomials. Hence,

$$f'(z) = \left(\frac{P}{Q}\right)' = \frac{P'Q - PQ'}{Q^2} = \frac{P_1}{Q_1},$$

and

$$f(z+c)=\frac{P(z+c)}{Q(z+c)},$$

where P_1 and Q_1 are two mutually prime polynomials. If $Q(z)$ is not a constant, then by the assumption that $f(z+c)=\infty \leftarrow f'(z)=\infty$, we have

$$Q_1(z)=0\rightarrow Q(z+c)=0.$$

Let z_1 is a zero of $Q(z)$, then we have $Q_1(z_1)=0$, and so $Q(z_1+c)=0$. From $Q(z_1+c)=0$, we have $Q_1(z_1+c)=0$, which implies that $Q(z_1+2c)=0$. Continuing inductively, we get that $Q(z_1+nc)=0$, which is impossible. Hence, $Q(z)$ is a constant. And so, $f(z)$ is a non-constant polynomial. Suppose $\deg f(z)=p\geq 1$, then we know the number of zeros of $f(z+c)$ is p and the number of zeros of f' is $p-1$, which contradicts the assumption $f(z+c)=0\rightarrow f'=0$. (Here, multiple zeros are counted to their multiplicities.) Therefore, $f(z)$ is transcendental.

Remarks. (1). Proposition 3.1 is an improvement of Theorem B and [2, Proposition 1]. Moreover, Proposition 3.1 leads us only to consider the condition that $f(z)$ is a transcendental meromorphic function in this paper.

(2). The main ideas of Proposition 3.1 and Theorem 3.1 come from Theorem B, however, the key way of proof is somewhat different. Hence, for the convenience of the reader, we provide the proof.

Theorem 3.1. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1, and let $a(\neq 0)\in\mathbb{C}$. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c)=0\rightarrow f'=0$, $f(z+c)=a\rightarrow f'=a$ and $f(z+c)=\infty\leftarrow f'=\infty$, then $f'(z)=f(z+c)$.*

Proof of Theorem 3.1. Suppose that $f'(z)\not\equiv f(z+c)$. Set

$$F(z)=\frac{f'}{f(z+c)}. \tag{3.1}$$

Then, we see $F(z)\not\equiv 1$. Further, from the assumption $f(z+c)=0\rightarrow f'=0$ and $f(z+c)=\infty\leftarrow f'=\infty$, we know that $F(z)$ is an entire function. Moreover, we have

$$m(r,F)=m\left(r,\frac{f'}{f}\frac{f}{f(z+c)}\right)=S(r,f). \tag{3.2}$$

Hence,

$$T(r,F)=S(r,f). \tag{3.3}$$

By the assumption that $f(z+c)=a\rightarrow f'=a$ and (3.3), it follows that

$$N\left(r,\frac{1}{f(z+c)-a}\right)\leq N\left(r,\frac{1}{F-1}\right)+S(r,f)=S(r,f). \tag{3.4}$$

From Lemmas 2.4-2.5, (3.4), and the sharing values assumption, we obtain that

$$\begin{aligned}T(r,f) &= T(r,f(z+c)) + S(r,f)\\ &\le \left(N\left(r,\frac{1}{f(z+c)}\right) - N\left(r,\frac{1}{f'}\right)\right) + (N(r,f') - N(r,f(z+c)))\\ &+ N\left(r,\frac{1}{f(z+c)-a}\right) + S(r,f) = S(r,f),\end{aligned}$$

which is a contradiction. Therefore, $f'(z) = f(z+c)$.

4. The improvement of Theorem C

When $f(z)$ is meromorphic, all the previous results were around the condition "$f'(z)$ and $f(z+c)$ share 0, a". What happens if f' and $f(z+c)$ share two arbitrary constants? In this part, we will give some results on the sharing value assumption that "2 IM" for meromorphic functions. As a corollary, we will get an improvement of Theorem C in Theorem 4.2.

