Известия НАН Армении, Математика, том 58, н. 5, 2023, стр. 17 – 32.

EFFECTIVENESS OF THE EVEN-TYPE DELAYED MEAN IN APPROXIMATION OF CONJUGATE FUNCTIONS

XH. Z. KRASNIQI

University of Prishtina, Prishtina, Republic of Kosovo
E-mails: *xhevat.krasniqi@uni-pr.edu*

Abstract. In this paper, using the even-type delayed mean of conjugate series, we have obtain the degree of approximation for a conjugate function in the metric of the generalized Höder class with weight. Involving two moduli of continuity, we have shown that the even-type delayed mean are streamlined to guarantee this degree to be of the Jackson order.

MSC2020 numbers: 42A10; 41A30; 26A16; 42A50.

Keywords: degree of approximation; Fourier series; conjugate function; Hölder class with weight; modulus of continuity; even-delayed mean.

1. Introduction

Looking into mathematical research articles, someone can encounter one class of functions introduced in the past and its generalizations defined in the last decades - the Hölder class. Focusing to our research interest, this provoking class for many researchers has been used to a research problem (question) which specifically is related to the trigonometric Fourier series and "more rarely" to their conjugate series. Factually, the problem consists in determining the degree of approximation of 2π-periodic integrable functions (which belongs to the Hölder class) and for the conjugate functions by various means of their Fourier series and their conjugate series, respectively.

Pertaining to the above problem, a lot of results have been published for dozens of decades. For instance, Prössdorf [25], Leindler [13]–[14], Chandra [1], Mohapatra and Chandra [17], Singh et al. [26]-[27], Das et al. [2]–[3], Mittal and Rhoades [19], Krasniqi [9] and [11], Krasniqi and Szal [10], Lenski at al. [15], Krasniqi et al. [12], Nayak at al. [21]–[22], Singh and Sonker [29], Değer and Kücükaslan [4], Değer [5], Pradhan et al. [24], and Kim [8], are among the researchers who have contributed to the present topic.

In the sequel, we do not recall all results in the mentioned papers, but for our purpose we are going to write some definitions and notations from [21]–[22] and the result in [8], which serve as preliminary materials.

Let $\omega(t)$ be a modulus of continuity, i.e., $\omega(t)$ is a positive nondecreasing continuous function with the properties

$$\omega(0)=0, \quad \omega(t_1+t_2)\le \omega(t_1)+\omega(t_2), \quad \omega(\lambda t)\le (\lambda+1)\omega(t),$$

where $0\le t_1\le t_2\le t_1+t_2\le 2\pi$ and λ is any nonnegative real number.

On one hand, Das, Nath and Ray [3] introduced the space H_p^ω as follows:

$$H_p^\omega := \left\{ f\in L^p[0,2\pi] : \sup_{h\neq 0} \frac{\|f(x+h)-f(x)\|_p}{\omega(|h|)} < \infty \right\}$$

with

$$\|f\|_p^{(\omega)} := \|f\|_p + \sup_{h\neq 0} \frac{\|f(x+h)-f(x)\|_p}{\omega(|h|)},$$

where $L^p[0,2\pi]$ denotes all integrable 2π-periodic functions and

$$\|f\|_p := \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right)^{\frac{1}{p}} & \text{for} \quad 1\le p<\infty \\ \text{ess} \sup\limits_{x\in(0,2\pi)} \{|f(x)|\} & \text{for} \quad p=\infty. \end{cases}$$

It is shown (see [22]) that $\|\cdot\|_p^{(\omega)}$ is a norm in the space H_p^ω and it is a Banach space as well. For an integrable 2π-periodic function $f(x)$, by

$$s_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx + b_k \sin kx$$

we denote the n-th partial sums of Fourier series of f (at the point x)

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos kx + b_k \sin kx. \tag{1.1}$$

We need to recall the delayed mean $\sigma_{n,q}(f;x)$ defined as follows ([33]):

$$\sigma_{n,q}(f;x) := \frac{1}{q}\sum_{i=n}^{n+q-1} s_i(f;x)$$

or

$$\sigma_{n,q}(f;x) := \frac{n+q}{q}\sigma_{n+q-1}(f;x) - \frac{n}{q}\sigma_{n-1}(f;x),$$

where $\sigma_m(f;x)$ denotes the well-known Fejer mean of $s_i(f;x)$.

In his result, Kim [8] used the (even–type) delayed means $\sigma_{n,q}(f;x)$ with $q=rn$, $r=2,4,6,\dots$, which can be expressed by the convolution

$$\sigma_{n,rn}(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_{rn}(t)dt,$$

where the kernel $K_{rn}(t)$ is defined by

$$K_{rn}(t) := \frac{2\sin\left(\frac{r}{2}+1\right)nt \sin\left(\frac{rnt}{2}\right)}{rn\left(2\sin\frac{t}{2}\right)^2}.$$

The following theorem already has been proved.

Theorem 1.1 ([8]). *Let $v(t)$ and $\omega(t)$ be moduli of continuity such that $\omega(t)/v(t)$ is nondecreasing. If $f \in H_p^\omega$, $p \geq 1$, then for an even positive integer r,*

$$\|\sigma_{n,rn}(f) - f\|_p^{(v)} = \mathcal{O}\left(\frac{1}{rn}\right) + \mathcal{O}\left(\frac{r}{n}\right)\int_{\frac{\pi}{n}}^{\pi} \frac{\omega(t)dt}{t^3 v(t)}.$$

In addition, if $\omega(t)/(tv(t))$ is non-increasing, then we have

$$\|\sigma_{n,rn}(f) - f\|_p^{(v)} = \mathcal{O}\left(\frac{r}{n}\right)\int_{\frac{\pi}{n}}^{\pi} \frac{\omega(t)dt}{t^3 v(t)}$$

and

$$\|\sigma_{n,rn}(f) - f\|_p^{(v)} = \mathcal{O}\left(r\frac{\omega\left(\frac{\pi}{n}\right)}{v\left(\frac{\pi}{n}\right)}\right).$$

In the other hand, to reveal our intention in this paper, we recall the weighted Lebesgue space $L_\beta^p[0, 2\pi]$. Let f be a 2π-periodic function and $f \in L_\beta^p := L_\beta^p[0, 2\pi]$ for $p \geq 1$, where $L_\beta^p[0, 2\pi]$ denotes all measurable functions and $\|f\|_{p;\beta}$ the weighted norm

$$\|f\|_{p;\beta} := \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p \left|\sin\left(\frac{x}{2}\right)\right|^{\beta p} dx\right)^{\frac{1}{p}} & \text{for} \quad 1 \leq p < \infty \\ \operatorname{ess}\sup\limits_{x\in(0,2\pi)} \left\{|f(x)| \left|\sin\left(\frac{x}{2}\right)\right|^{\beta}\right\} & \text{for} \quad p = \infty \end{cases}$$

with $\beta \geq 0$ a real number (see [30], [32]).

