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OPERATOR PRESERVING BERNSTEIN-TYPE INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. In this paper, we establish some operator preserving inequalities of Bernstein-type in the uniform-norm between univariate complex coefficient polynomials while taking into account the placement of their zeros. The obtained results produce a variety of interesting results as special cases.

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1. Introduction

Let $P_n := \{P \in \mathbb{C}[z];\ \deg P \leq n\}$ be the space of all univariate complex coefficient polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n and let $P'(z)$ be the derivative of $P(z)$. The famous Bernstein inequality for the uniform-norm on the unit circle states that if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq nM_1, \tag{1.1}$$

where here and throughout $M_1 = \max_{|z|=1} |P(z)|$ is the uniform-norm of P on the unit circle. On the other hand, concerning the maximum modulus of $P(z)$ on the circle $|z| = R \geq 1$, we have

$$\max_{|z|=R} |P(z)| \leq R^n M_1. \tag{1.2}$$

Inequality (1.1) is due to Bernstein [4], while as inequality (1.2) is a simple deduction from the Maximum Modulus Principle, for reference see ([16], page 346). Equality holds in (1.1) and (1.2) if and only if $P(z)$ is a non-zero multiple of z^n. For $P \in P_n$, it is known (see [2] and [12]) that

$$|P(Rz)| + |Q(Rz)| \leq (R^n + 1)M_1, \tag{1.3}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

For the class of polynomials $P \in P_n$ not vanishing in the interior of the unit circle, the inequalities (1.1) and (1.2) have been respectively replaced by the following inequalities:

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}M_1 \tag{1.4}$$

and

$$\max_{|z|=R}|P(z)| \leq \frac{R^n+1}{2}M_1, \quad R \geq 1. \tag{1.5}$$

Inequalities (1.4) and (1.5) are sharp and equality holds for polynomials having all their zeros on the unit circle. As is well known, inequality (1.4) was conjectured by Erdös and later proved by Lax [8], while inequality (1.5) is due to Ankeny and Rivlin [1]. As an generalization of (1.3), Jain [6] proved the following interesting result:

Theorem A. If $P \in P_n$, then for every β with $|\beta| \leq 1$, $R \geq 1$ and $|z| = 1$, we have

$$\begin{aligned} &\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| + \left|Q(Rz) + \beta\left(\frac{R+1}{2}\right)^n Q(z)\right| \\ &\leq \left\{\left|1 + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|\right\}M_1. \end{aligned} \tag{1.6}$$

In 2012, Zireh ([17], Lemma 2.6) proved a more general result, which in particular gives the following generalization of (1.6).

Theorem B. If $P \in P_n$, then for every β with $|\beta| \leq 1$, $R > r \geq k,\ k \leq 1$ and $|z| = 1$, we have

$$\begin{aligned} &\left|P(Rz) + \beta\left(\frac{R+k}{r+k}\right)^n P(rz)\right| + \left|Q(Rz) + \beta\left(\frac{R+k}{r+k}\right)^n Q(rz)\right| \\ &\leq \left\{\left|1 + \beta\left(\frac{R+k}{r+k}\right)^n\right| + k^{-n}\left|R^n + \beta\left(\frac{R+k}{r+k}\right)^n\right|\right\}M_1, \end{aligned}$$

where $Q(z) = \left(\frac{z}{k}\right)^n \overline{P\left(\frac{k^2}{\overline{z}}\right)}$.

It is topical in the geometric function theory to study the extremal problems of functions of a complex variable and generalizing the classical polynomial inequalities in various directions. Although the literature on polynomial inequalities is vast and growing and over the years, many authors produced an abundance of various versions and generalizations of the above inequalities by introducing various operators that preserve such type of inequalities between polynomials (for example, see [5], [11] and [12]). It is an interesting problem, as pointed out by Rahman to characterize all such operators, and as part of this characterization Rahman in [12] (see also [9] or Rahman and Schmeisser [[14], pp. 538-551]) introduced a class B_n of operators B that maps $P \in P_n$ to $B[P] \in P_n$.