Proposition 4.1. Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. Suppose $f'(z)$ and $f(z+c)$ share a, b IM and satisfy $f(z+c) = \infty \leftarrow f' = \infty$. If $f'(z) \not\equiv f(z+c)$ and $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$. Then,

(1).

$$T(r,f') = T(r,f(z+c)) + S(r,f).$$

(2).

$$T(r,f(z+c)) = \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + S(r,f).$$

(3).

$$m\left(r,\frac{f(z+c)-d}{f'-d}\right) = S(r,f), \quad \text{where } d(\ne a,b) \in \mathbb{C}.$$

Proof. (1). Since $f(z+c) = \infty \leftarrow f' = \infty$, we have $\overline{N}(r,f(z+c)) = \overline{N}(r,f') = \overline{N}(r,f) = S(r,f)$, from Lemma 2.6. And $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$ means that

$$N(r,f) \le k\overline{N}(r,f) + S(r,f) = S(r,f),$$

where k is a positive number. Hence,

$$N(r,f(z+c)) = N(r,f) + S(r,f) = S(r,f), \tag{4.1}$$

and

$$N(r,f') \le N(r,f) + \overline{N}(r,f) = S(r,f). \tag{4.2}$$

Set

$$H(z) = \frac{f'(z+c)(f(z+c)-f')}{(f(z+c)-a)(f(z+c)-b)}. \tag{4.3}$$

Noting $f'(z)$ and $f(z+c)$ share a, b IM, we obtain the zeros of $f(z+c)-a$ or $f(z+c)-b$ are not poles of $H(z)$, by using an elementary computation. Hence, it follows from (4.1) and (4.2) that

$$N(r,H) \le 2N(r,f(z+c)) + N(r,f') + S(r,f) = S(r,f). \tag{4.4}$$

Rewrite (4.3) as

$$H(z) = \frac{f'(z+c)f(z+c)}{(f(z+c)-a)(f(z+c)-b)} \frac{f(z+c)-f'}{f(z+c)}.$$

It follows from Lemma 2.5 and Lemma 2.7 that

$$m\left(r, \frac{f'(z+c)f(z+c)}{(f(z+c)-a)(f(z+c)-b)}\right) = S(r,f(z+c)) = S(r,f).$$

Moreover, by Lemma 2.1 and the lemma on the logarithmic derivative, we obtain

$$m\left(r, \frac{f(z+c)-f'}{f(z+c)}\right) \le m\left(r, \frac{f'}{f(z+c)}\frac{f}{f}\right) + S(r,f) = S(r,f).$$

Therefore,

$$T(r,H) = S(r,f). \tag{4.5}$$

Rewrite (4.3) as

$$H(z)f^2(z+c) - (a+b)H(z)f(z+c) + abH(z) = f'(z+c)f(z+c) - f'(z+c)f'.$$

Note $f'(z) \ne f(z+c)$, we have $H(z) \not\equiv 0$. Further, from (4.1) and (4.2), we get

$$\begin{aligned}
&2T(r,f(z+c)) = T(r,f'(z+c)f(z+c) - f'(z+c)f') + S(r,f) \\
&= m(r,f'(z+c)f(z+c) - f'(z+c)f') + S(r,f) \\
&\le m\left(r, \frac{f'(z+c)f(z+c) - f'(z+c)f'}{f(z+c)f'}\right) + m(r,f(z+c)) + m(r,f') + S(r,f) \\
&\le T(r,f(z+c)) + T(r,f') + S(r,f),
\end{aligned}$$

which means that

$$T(r,f(z+c)) \le T(r,f') + S(r,f). \tag{4.6}$$

On the other hand, from Lemma 2.5 and Lemma 2.6, we conclude that

$$T(r,f') \le T(r,f) + \overline{N}(r,f) + S(r,f) = T(r,f(z+c)) + S(r,f). \tag{4.7}$$