We define the generalized Hölder space with weight by

$$H_{p;\beta}^{(w)} := \left\{f \in L_\beta^p[0, 2\pi] : \sup_{t\neq 0} \frac{\|f(x+t) - f(x-t)\|_{p;\beta}}{w(|t|)} < \infty\right\}$$

and

$$\|f\|_{p;\beta}^{(w)} := \|f\|_{p;\beta} + \sup_{t\neq 0} \frac{\|f(x+t) - f(x-t)\|_{p;\beta}}{w(|t|)},$$

where $1 \leq p < \infty$, $\beta \geq 0$, and $w(t)$ is a function of modulus continuity type.

Note that $\|\cdot\|_{p;\beta}^{(w)}$ is a norm in $H_{p;\beta}^{(w)}$. The completeness of the space $H_{p;\beta}^{(w)}$ can be debated as long as the completeness of L^p space, and thus the space $H_{p;\beta}^{(w)}$ is a Banach space under the norm $\|\cdot\|_{p;\beta}^{(w)}$.

Assume that the functions $w_1(t)$ and $w_2(t)$ are two moduli of continuity, and $\frac{w_1(t)}{w_2(t)}$ is a non-negative and non-decreasing function. Then,

$$\|f\|_{p;\beta}^{(w_2)} \leq \max\left(1, \frac{w_1(2\pi)}{w_2(2\pi)}\right)\|f\|_{p;\beta}^{(w_1)},$$

which shows that

$$H_{p;\beta}^{(w_1)} \subseteq H_{p;\beta}^{(w_2)} \subseteq L_\beta^p, \quad p \geq 1.$$

The series

$$\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx) \tag{1.2}$$

is the conjugate series of its Fourier series (1.1).

If f is integrable functions in the sens of Lebesgue, then it known that

$$\widetilde{f}(x) := -\frac{1}{\pi}\int_0^\pi \psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt = \lim_{\varepsilon\to 0}\widetilde{f}(x;\varepsilon)$$

exists for almost all x (see [34]), where

$$\widetilde{f}(x;\varepsilon) := -\frac{1}{\pi}\int_\varepsilon^\pi \psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt$$

with

$$\psi(x;t) := f(x+t) - f(x-t).$$

Regrading to L^p integrability of the function $\widetilde{f}$, conjugate to the function f, we are based on a theorem of M.Riesz which states that ([34]): If $f \in L^p$ for $1 < p < \infty$, then $\widetilde{f} \in L^p$.

We have to mentioned here that, as a whole, the conjugate series of a Fourier series is not necessarily a Fourier series. For example, the series $\sum_{k=2}^{\infty}(\log n)^{-1}\sin kx$ is the conjugate series of the Fourier series $\sum_{k=2}^{\infty}(\log n)^{-1}\cos kx$, however it is not itself a Fourier series (see [34], p. 186). This fact has provoked and motivated us to determine the degree of approximation of the function $\widetilde{f}$, conjugate to the function f, in the metric of the space $H_{p;\beta}^{(w)}$, by using the even–type delayed arithmetic mean $\widetilde{\sigma}_{n,rn}(f;x)$, $(r = 2,4,6,\dots)$ of the series (1.2), which is the aim of the present paper.

More results, in reference to determining the degree of approximation of the function $\widetilde{f}$, conjugate to the function f, in the metrics of the Hölder spaces, the interested reader could find in the work of Chandra [1], Nigam and Hadish [23], Mishra and Khatri [18]-[7], Singh [31], and London et al. [16].

Even though we adopt the same technique (as other authors) for the proof of our result, this last one is new and includes its application for a wide class of functions, which at least is not narrower than classes of functions defined by others.

Our paper is organized as follows. The second section contains some helpful lemmas which play a key role for the proof of the new result, the third section is devoted to the main result, and in the forth section we give a conclusion.

2. Auxiliary lemmas

We need four auxiliary statements. The first one and the third one previously are known, the second one and the fourth one will be proved in sequel (their play a key role in the proof of the main result).

Lemma 2.1 (Generalized Minkowski inequality, [6], [34])**.** *If $z(x,t)$ is a function in two variables defined for $c \le t \le d$, $a \le x \le b$, then*

$$\left\{\int_a^b \left|\int_c^d z(x,t)dt\right|^p dx\right\}^{\frac{1}{p}} \le \int_c^d \left\{\int_a^b |z(x,t)|^p dx\right\}^{\frac{1}{p}} dt, \quad p \ge 1.$$

Lemma 2.2. *Let $w_{rm}(t) := \frac{\sin(r+1)mt - \sin mt}{\left(2\sin\frac{t}{2}\right)^2}$ and $0 \le t \le \frac{1}{n+1}$. Then,*

(i) $w_{rm}(t) = \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{\left(2\sin\frac{t}{2}\right)^2}$

(ii) $|w_{rm}(t)| \le \frac{rm\pi^2}{4t}$, $(r = 2, 4, 6, \dots)$.

Proof. (i) The equality follows by the formula for converting the difference into product.