The class of B_n-operators: For fixed $n \in \mathbb{N}$, Marden ([9], pp. 65) in 1966 defined and studied the differential operator B that to each polynomial $P(z)$ of degree at most n assigns the polynomial

$$B[P](z) := \lambda_0 P(z) + \lambda_1\Big(\frac{nz}{2}\Big)\frac{P'(z)}{1!} + \lambda_2\Big(\frac{nz}{2}\Big)^2\frac{P''(z)}{2!},$$

where $\lambda_0,\ \lambda_1$ and λ_2 are such that all the zeros of

$$\phi(z) = \lambda_0 + \binom{n}{1}\lambda_1 z + \binom{n}{2}\lambda_2 z^2$$

lie in the half plane

$$Re(z) \leq \frac{n}{4}. \tag{1.7}$$

In fact, Marden proved that this operator preserves the zeros of the polynomial in a closed disk, i.e., if all the zeros of $P(z)$ lie in the closed unit disk, then all the zeros of $B[P](z)$ also lie in the same disk. Usually, such operators are called B_n-operators (see [14], page 538) and were also extensilvely studied by Rahman [12]. For more information regarding the B_n-operators (see [10], [13] and [14]). The study of such operators preserving inequalities between polynomials in the geometric function theory is a problem of interest both in mathematics and in the application areas such as physical systems. In addition to having numerous applications, this study has been the inspiration for much more research both from the theoretical point of view, as well as from the practical point of view. Recently, Rather et al. [15] considered the generalized B_n-operator N_v which carries $P \in P_n$ into $N_v[P] \in P_n$ defined by

$$N_v[P](z) := \sum_{v=0}^{m} \lambda_v \left(\frac{nz}{2}\right)^v \frac{P^{(v)}(z)}{v!}, \tag{1.8}$$

where $\lambda_v;\ v = 0, 1, 2, ..., m$, are such that the zeros of the polynomial

$$\phi_v(z) = \sum_{v=0}^{m} \binom{n}{v} \lambda_v z^v,\ m \leq n, \tag{1.9}$$

lie in the half plane (1.7).

It is easy to observe that if we take $\lambda_v = 0$ in (1.8) and (1.9) for $3 \leq v \leq m$, then N_v reduces to the B-operator. They established certain results concerning the upper bound of $|N_v[P]|$ for $|z| \geq 1$. More precisely, they proved the following results:

Theorem C. If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then

$$|N_v[P](z)| \leq |N_v[f](z)| \quad \text{for } |z| \geq 1. \tag{1.10}$$

Equality in (1.10) holds for $P(z) = e^{i\gamma} f(z),\ \gamma \in \mathbb{R}$.

Theorem D. If $P \in P_n$, and $P(z) \neq 0$ in $|z| < 1$, then

$$|N_v[P](z)| \leq \frac{1}{2}\Big\{\big|N_v[\rho_n](z)\big| + |\lambda_0|\Big\} M_1 \quad \text{for } |z| \geq 1, \tag{1.11}$$

where here and throughout $\rho_n = z^n$.
Equality in (1.11) holds for $P(z) = az^n + b,\ |a| = |b| \neq 0$.
Very recently, Mir [11] obtained the following generalizations of the above inequalities by considering a more general problem of investigating the dependence of $|N_v[P(Rz)] - \beta N_v[P(rz)]|$ on the maximum of $|P(z)|$ on $|z| = 1$ for every $|\beta| \leq 1$, $R \geq r \geq 1$, and developed a unified method for arriving at these results. More precisely, Mir proved the following results:

Theorem E. If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and if $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then for any complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$\Big|N_v[P](Rz) - \beta N_v[P](rz)\Big| \leq \Big|N_v[f](Rz) - \beta N_v[f](rz)\Big| \text{ for } |z| \geq 1. \tag{1.12}$$

Equality in (1.12) holds for $P(z) = e^{i\alpha} f(z), \alpha \in \mathbb{R}$.

Theorem F. If If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$\begin{aligned}
&\Big|N_v[P](Rz) - \beta N_v[P](rz)\Big| \\
&\qquad \leq \frac{1}{2}\Big\{\Big(|R^n - \beta r^n||N_v[\rho_n](z)| + |1-\beta||\lambda_0|\Big)M_1 \\
&\qquad - \Big(|R^n - \beta r^n||N_v[\rho_n](z)| - |1-\beta||\lambda_0|\Big)m_1\Big\} \text{ for } |z| \geq 1,
\end{aligned} \tag{1.13}$$

where here and throughout $m_1 = \min_{|z|=1} |P(z)|$.
Equality in (1.13) holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta| \neq 0$.
The Bernstein-type inequalities are seminal in the field of classical analysis, and over a period, these inequalities have been studied for different operators, in different norms, and for different classes of functions. The present paper is mainly motivated by the desire to establish some new inequalities concerning the N_v-operator in the uniform-norm between polynomials, which in turn yield compact generalizations of inequalities (1.10)-(1.13) and other related results. The essence in the papers of Jain ([6], [7]) and Zireh [17] is the origin of thought for the new inequalities presented in this paper.