Combining (4.6) and (4.7), it follows that

$$T(r,f') = T(r,f(z+c)) + S(r,f). \tag{4.8}$$

(2). From (4.1), (4.2), (4.8), the second fundamental theorem and the value sharing condition, we have

$$\begin{aligned}
&T(r,f') = T(r,f(z+c)) + S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{f'-a}\right) + \overline{N}\left(r,\frac{1}{f'-b}\right) + S(r,f) \leq \overline{N}\left(r,\frac{1}{f'-f(z+c)}\right) + S(r,f)\\
&\leq T(r,f'-f(z+c)) + S(r,f) = m(r,f'-f(z+c)) + S(r,f)\\
&\leq m\left(r,\frac{f'-f(z+c)}{f(z+c)}\right) + m(r,f(z+c)) + S(r.f)\\
&\leq m\left(r,\frac{f'}{f(z+c)}\frac{f}{f}\right) + T(r,f(z+c)) + S(r,f)\\
&\leq T(r,f(z+c)) + S(r,f) = T(r,f') + S(r,f),
\end{aligned}$$

which means that

$$T(r,f(z+c)) = \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + S(r,f), \tag{4.9}$$

and

$$T(r,f') = \overline{N}\left(r,\frac{1}{f'-a}\right) + \overline{N}\left(r,\frac{1}{f'-b}\right) + S(r,f).$$

(3). From the second fundamental theorem, Lemma 2.6 and (4.9), we obtain that

$$\begin{aligned}
&2T(r,f(z+c))\\
&\leq \overline{N}\left(r,\frac{1}{f(z+c)-a}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-b}\right) + \overline{N}\left(r,\frac{1}{f(z+c)-d}\right)\\
&+ \overline{N}(r,f(z+c)) + S(r,f)\\
&= T(r,f(z+c)) + \overline{N}\left(r,\frac{1}{f(z+c)-d}\right) + S(r,f) \leq 2T(r,f(z+c)) + S(r,f).
\end{aligned}$$

Hence, we have

$$T(r,f(z+c)) = \overline{N}\left(r,\frac{1}{f(z+c)-d}\right) + S(r,f),$$

which means that

$$m\left(r,\frac{1}{f(z+c)-d}\right) = S(r,f). \tag{4.10}$$

Further, we know

$$\begin{aligned}
&m\left(r,\frac{f'-d}{f(z+c)-d}\right)\\
&\leq m\left(r,\frac{f'}{f(z+c)-d}\right) + m\left(r,\frac{d}{f(z+c)-d}\right) + S(r,f)\\
&\leq m\left(r,\frac{f'}{f-d}\frac{f-d}{f(z+c)-d}\right) + S(r,f) = S(r,f).
\end{aligned} \tag{4.11}$$

Similarly, we have

$$m\left(r, \frac{1}{f'-d}\right) = S(r,f). \tag{4.12}$$

From (4.1), (4.2), (4.8), (4.10), (4.12) and Lemma 2.2, we have

$$\begin{aligned}
& m\left(r, \frac{f(z+c)-d}{f'-d}\right) - m\left(r, \frac{f'-d}{f(z+c)-d}\right) \\
&= T\left(r, \frac{f(z+c)-d}{f'-d}\right) - N\left(r, \frac{f(z+c)-d}{f'-d}\right) \\
&- T\left(r, \frac{f'-d}{f(z+c)-d}\right) + N\left(r, \frac{f'-d}{f(z+c)-d}\right) \\
&= N\left(r, \frac{f'-d}{f(z+c)-d}\right) - N\left(r, \frac{f(z+c)-d}{f'-d}\right) + S(r,f) \\
&= N\left(r, \frac{1}{f(z+c)-d}\right) - N\left(r, \frac{1}{f'-d}\right) + S(r,f) \\
&= T\left(r, \frac{1}{f(z+c)-d}\right) - m\left(r, \frac{1}{f(z+c)-d}\right) \\
&- T\left(r, \frac{1}{f'-d}\right) + m\left(r, \frac{1}{f'-d}\right) + S(r,f) \\
&= T(r, f(z+c)) - T(r,f') + S(r,f) = S(r,f).
\end{aligned}$$

Combining this equation and (4.11), we get

$$m\left(r, \frac{f(z+c)-d}{f'-d}\right) = m\left(r, \frac{f'-d}{f(z+c)-d}\right) + S(r,f) = S(r,f). \tag{4.13}$$