(ii) Using the well-known inequalities $|\sin(m\alpha)| \le m|\sin(\alpha)|$ and $\pi\sin(\alpha) \ge 2\alpha$ for $\alpha \in [0, \pi/2]$, we obtain

$$|w_{rm}(t)| = \frac{\left|2\sin\frac{rmt}{2}\right|\left|\cos\frac{(r+2)mt}{2}\right|}{\left(2\sin\frac{t}{2}\right)^2} \le \frac{2rm\left|\sin\frac{t}{2}\right|}{\left(\frac{2t}{\pi}\right)^2} \le \frac{2\pi^2 rm\frac{t}{2}}{4t^2} = \frac{\pi^2 rm}{4t}.$$

The proof is completed. □

Lemma 2.3 ([34])**.** *A function $w(t)$ of modulus of continuity type on the interval $[0, 2\pi]$ satisfies the following condition $\delta_2^{-1}w(\delta_2) \le 2\delta_1^{-1}w(\delta_1)$ for $0 < \delta_1 \le \delta_2$.*

Lemma 2.4. *Let $\frac{w_1(t)}{w_2(t)}$ be a positive and a non-decreasing function, $f \in H_{p;\beta}^{(w_1)}$, $p \ge 1$, and $\beta \ge 0$ a real number. Then,*

(a)

$$\|\psi(x+y;t) - \psi(x-y;t)\|_{p;\beta} = \mathcal{O}\left(w_1(t)\right).$$

(b)

$$\|\psi(x+y;t) - \psi(x-y;t)\|_{p;\beta} = \mathcal{O}\left(w_1(y)\right).$$

(c)

$$\|\psi(x+y;t) - \psi(x-y;t)\|_{p;\beta} = \mathcal{O}\left(\frac{w_1(t)}{w_2(t)}w_2(y)\right).$$

(d)

$$\|\psi(x+y;t) - \psi(x-y;t) - \psi(x+y;t+\pi/m) + \psi(x-y;t+\pi/m)\|_{p;\beta}$$
$$= \mathcal{O}\left(\frac{w_1(t)}{w_2(t)}w_2(y)\right).$$

Proof. (a) Because of

$$\psi(x+y;t) - \psi(x-y;t) = f(x+y+t) - f(x+y-t) - f(x-y+t) + f(x-y-t),$$

then (applying the Minkowski inequality) we have

$$\begin{aligned}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta} &\leq \|f(x+y+t)-f(x+y-t)\|_{p;\beta} \\ &+\|f(x-y+t)-f(x-y-t)\|_{p;\beta}=\mathcal{O}\left(w_1(t)\right).\end{aligned}$$

(b) Very similarly, rearrangement of the terms into the brackets of the case (a) implies

$$\begin{aligned}\psi(x+y;t)-\psi(x-y;t) &= f(x+t+y)-f(x+t-y) \\ &- f(x-t+y)+f(x-t-y),\end{aligned}$$

and thus, we have

$$\begin{aligned}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta} &\leq \|f(x+t+y)-f(x+t-y)\|_{p;\beta} \\ &+\|f(x-t+y)-f(x-t-y)\|_{p;\beta}=\mathcal{O}\left(w_1(y)\right).\end{aligned}$$

(c) Let $w_2(t)$ be positive and non-decreasing function. Then, for $t\leq y$ and (a), we get

$$\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}=\mathcal{O}\left(w_2(t)\frac{w_1(t)}{w_2(t)}\right)=\mathcal{O}\left(w_2(y)\frac{w_1(t)}{w_2(t)}\right).$$

Now, let $t\geq y$. Since $\frac{w_1(t)}{w_2(t)}$ is positive and non-decreasing function, then based on (b) we obtain

$$\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}=\mathcal{O}\left(w_2(y)\frac{w_1(y)}{w_2(y)}\right)=\mathcal{O}\left(w_2(y)\frac{w_1(t)}{w_2(t)}\right).$$

(d) The proof can be done similarly. □

3. Main results

We managed to prove the following.

Theorem 3.1. *Let $w_1(t)$ and $w_2(t)$ be two moduli of continuity such that $\frac{w_1(t)}{w_2(t)}$ is positive and non-decreasing function. In addition, let f be a 2π-periodic function, Lebesgue integrable on $[0,2\pi]$, belonging to the generalized Hölder class with weight $H_{p;\beta}^{(w_1)}$, $p\geq 1$, and $\beta\geq 0$. Then for the function $\widetilde{f}$, conjugate to the function f, and for an even positive integer r*

$$\begin{aligned}&\|\widetilde{\sigma}_{m,rm}(f)-\widetilde{f}\|_{p;\beta}^{(w_2)} \\ &\quad=\mathcal{O}\left(\frac{1}{r}\left(\frac{1}{m}+\left(1+r+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right)\end{aligned}$$

provided that

$$\int_0^{\eta} t^{-1}w_1(t)dt=\mathcal{O}\left(w_1(\eta)\right) \tag{3.1}$$

and

$$\int_\eta^\pi t^{-2} w_1(t)dt = \mathcal{O}\left(\eta^{-1} w_1(\eta)\right) \tag{3.2}$$

for $0 < \eta < \pi$.

Proof. It is a well-known fact that the Cesàro mean $\widetilde{\sigma}_m(f;x)$ of the partial sums $\widetilde{s}_j(f;x)$ of the series (1.2) can be expressed as follows (see [34])

$$\widetilde{\sigma}_m(f;x) = -\frac{2}{\pi}\int_0^\pi [f(x+t) - f(x-t)]\widetilde{K}_m(t)dt,$$

where

$$\widetilde{K}_m(t) := \frac{(m+1)\sin t - \sin(m+1)t}{(m+1)\left(2\sin\frac{t}{2}\right)^2}.$$

The above equality can be rewritten as follows

$$\widetilde{\sigma}_m(f;x) - \widetilde{f}(x) = \frac{2}{\pi(m+1)}\int_0^\pi [f(x+t) - f(x-t)]\frac{\sin(m+1)t}{\left(2\sin\frac{t}{2}\right)^2}dt.$$

Whence, for the even-type delayed arithmetic mean $\widetilde{\sigma}_{m,rm}(f;x)$ of the series (1.2), we can write

$$\begin{aligned}
\widetilde{\tau}_m(x) &:= \widetilde{\sigma}_{m,rm}(f;x) - \widetilde{f}(x)\\
&= \frac{r+1}{r}\left[\sigma_{(r+1)m-1}(f;x) - \widetilde{f}(x)\right] - \frac{1}{r}\left[\sigma_{m-1}(f;x) - \widetilde{f}(x)\right]\\
&= \frac{r+1}{r}\frac{2}{\pi(r+1)m}\int_0^\pi [f(x+t) - f(x-t)]\frac{\sin(r+1)mt}{\left(2\sin\frac{t}{2}\right)^2}dt\\
&- \frac{1}{r}\frac{2}{\pi m}\int_0^\pi [f(x+t) - f(x-t)]\frac{\sin mt}{\left(2\sin\frac{t}{2}\right)^2}dt\\
&= \frac{2}{rm\pi}\int_0^\pi [f(x+t) - f(x-t)]w_{rm}(t)dt, \qquad (3.3)
\end{aligned}$$

where

$$w_{rm}(t) = \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{\left(2\sin\frac{t}{2}\right)^2}.$$

By definition of the norm $\|\cdot\|^{(w_2)}_{p;\beta}$, we have

$$\|\widetilde{\tau}_m(x)\|^{(w_2)}_{p;\beta} := \|\widetilde{\tau}_m(x)\|_{p;\beta} + \sup_{y\neq 0}\frac{\|\widetilde{\tau}_m(x+y) - \widetilde{\tau}_m(x-y)\|_{p;\beta}}{w_2(y)}. \tag{3.4}$$

Further, we will find the upper bound of $\|\widetilde{\tau}_m(x)\|_{p;\beta}$. First of all, it clear that

$$\begin{aligned}
\widetilde{\tau}_m(x) = & \frac{2}{rm\pi}\int_0^{\frac{\pi}{m}}[f(x+t)-f(x-t)]w_{rm}(t)dt\\
& + \frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}[f(x+t)-f(x-t)]\\
& \times\left(2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right)\left(\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right)dt\\
& + \frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}[f(x+t)-f(x-t)]\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt.
\end{aligned}$$

Hence, applying Lemma 2.1, we have

$$\begin{aligned}
\|\widetilde{\tau}_m(x)\|_{p;\beta} \leq & \frac{2}{rm\pi}\int_0^{\frac{\pi}{m}}\|f(x+t)-f(x-t)\|_{p;\beta}\,|w_{rm}(t)|\,dt\\
& + \frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|f(x+t)-f(x-t)\|_{p;\beta}\\
& \times\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\left|\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right|dt\\
& + \frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|f(x+t)-f(x-t)\|_{p;\beta}\frac{\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|}{t^2}dt\\
:= & P_1+P_2+P_3. \qquad (3.5)
\end{aligned}$$

Since $f\in H_{p;\beta}^{(w_1)}$, for $p\geq 1$ and $\beta\geq 0$, then

$$\|f(x+t)-f(x-t)\|_{p;\beta}=\mathcal{O}\left(w_1(t)\right), \qquad (3.6)$$

and consequently by Lemma 2.2 we have

$$\begin{aligned}
P_1 = & \frac{\mathcal{O}(1)}{rm}\int_0^{\frac{\pi}{m}}w_1(t)\frac{rm}{t}dt=\mathcal{O}(1)\int_0^{\frac{\pi}{m}}\frac{w_1(t)}{t}dt\\
= & \mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2\left(\frac{\pi}{m}\right)=\mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2\left(\pi\right)=\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right) \qquad (3.7)
\end{aligned}$$

taking into account the condition (3.1).

Noticing that the function

$$t\to\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}$$

is bounded for $t\in[\pi/(n+1),\pi]$, and using Lemma 2.3 and (3.6) we can write

$$P_2=\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|f(x+t)-f(x-t)\|_{p;\beta}\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\left|\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right|dt$$

(3.8)

$$=\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\mathcal{O}\left(\frac{tw_1(t)}{t}\right)dt=\frac{4w_2(\pi)}{r\pi^2}\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right)=\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}\right).$$

In

$$P_0:=\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\psi(x;t)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt$$

we substitute t with $t+\pi/m$ and since r is an even natural number, we get

$$\begin{aligned}P_0&:=-\frac{2}{rm\pi}\int_0^{\pi-\frac{\pi}{m}}\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt\\&=-\frac{2}{rm\pi}\left(\int_0^{\frac{\pi}{m}}+\int_{\frac{\pi}{m}}^{\pi}-\int_{\pi-\frac{\pi}{m}}^{\pi}\right)\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt\end{aligned}$$

Adding these two expressions side by side we obtaining

$$\begin{aligned}P_0&:=\frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\left[\frac{\psi(x;t)}{t^2}-\frac{\psi(x;t+\pi/m)}{(t+\pi/m)^2}\right]2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt\\&+\int_{\pi-\frac{\pi}{m}}^{\pi}\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt\\&-\frac{1}{rm\pi}\int_0^{\frac{\pi}{m}}\psi(x;t+\pi/m)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt:=P_{00}+P_{01}-P_{02}.\end{aligned}\tag{3.9}$$

The quantity P_{00} can be rewritten as follows

$$\begin{aligned}P_{00}&=\frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\left(\psi(x;t)-\psi(x;t+\pi/m)\right)\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt\\&+\frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\psi(x;t+\pi/m)\left(\frac{1}{t^2}-\frac{1}{(t+\pi/m)^2}\right)2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt\end{aligned}$$

(3.10)

$$:=P_{000}+P_{001}.$$

Now, applying Lemma 2.1 we have

$$\begin{aligned}\|P_{000}\|_{p;\beta}&=\frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|\psi(x;t)-\psi(x;t+\pi/m)\|_{p;\beta}\frac{\left|\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|dt}{t^2}\\&=\frac{\mathcal{O}(1)}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\left(w_1(t)+w_1(t+\pi/m)\right)\frac{dt}{t^2}\\&=\frac{\mathcal{O}(1)}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^2}dt=\frac{w_2(\pi)}{r\pi^2}\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right)=\mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}\right)\end{aligned}\tag{3.11}$$

since $\left|\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\le 1$, $w_1(t+\pi/m)\le 2w_1(t)$ for $t\ge\pi/m$, and condition (3.2).