Theorem 4.1. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z+c)$ share a, b IM and satisfy $f(z+c) = \infty \leftarrow f' = \infty$, and if $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$, then $f'(z) = f(z+c)$.*

As a corollary of Theorem 4.1, we are easy to get the following result:

Theorem 4.2. *Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, and let a, b be two distinct constants. If $f'(z)$ and $f(z+c)$ share ∞ CM and a, b IM, and if $N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$, then $f'(z) = f(z+c)$.*

Question. If we omit the condition that ¡°$N(r,f) = O(\overline{N}(r,f)), (r \to \infty)$¡±, would Theorems 4.1 and 4.2 still valid?

Proof of Theorem 4.1. Suppose that $f'(z) \not\equiv f(z+c)$. Set

$$U(z) = \frac{f''(f(z+c)-f')}{(f'-a)(f'-b)}. \tag{4.14}$$

Using the same argument of $H(z)$, we have that $U(z) \not\equiv 0$ and $N(r,U) = S(r,f)$. Further, from the lemma on the logarithmic derivative and the conclusion (3) of

Proposition 4.1, we have

$$\begin{aligned}
m(r,U) &= m\left(r, \left(\frac{(a-d)f''}{(a-b)(f'-a)} - \frac{(b-d)f''}{(a-b)(f'-b)}\right)\left(\frac{f(z+c)-d}{f'-d} - 1\right)\right) \\
&\le m\left(r, \frac{f(z+c)-d}{f'-d}\right) + S(r,f) \\
&= S(r,f).
\end{aligned}$$

Hence,

$$T(r,U) = S(r,f). \tag{4.15}$$

Define $S_{F\sim G(m,n)}(\alpha)$ for the set of those points $z \in \mathbb{C}$ such that z is an α-point of F with multiplicity m and an α-point of G with multiplicity n. Let $N_{(m,n)}(r, \frac{1}{F-\alpha})$ and $\overline{N}_{(m,n)}(r, \frac{1}{F-\alpha})$ denote the counting function and reduced counting function of F with respect to the set $S_{F\sim G(m,n)}(\alpha)$, respectively.

Let $z_1 \in S_{f'\sim f(z+c)(m,n)}(a)$. Substituting the Taylor expansion of f' and $f(z+c)$ at z_1 into (4.3), (4.14), by calculating carefully, we conclude that $mH(z_1) - nU(z_1) = 0$.

If $mH = nU$ for some m, n, then we have

$$m\left(\frac{f'(z+c)}{f(z+c)-a} - \frac{f'(z+c)}{f(z+c)-b}\right) = n\left(\frac{f''}{f'-a} - \frac{f''}{f'-b}\right). \tag{4.16}$$

Hence,

$$\left(\frac{f(z+c)-a}{f(z+c)-b}\right)^m = A\left(\frac{f'-a}{f'-b}\right)^n,$$

where A is a non-zero constant. Suppose $m \ne n$, then from Lemma 2.3, we get

$$nT(r,f') = mT(r, f(z+c)) + S(r,f),$$

which contradicts the conclusion (1) of Proposition 4.1. Hence, $m = n$. From (4.1), (4.2), (4.16) and Lemma 2.8, it follows that

$$2T(r, f(z+c)) \le \overline{N}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b}\right) + S(r,f),$$

which contradicts the conclusion (2) of Proposition 4.1.