By the same manner, we obtain

$$
\begin{aligned}
\|P_{001}\|_{p;\beta} &= \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi} \|\psi(x;t+\pi/m)\|_{p;\beta}\left|\frac{1}{t^2}-\frac{1}{(t+\pi/m)^2}\right| dt \\
&= \frac{\mathcal{O}(1)}{rm\pi}\int_{\frac{\pi}{m}}^{\pi} \left(w_1(t+\pi/m)\right)\frac{(t+\pi/m)^2-t^2}{t^2(t+\pi/m)^2}dt \\
&= \frac{\mathcal{O}(1)}{rm^2}\int_{\frac{\pi}{m}}^{\pi} w_1(t)\frac{2t+\pi/m}{t^2(t+\pi/m)^2}dt \\
&= \frac{\mathcal{O}(1)}{rm^2}\int_{\frac{\pi}{m}}^{\pi} \frac{w_1(t)}{t^3w_2(t)}w_2(t)dt = \frac{\mathcal{O}(1)}{rm^2}\int_{\frac{\pi}{m}}^{\pi} \frac{w_1(t)}{t^3w_2(t)}dt. \qquad (3.12)
\end{aligned}
$$

Thus, from (3.11) and (3.12), we get

$$
\|P_{00}\|_{p;\beta} = \mathcal{O}(1)\left(\frac{w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}+\frac{1}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right). \tag{3.13}
$$

Since

$$
\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right| \le rmt, \quad t\in(0,\pi]
$$

then

$$
\begin{aligned}
\|P_{02}\|_{p;\beta} &= \frac{\mathcal{O}(1)}{rm\pi}\int_0^{\frac{\pi}{m}} \|\psi(x;t+\pi/m)\|_{p;\beta}\frac{rmtdt}{(t+\pi/m)^2} \\
&= \frac{\mathcal{O}(1)}{\pi}\int_0^{\frac{\pi}{m}}\frac{w_1(t)}{t}dt = \mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2\left(\frac{\pi}{m}\right) \\
&= \mathcal{O}(1)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}w_2(\pi) = \mathcal{O}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}\right) \qquad (3.14)
\end{aligned}
$$

taking into account the condition (3.1).

Moreover, applying Lemma 2.1, we have

$$
\begin{aligned}
\|P_{01}\|_{p;\beta} &= \frac{1}{rm\pi}\int_{\pi-\frac{\pi}{m}}^{\pi} \|\psi(x;t+\pi/m)\|_{p;\beta}\frac{dt}{(t+\pi/m)^2} \\
&= \frac{\mathcal{O}(1)}{rm\pi}\int_{\pi-\frac{\pi}{m}}^{\pi}\frac{w_1(t+\pi/m)}{(t+\pi/m)^2w_2(t+\pi/m)}w_2(t+\pi/m)dt \\
&= \frac{\mathcal{O}(2w_2(\pi))}{rm\pi}\int_{\pi}^{\pi+\frac{\pi}{m}}\frac{w_1(t)}{t^2w_2(t)}dt = \mathcal{O}\left(\frac{1}{rm^2}\right). \qquad (3.15)
\end{aligned}
$$

So, from (3.9) we obtain

$$
\|P_0\|_{p;\beta} = \mathcal{O}(1)\left(\frac{1}{rm^2}+\left(1+\frac{1}{r}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right) \tag{3.16}
$$

based on (3.13), (3.14), and (3.15).

Combining (3.5), (3.7), (3.8), and (3.16) we get

$$
\|\widetilde{\tau}_m(x)\|_{p;\beta} = \mathcal{O}(1)\left(\frac{1}{rm^2}+\left(1+\frac{1}{r}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{rm^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right). \tag{3.17}
$$

Now we are going to estimate the second term in (3.4). For the sake of the interested reader, we will sketch the proof in details and along the same lines. Based on (3.3) we have

$$\widetilde{\tau}_m(x+y)-\widetilde{\tau}_m(x-y)=\frac{2}{rm\pi}\int_0^{\pi}\left[\psi(x+y;t)-\psi(x-y;t)\right]w_{rm}(t)dt. \tag{3.18}$$

We split the integral as follows

$$\begin{aligned}
&\|\widetilde{\tau}_m(x+y)-\widetilde{\tau}_m(x-y)\|_{p;\beta}\\
&\quad\le\frac{2}{rm\pi}\int_0^{\frac{\pi}{m}}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}\,|w_{rm}(t)|\,dt\\
&\qquad+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}\\
&\qquad\times\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|\left|\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}\right|dt\\
&\qquad+\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\|\psi(x+y;t)-\psi(x-y;t)\|_{p;\beta}\frac{\left|2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\right|}{t^2}dt\\
&\quad:=R_1+R_2+R_3.
\end{aligned}\tag{3.19}$$

Inasmuch as

$$\left|\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{\left(2\sin\frac{t}{2}\right)^2}\right|=\mathcal{O}\left(rm\right),\quad t\in(0,\pi),$$

then using Lemma 2.4 (c), we have

$$R_1=\mathcal{O}\left(w_2(y)\right)\int_0^{\frac{\pi}{m}}\frac{w_1(t)}{w_2(t)}dt=\mathcal{O}\left(\frac{w_2(y)}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}. \tag{3.20}$$

The function

$$t\to\frac{1}{4\sin^2\frac{t}{2}}-\frac{1}{t^2}$$

is bounded for $t\in[\pi/(n+1),\pi]$, therefore and using Lemma 2.4 (c) we get

$$R_2=\mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{w_2(t)}dt=\mathcal{O}\left(\frac{w_2(y)}{rm}\right). \tag{3.21}$$