If $mH \not\equiv nU$ for all m, n, then we get

$$\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) \le N\left(r, \frac{1}{mH-nU}\right) = S(r,f).$$

Similarly, we also get

$$\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right) \le N\left(r, \frac{1}{mH-nU}\right) = S(r,f).$$

Hence,

$$\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right) = S(r,f). \tag{4.17}$$

From (4.17) and the conclusions (1)-(2) of Proposition 4.1, we get

$$\begin{aligned}
&T(r, f(z+c)) \\
&= \overline{N}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b}\right) + S(r,f) \\
&= \sum_{m,n}\left(\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right)\right) + S(r,f) \\
&= \sum_{m+n\geq 5}\left(\overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right)\right) + S(r,f) \\
&\leq \frac{1}{5}\sum_{m+n\geq 5}\left(N_{(m,n)}\left(r, \frac{1}{f(z+c)-a}\right) + N_{(m,n)}\left(r, \frac{1}{f(z+c)-b}\right)\right. \\
&\left.+N_{(m,n)}\left(r, \frac{1}{f'-a}\right) + N_{(m,n)}\left(r, \frac{1}{f'-b}\right)\right) + S(r,f) \\
&\leq \frac{4}{5}T(r, f(z+c)) + S(r,f),
\end{aligned}$$

which is a contradiction. Therefore, $f'(z) = f(z+c)$.

5. Some partially shared values results

In this part, we will give two partially shared values results related to theorems B and C.

Theorem 5.1. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM, and satisfy $f(z+c) = 0 \to f' = 0$, $f(z+c) = \infty \leftarrow f' = \infty$, then $f'(z) = f(z+c)$.*

Proof of Theorem 5.1. Set

$$F(z) = \frac{f'}{f(z+c)}. \tag{5.1}$$

If $F(z) \equiv 1$, then we have $f' = f(z+c)$. In the following, we suppose that $F(z) \not\equiv 1$. Then, by the the same argument of Theorem 3.1, we know $F(z)$ also satisfy

$$T(r, F) = S(r, f). \tag{5.2}$$

In addition, from (5.1)-(5.2), and Lemma 2.5, it follows that

$$T(r, f') = T(r, f(z+c)) + S(r,f) = T(r,f) + S(r,f). \tag{5.3}$$

Hence,

$$S(r, f') = S(r, f(z+c)) = S(r,f). \tag{5.4}$$

Further, since f' and $f(z+c)$ share a IM, we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f'-a}\right)&=\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)\le\overline{N}\left(r,\frac{1}{\frac{f'}{f(z+c)}-1}\right)+S(r,f)\\&=\overline{N}\left(r,\frac{1}{F-1}\right)+S(r,f)\le T(r,F)+S(r,f)=S(r,f).\end{aligned}\tag{5.5}$$

Set

$$G(z)=\frac{f''}{f'-a}-\frac{f'(z+c)}{f(z+c)-a}.\tag{5.6}$$

In the following, we distinguish two cases.

Case 1. If $G(z)\equiv 0$, then we have

$$f'-a=A(f(z+c)-a),\tag{5.7}$$

where A is a non-zero constant.

If $A=1$, then $f'=f(z+c)$. In the following, we suppose that $A\neq 1$, then by (5.7) and $A\neq 1$, we can immediately get $f(z+c)\neq 0$. Hence, 0 is a Picard value of $f(z+c)$, then it follows from (5.4)-(5.5), Lemma 2.6 and the second main theorem that

$$\begin{aligned}T(r,f(z+c))&\le\overline{N}\left(r,\frac{1}{f(z+c)}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)\\&+\overline{N}(r,f(z+c))+S(r,f(z+c))=S(r,f(z+c)),\end{aligned}$$

which is a contradiction.

Case 2. If $G(z)\not\equiv 0$, then by (5.4) and the lemma of logarithmic derivative, we obtain

$$m(r,G)=S(r,f).$$

Further, by (5.5) and Lemma 2.6, we get

$$\begin{aligned}N(r,G)&\le\overline{N}(r,f')+\overline{N}\left(r,\frac{1}{f'-a}\right)\\&+\overline{N}(r,f(z+c))+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+S(r,f)=S(r,f).\end{aligned}$$