Taking into account that r is an even positive integer number and substituting t with $t+\pi/m$ in

$$R_3^{(0)}:=\frac{2}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\left[\psi(x+y;t)-\psi(x-y;t)\right]\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{t^2}dt$$

we substitute t with $t+\pi/m$ and since r is an even natural number, we get

$$R_3^{(0)} = -\frac{2}{rm\pi}\int_0^{\pi-\frac{\pi}{m}}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \times \frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt = -\frac{2}{rm\pi}\left(\int_0^{\frac{\pi}{m}}+\int_{\frac{\pi}{m}}^{\pi}-\int_{\pi-\frac{\pi}{m}}^{\pi}\right) \times[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)]\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt$$

Adding these two latest equalities we have

$$\begin{aligned} R_3^{(0)} &= \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\Bigg[\frac{\psi(x+y;t)-\psi(x-y;t)}{t^2} \\ &\quad -\frac{\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)}{(t+\pi/m)^2}\Bigg]2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt \\ &\quad +\frac{1}{rm\pi}\int_{\pi-\frac{\pi}{m}}^{\pi}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \\ &\quad \times\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt \\ &\quad -\frac{1}{rm\pi}\int_0^{\frac{\pi}{m}}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \\ &\quad \times\frac{2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}}{(t+\pi/m)^2}dt := R_3^{(00)}+R_3^{(01)}-R_3^{(02)}. \end{aligned} \tag{3.22}$$

For $R_3^{(00)}$ we can write

$$\begin{aligned} R_3^{(00)} &= \frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}\frac{\psi(x+y;t)-\psi(x-y;t)-\psi(x+y;t+\pi/m)+\psi(x-y;t+\pi/m)}{t^2} \\ &\quad \times 2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}dt \\ &+\frac{1}{rm\pi}\int_{\frac{\pi}{m}}^{\pi}[\psi(x+y;t+\pi/m)-\psi(x-y;t+\pi/m)] \\ &\quad \times 2\sin\frac{rmt}{2}\cos\frac{(r+2)mt}{2}\left(\frac{1}{t^2}-\frac{1}{(t+\pi/m)^2}\right)dt := R_{31}^{(00)}+R_{32}^{(00)}. \end{aligned} \tag{3.23}$$

Now, using Lemma 2.1, Lemma 2.4 (d), and condition (3.2), we get

$$\begin{aligned} \|R_{31}^{(00)}\|_{p;\beta} &= \mathcal{O}\left(\frac{w_2(y)}{rm}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^2w_2(t)}dt\right) \\ &= \mathcal{O}\left(\frac{w_2(y)}{rmw_2\left(\frac{\pi}{m}\right)}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^2}dt = \mathcal{O}\left(\frac{w_2(y)w_1\left(\frac{\pi}{m}\right)}{rw_2\left(\frac{\pi}{m}\right)}\right). \end{aligned} \tag{3.24}$$

Moreover, using Lemma 2.1 and Lemma 2.4 (c), we also get

$$\begin{aligned} \|R_{32}^{(00)}\|_{p;\beta} &= \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1\left(t+\frac{\pi}{m}\right)}{w_2\left(t+\frac{\pi}{m}\right)}\frac{2t+\pi/m}{t^2(t+\pi/m)^2}dt \\ &= \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{2w_1\left(t\right)}{w_2\left(t\right)}\frac{3t}{t^4}dt = \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right)\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt. \end{aligned} \tag{3.25}$$

Whence, using (3.23), (3.24), and (3.25), we obtain

$$\begin{aligned} \|R_{3}^{(00)}\|_{p;\beta} &= \mathcal{O}(1)\left(\|R_{32}^{(00)}\|_{p;\beta}+\|R_{32}^{(00)}\|_{p;\beta}\right) \\ &= \mathcal{O}\left(\frac{w_2(y)}{r}\left(\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right). \end{aligned} \tag{3.26}$$

By similar reasoning we also have

$$\begin{aligned} \|R_{3}^{(01)}\|_{p;\beta} &= \mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{\pi-\frac{\pi}{m}}^{\pi}\frac{w_1\left(t+\frac{\pi}{m}\right)}{w_2\left(t+\frac{\pi}{m}\right)(t+\pi/m)^2}dt \\ &= \mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{\pi}^{\pi+\frac{\pi}{m}}\frac{w_1(t)}{t^2w_2(t)}dt = \mathcal{O}\left(\frac{w_2(y)}{rm^2}\right). \end{aligned} \tag{3.27}$$

Once more, using Lemma 2.1 and Lemma 2.4 (c), we proceed as follows

$$\begin{aligned} \|R_{3}^{(02)}\|_{p;\beta} &= \mathcal{O}\left(\frac{w_2(y)}{rm}\right)\int_{0}^{\frac{\pi}{m}}\frac{w_1\left(t+\frac{\pi}{m}\right)}{w_2\left(t+\frac{\pi}{m}\right)}rmdt \\ &= \mathcal{O}\left(w_2(y)\right)\int_{0}^{\frac{\pi}{m}}\frac{2w_1\left(t\right)}{w_2\left(t\right)}dt = \mathcal{O}\left(\frac{w_2(y)w_1\left(\frac{\pi}{m}\right)}{mw_2\left(\frac{\pi}{m}\right)}\right). \end{aligned} \tag{3.28}$$

Now, from (3.22), (3.26), (3.27), and (3.28), we find that

$$\begin{aligned} \|R_3\|_{p;\beta} &= \mathcal{O}(1)\left(\|R_{3}^{(00)}\|_{p;\beta}+\|R_{3}^{(01)}\|_{p;\beta}+\|R_{3}^{(02)}\|_{p;\beta}\right) \\ &= \mathcal{O}\left(\frac{w_2(y)}{r}\left(\frac{1}{m^2}+\left(1+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right). \end{aligned} \tag{3.29}$$

Thus, using (3.19), (3.20), (3.21), and (3.29), we have

$$\begin{aligned} &\frac{\|\widetilde{\tau}_m(x+y)-\widetilde{\tau}_m(x-y)\|_{p;\beta}}{w_2(y)} \\ &\qquad = \mathcal{O}\left(\frac{1}{r}\left(\frac{1}{m}+\left(r+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right). \end{aligned} \tag{3.30}$$

Finally, inserting (3.17) and (3.30) in (3.4), we obtain

$$\|\widetilde{\tau}_m(x)\|_{p;\beta}^{(w_2)} = \mathcal{O}\left(\frac{1}{r}\left(\frac{1}{m}+\left(1+r+\frac{1}{m}\right)\frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)}+\frac{1}{m^2}\int_{\frac{\pi}{m}}^{\pi}\frac{w_1(t)}{t^3w_2(t)}dt\right)\right).$$

The proof is completed. □

Next, from the main result we are going to extract only one of its particular case. To begin with, for $\gamma \in (0,1]$ and $w(t) = |t|^{\gamma}$ in $H_{p;\beta}^{(w)}$ class, we obtain the Hölder class with weight

$$H_{p;\beta}^{(\gamma)} := \left\{ f \in L_{\beta}^{p}[0,2\pi] : \sup_{t\neq 0} \frac{\|f(x+t)-f(x-t)\|_{p;\beta}}{|t|^{\gamma}} < \infty \right\}$$

endowed with the norm

$$\|f\|_{p;\beta}^{(\gamma)} := \|f\|_{p;\beta} + \sup_{t\neq 0} \frac{\|f(x+t)-f(x-t)\|_{p;\beta}}{|t|^{\gamma}},$$

where $1 \leq p < \infty$ and $\beta \geq 0$.

Remark 3.1. *Note that the functions* $\frac{w_1(t)}{w_2(t)} = |t|^{\gamma_1-\gamma_2}$ *is positive and non-decreasing, and* $\frac{w_1(t)}{tw_2(t)} = |t|^{\gamma_1-\gamma_2-1}$ *is positive and non-increasing for* $0 \leq \gamma_2 < \gamma_1 \leq 1$, *and* $t \in (0,\pi]$. *Furthermore, the condition (3.1) and (3.2) automatically are satisfied. It worth to mentioned here that is said* $w_1(t)$ *to be of the first kind (see* [20]*) if it satisfies condition (3.2).*

Corollary 3.1. *Let* $w_1(t) = |t|^{\gamma_1}$, $w_2(t) = |t|^{\gamma_2}$ *and* $0 \leq \gamma_2 < \gamma_1 \leq 1$. *Additionally, let* $f \in H_{p;\beta}^{(\gamma_1)}$ *be a* 2π*-periodic function and Lebesgue integrable on* $[0,2\pi]$, *with* $p \geq 1$ *and* $\beta \geq 0$. *Then for the function* $\widetilde{f}$, *conjugate to the function* f, *and for an even positive integer* r

$$\|\widetilde{\sigma}_{m,rm}(f) - \widetilde{f}\|_{p;\beta}^{(\gamma_2)} = \mathcal{O}\left(\left(3 + \frac{1}{r}\right) \frac{1}{m^{\gamma_1-\gamma_2}} \right).$$

Proof. Since the function $\frac{w_1(t)}{tw_2(t)} = |t|^{\gamma_1-\gamma_2-1}$ is positive and non-increasing for $0 \leq \gamma_2 < \gamma_1 \leq 1$, then we have

$$\begin{aligned}
\|\widetilde{\tau}_m(x)\|_{p;\beta}^{(w_2)} &= \mathcal{O}\left(\frac{1}{r} \left(\frac{1}{m} + \frac{1}{m^2} + \left(1 + r + \frac{1}{m}\right) \frac{w_1\left(\frac{\pi}{m}\right)}{w_2\left(\frac{\pi}{m}\right)} + \frac{1}{m^2} \int_{\frac{\pi}{m}}^{\pi} \frac{w_1(t)}{t^3 w_2(t)} dt \right) \right) \\
&= \mathcal{O}\left(\frac{1}{r} \left(\frac{2}{m} + (2+r) \frac{\left(\frac{\pi}{m}\right)^{\gamma_1}}{\left(\frac{\pi}{m}\right)^{\gamma_2}} + \frac{1}{m^2} \frac{\left(\frac{\pi}{m}\right)^{\gamma_1}}{\frac{\pi}{m}\left(\frac{\pi}{m}\right)^{\gamma_2}} \int_{\frac{\pi}{m}}^{\pi} \frac{dt}{t^2} \right) \right) \\
&= \mathcal{O}\left(\frac{1}{r} \left(\frac{2}{m} + \frac{2r}{m^{\gamma_1-\gamma_2}} + \frac{1}{m^{1+\gamma_1-\gamma_2}} \left(\frac{m}{\pi} - \frac{1}{\pi} \right) \right) \right) \\
&= \mathcal{O}\left(\frac{1}{r} \left(\frac{3r}{m^{\gamma_1-\gamma_2}} + \frac{1}{m^{\gamma_1-\gamma_2}} \right) \right) = \mathcal{O}\left(\left(3 + \frac{1}{r}\right) \frac{1}{m^{\gamma_1-\gamma_2}} \right).
\end{aligned}$$

The proof has ended. □

4. Conclusion

Using the even-type delayed mean of conjugate series, we have obtained the degree of approximation for a conjugate function in the metric of generalized Höder class with weight. Involving two moduli of continuity and two condition on them, we have shown that this mean are streamlined to guarantee this degree to be of the Jackson order.

Список литературы

[1] P. Chandra, "On the generalised Fejér means in the metric of Hölder space", Math. Nachr. **109**, 39 – 45 (1982).

[2] G. Das, T. Ghosh and B. K. Ray, "Degree of approximation of functions by their Fourier series in the generalized Hölder metric", Proc. Indian Acad. Sci. Math. Sci. **106**, no. 2, 139 – 153 (1996).

[3] G. Das, A. Nath, B. K. Ray, "An estimate of the rate of convergence of Fourier series in the generalized Hölder metric", Analysis and applications (Ujjain, 1999), 43 – 60, Narosa, New Delhi (2002).

[4] U. Değer and M. Kücükaslan, "A generalization of deferred Cesàro means and some of their applications", J. Inequal. Appl., 2015:14, 16 pp. (2015).