Therefore,

$$T(r,G)=S(r,f).$$

According to (5.1), we have

$$f''=F'f(z+c)+Ff'(z+c).\tag{5.8}$$

Substituting (5.1) and (5.8) into (5.6), we get

$$G(z)=\frac{F'f(z+c)+Ff'(z+c)}{Ff(z+c)-a}-\frac{f'(z+c)}{f(z+c)-a},$$

which means that

$$(5.9)\quad (FG-F')f^2(z+c)+(aF'-aG(1+F))f(z+c)+a^2G=a(1-F)f'(z+c).$$

If $FG-F'\not\equiv 0$, then by Lemma 2.6 and (5.9), we have

$$\begin{aligned}2T(r,f(z+c))&=T(r,f'(z+c))+S(r,f(z+c))\\&\le T(r,f(z+c))+\overline{N}(r,f(z+c))+S(r,f(z+c))\\&\le T(r,f(z+c))+S(r,f(z+c)),\end{aligned}$$

and so, $T(r,f(z+c))=S(r,f(z+c))$, which is impossible. Hence, $FG-F'\equiv 0$. Namely,

$$\frac{F'}{F}=G=\frac{f''}{f'-a}-\frac{f'(z+c)}{f(z+c)-a},$$

which implies that

$$(5.10)\qquad \frac{f'}{f(z+c)}=B\frac{f'-a}{f(z+c)-a},$$

where B is a non-zero constant. Note that f' and $f(z+c)$ share a IM.

If a is a picard value of f' and $f(z+c)$, then f' and $f(z+c)$ share a CM. From Theorem B, we have $f'=f(z+c)$.

If a is not a picard value of f' and $f(z+c)$, then compare both side of (5.10), we also get f' and $f(z+c)$ share a CM. Otherwise, suppose z_2 is a common zero of $f'-a$ and $f(z+c)-a$, then from (5.10), we have $1=\frac{f'(z_2)}{f(z_2+c)}=0$ or $1=\frac{f'(z_2)}{f(z_2+c)}=\infty$, which is impossible. Hence, from Theorem B, the conclusion holds as well.

Theorem 5.2. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1, and let a,b be two distinct non-zero constants. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c)=a\to f'=a$, $f(z+c)=b\to f'=b$, $f(z+c)=\infty\leftarrow f'=\infty$ and $\delta(0,f)>0$, then $f'(z)=f(z+c)$.*

Here, we define $\delta(0,f)$ as following

$$\delta(0,f)=1-\limsup_{r\to\infty}\frac{N(r,\frac{1}{f})}{T(r,f)}.$$

Question. If we omit the condition that ¡°$\delta(0,f)>0$¡±, is Theorem 5.2 still valid?

Proof of Theorem 5.2. Suppose that $f'(z)\not\equiv f(z+c)$. Set

$$(5.11)\qquad F(z)=\frac{f'}{f(z+c)}.$$

Similarly as above, we know $F(z)\not\equiv 1$. And the equation (3.2) also holds, namely,

$$m(r,F)=S(r,f).$$

Combining this equation, Lemma 2.5 and the assumption that $f(z+c)=\infty \leftarrow f'=\infty$, it follows that

$$(5.12)\qquad \begin{aligned} T(r,F) =& N(r,F)+S(r,f) \leq N\left(r, \frac{1}{f(z+c)}\right)+S(r,f) \\ &= N\left(r, \frac{1}{f}\right)+S(r,f). \end{aligned}$$

Moreover, by the assumption that $f(z+c)=a \rightarrow f'=a$, $f(z+c)=b \rightarrow f'=b$ and (5.12), we get that

(5.13)

$$\begin{aligned} &N\left(r, \frac{1}{f(z+c)-a}\right)+N\left(r, \frac{1}{f(z+c)-b}\right) \leq N\left(r, \frac{1}{\frac{f'}{f(z+c)}-1}\right)+S(r,f) \\ &= N\left(r, \frac{1}{F-1}\right)+S(r,f) \leq T(r,F)+S(r,f) \leq N\left(r, \frac{1}{f}\right)+S(r,f). \end{aligned}$$

Therefore, from Lemmas 2.4-2.5, the assumption that $f(z+c)=\infty \leftarrow f'=\infty$ and (5.13), we get

$$\begin{aligned} T(r,f) = T(r,f(z+c))+S(r,f) &\leq (N(r,f')-N(r,f(z+c))) \\ &+ N\left(r, \frac{1}{f(z+c)-a}\right)+N\left(r, \frac{1}{f(z+c)-b}\right)+S(r,f) \\ &\leq N(r, \frac{1}{f})+S(r,f), \end{aligned}$$

which contradicts the assumption that $\delta(0,f)>0$. Therefore, $f'(z)=f(z+c)$.