[5] U. Değer, "A note on the degree of approximation by matrix means in the generalized Hölder metric", Ukr. Math. J. **68**, 545 – 556 (2016).

[6] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, London (1967).

[7] K. Khatri and V. N. Mishra, "Degree of approximation by the $(T.E^1)$ means of conjugate series of Fourier series in the Hölder metric", Iran. J. Sci. Technol. Trans. A Sci. **43**, no. 4, 1591 – 1599 (2019).

[8] J. Kim, "Degree of approximation in the space H_p^ω by the even-type delayed arithmetic mean of Fourier series", Georgian Math. J., **28**, no. 5, 747 – 753 (2021).

[9] Xh. Z. Krasniqi, "On the degree of approximation by Fourier series of functions from the Banach space $H_p^{(\omega)}$, $p \geq 1$, in generalized Hölder metric", Int. Math. Forum **6**, no. 13 – 16, 613 – 625 (2011).

[10] Xh. Z. Krasniqi and B. Szal, "On the degree of approximation of continuous functions by means of Fourier series in the Hölder metric", Anal. Theory Appl. **35**, no. 4, 392 – 404 (2019).

[11] Xh. Z. Krasniqi, "Approximation by sub-matrix means of multiple Fourier series in the Hölder metric", Palest. J. Math. **9**, no. 2, 761 – 770 (2020).

[12] Xh. Z. Krasniqi, W. Lenski, and B. Szal, "Trigonometric approximation of functions in seminormed spaces", Results Math. **77**, no. 4, no. 145 (2022).

[13] L. Leindler, "Generalizations of Prössdorf's theorems", Studia Sci. Math. Hungar. **14**, no. 4, 431 – 439 (1979).

[14] L. Leindler, "A relaxed estimate of the degree of approximation by Fourier series in generalized Hölder metric", Anal. Math. **35**, no. 1, 51 – 60 (2009).

[15] W. Lenski, U. Singh, and B. Szal, "Trigonometric approximation of functions in seminormed spaces", Math. Inequal. Appl. **24**, no. 1, 89 – 101 (2021).

[16] B. London, H. Carley and R. N. Mohapatra, "Approximation by the K^λ means of Fourier series and conjugate series of functions in $H_{\alpha,p}$", Appl. Anal. Discrete Math. **14**, no. 3, 800 – 818 (2020).

[17] R. N. Mohapatra and P. Chandra, "Degree of approximation of functions in the Hölder metric", Acta Math. Hungar. **41**, no. 1 - 2, 67 – 76 (1983).

[18] V. N. Mishra and K. Khatri, "Degree of approximation of functions $\tilde{f} \in H_\omega$ class by the $(N_p \cdot E^1)$ means in the Hölder metric", Int. J. Math. Math. Sci. **2014**, Art. ID 837408, 9 pp. (2014).

[19] M. L. Mittal and B. E. Rhoades, "Degree of approximation to functions in a normed space", J. Comput. Anal. Appl. **2**, no. 1, 1 – 10 (2000).

[20] F. Móricz and B. E. Rhoades, "Approximation by Nörlund means of double Fourier series to continuous functions in two variables", Constr. Approx. **3**, no. 3, 281 – 296 (1987).

[21] L. Nayak, G. Das, and B. K. Ray, "An estimate of the rate of convergence of Fourier series in the generalized Hölder metric by deferred Cesàro mean", J. Math. Anal. Appl. **420**, no. 1, 563 – 575 (2014).

[22] L. Nayak, G. Das, and B. K. Ray, "An estimate of the rate of convergence of the Fourier series in the generalized Hölder metric by delayed arithmetic mean", Int. J. Anal., Art. ID 171675, 7 pp. (2014).

[23] H. K. Nigam and Md. Hadish, "Best approximation of functions in generalized Hölder class", J. Inequal. Appl. No. 276, 15 pp. (2018).

[24] T. Pradhan et al., “On approximation of the rate of convergence of Fourier series in the generalized Hölder metric by deferred Nörlund mean”, Afr. Mat. **30**, no. 7 - 8, 1119 – 1131 (2019).

[25] S. Prössdorf, “Zur Konvergenz der Fourierreihen hölderstetiger Funktionen” [in German], Math. Nachr. **69**, 7 – 14 (1975).

[26] T. Singh, “Degree of approximation to functions in a normed space”, Publ. Math. Debrecen **40**, no. 3 – 4, 261 – 271 (1992).

[27] T. Singh, S. K. Jain, and P. Mahajan, “Error bound of periodic signal in normed space by deferred Cesàro-transform”, Aligarh Bull. Math. **27**, no. 2, 101 – 107 (2008).

[28] T. Singh and P. Mahajan, “Error bound of periodic signals in the Hölder metric”, Int. J. Math. Math. Sci., Art. ID 495075, 9 pp. (2008).

[29] U. Singh and S. Sonker, “Degree of approximation of function $f \in H_p^{(w)}$ class in generalized Hölder metric by matrix means”, Mathematical modelling and scientific computation, 1 – 10, Commun. Comput. Inf. Sci., **283**, Springer, Heidelberg (2012).

[30] U. Singh and A. Rathore, “A note on the degree of approximation of functions belonging to certain Lipschitz class by almost Riesz means”, Stud. Univ. Babeş–Bolyai Math. **63**, no. 3, 371 – 379 (2018).

[31] U. Singh and S. Saini, “Approximation of periodic functions in certain sub-classes of $L^p[0, 2\pi]$”, Asian-Eur. J. Math. **10**, no. 3, 1750046, 12 pp. (2017).

[32] U. Singh, “On the trigonometric approximation of functions in a weighted Lipschitz class”, J. Anal. **29**, 325 – 335 (2021).

[33] C. de la Vallée Poussin, “Sur la meilleure approximation des fonctions d’une variable réelle par des expressions d’ordre donné”, C. R. Acad. Sci. Paris **166**, 799 – 802 (1918).

[34] A. Zygmund, Trigonometric series, Cambridge Univ. Press, Cambridge, 3rd rev. ed. (2002).

Поступила 12 сентября 2022

После доработки 12 февраля 2023

Принята к публикации 21 февраля 2023