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Список литературы

[1] R. Bellman and K. L. Cooke, Differential-Difference Equations, Academic Press, New York (1963).

[2] S. J. Chen and A. Z. Xu, "Uniqueness of derivatives and shifts of meromorphic functions", Comput. Methods Funct. Theory, **22**, 197 – 205 (2022).

[3] Z. X. Chen, Complex Differences and Difference Equations, Mathematics Monograph Series 29, Science Press, Beijing (2014).

[4] Y. M. Chiang and S. J. Feng, "On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane", Ramanujan J. **16**, 105 – 129 (2008).

[5] L. K. Gao, K. Liu and X. L. Liu, "Exponential Polynomials as solutions of nonlinear differential-difference equations", J. Contemp. Math. Anal. **57**, 14 – 29 (2022).

[6] A. Gylys and A. Naftalevich, "On meromorphic solutions of a linear differential-difference equation with constant coefficients", Mich. Math. J., **27**, 195 – 213 (1980).

[7] R. G. Halburd and R. J. Korhonen, "Difference analogue of the lemma on the logarithmic derivative with applications to difference equations", J. Math. Anal. Appl. **314**, 477 – 487 (2006).

[8] R. G. Halburd and R. J. Korhonen and K. Tohge, "Holomorphic curves with shift-invariant hyperplane preimages", Trans. Amer. Math. Soc., **366**, 4267 – 4298 (2014).

[9] R. G. Halburd and R. J. Korhonen, "Growth of meromorphic solutions of delay differential equation", Proc. Am. Math. Soc., **145**, 2513 – 2526 (2017).

[10] X. H. Huang and M. L. Fang, "Unicity of entire functions concerning their shifts and derivatives", Comput. Methods Funct. Theory, **21**, 523 – 532 (2021).

[11] K. Liu and L. Z. Yang, “On entire solutions of some differential-difference equations”, Comput. Methods Funct. Theory, **13**, 433 – 447 (2013).
[12] K. Liu and X. J. Dong, “Some results related to complex differential-difference equations of certain types”, Bull. Korean Math. Soc., **51**, 1453 – 1467 (2014).
[13] K. Liu and C. J. Song, “Meromorphic solutions of complex differential-difference equations”, Results Math., **72**, 1759 – 1771 (2017).
[14] F. Lü, W. R. Lü, C. P. Liu and J. F. Xu, “Growth and uniqueness related to complex differential and difference equations”, Results Math., **74**, 30 (2019).
[15] A. Z. Mokhon’ko, “The Nevanlinna characteristics of certain meromorphic function”, Teor. Funkts., Funkts. Anal. Prilozh. **14**, 83 – 87 (1971).
[16] A. Naftalevich, “Meromorphic solutions of a differential-difference equation”, Uspekhi Mat. Nauk. **99**, 191 – 196 (1961).
[17] A. Naftalevich, “On a differential-difference equation”, Mich. Math. J. **22**, 205 – 223 (1976).
[18] X. G. Qi, N. Li and L. Z. Yang, “Uniqueness of meromorphic functions concerning their differences and solutions of difference Painlevé equations”, Comput. Methods Funct. Theory, **18**, 567 – 582 (2018).
[19] X. G. Qi and L. Z. Yang, “Uniqueness of meromorphic functions concerning their shifts and derivatives”, Comput. Methods Funct. Theory, **20**, 159 – 178 (2020).
[20] Z. J. Wang, X. G. Qi and L. Z. Yang, “On shared-value properties of $f'(z) = f(z + c)$”, J. Contemp. Math. Anal. **56**, 245 – 253 (2021).
[21] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, Dordrecht (2003).
[22] L. Yang, Value Distribution Theory, Springer-Verlag, Berlin (1993).

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