2. Main results

In this section, we state our main results. Their proofs are given in the next section. We begin by proving the following inequality giving compact generalizations of Theorems A and B.

Theorem 2.1. *If $P \in P_n$, then for $|\beta| \leq 1$, $R > r \geq k$, $k > 0$, and $|z| \geq 1$ with $Q(z) = z^n\overline{P(1/\overline{z})}$ we have*

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| + k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|$$

$$\leq \left[\frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right| + \left|1 + \beta\left(\frac{R+k}{r+k}\right)^n\right||\lambda_0|\right]M_k, \tag{2.1}$$

where here and throughout $M_k = \max_{|z|=k}|P(z)|$.

Remark 2.1. *One can observe that Theorem 2.1 provides an interesting generalization of Theorem A. For instance, if in* (2.1)*, after substituting the value of $N_v[\rho_n](z)$ and taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$, and noting that $N_v[P](z) = \lambda_0 P(z)$, we get Theorem A as a special case when $k = r = 1$.*

Theorem 2.2. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \leq k$, $k > 0$, then for every $|\beta| \leq 1$ and $R > r \geq k$, we have*

$$\min_{|z|=1}\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \geq \frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right|m_k, \tag{2.2}$$

where here and throughout $m_k = \min_{|z|=k}|P(z)|$.

As in remark 2.1, after substituting the value of $N_v[\rho_n](z)$ in (2.2) and taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$, and noting that $N_v[P](z) = \lambda_0 P(z)$, we get a result of (Zireh [17], when $\alpha = 0$), see also Dewan and Hans ([5], Theorem 1).

Theorem 2.3. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq k$, $k \leq 1$ then for every $|\beta| \leq 1$ and $R > r \geq k$, we have for $|z| \geq 1$,*

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq \frac{1}{2}\left[\frac{1}{k^n}\left|R^n + \beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right| + |\lambda_0|\left|1 + \beta\left(\frac{R+k}{r+k}\right)^n\right|\right]M_k. \tag{2.3}$$

Remark 2.2. *By taking $\beta = 0$ and $k = 1$, Theorem 2.3 in particular gives Theorem D and for suitable choices of λ_v; $0 \leq v \leq m$, it yields inequalities* (1.4) *and* (1.5) *as well.*

The above inequality (2.3) will be a consequence of a more fundamental inequality presented by the following theorem.

Theorem 2.4. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq k,\ k \leq 1$ then for every $|\beta| \leq 1$ and $R > r \geq k$, we have*

$$\begin{aligned}\Big|N_v[P](Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^n N_v[P](rz)\Big| \leq \frac{1}{2}\Bigg[\Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\big|N_v[\rho_n](z)\big| \\ + |\lambda_0|\Big|1 + \beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)M_k - \Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\big|N_v[\rho_n](z)\big| \\ - |\lambda_0|\Big|1 + \beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)m_k\Bigg]. \end{aligned} \tag{2.4}$$

Equality in (2.4) holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$. We shall now discuss some consequences of Theorem 2.4. If in (2.4), after substituting the value of $N_v[\rho_n](z)$, we get for every $|\beta| \leq 1$ and $R > r \geq k$,

$$\begin{aligned}&\Big|N_v[P](Rz) + \beta\Big(\frac{R+k}{r+k}\Big)^n N_v[P](rz)\Big| \\ &\leq \frac{1}{2}\Bigg[\Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big||z|^n\Big|\sum_{v=0}^{m}\lambda_v \binom{n}{v}\Big(\frac{n}{2}\Big)^v\Big| \\ &+ |\lambda_0|\Big|1 + \beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)M_k \\ &- \Bigg(\frac{1}{k^n}\Big|R^n + r^n\beta\Big(\frac{R+k}{r+k}\Big)^n\Big||z|^n\Big|\sum_{v=0}^{m}\lambda_v \binom{n}{v}\Big(\frac{n}{2}\Big)^v\Big| \\ &- |\lambda_0|\Big|1 + \beta\Big(\frac{R+k}{r+k}\Big)^n\Big|\Bigg)m_k\Bigg] \quad \text{for } |z| \geq 1,\end{aligned} \tag{2.5}$$

where $\lambda_v;\ 0 \leq v \leq m$ are such that all the zeros of $\phi_v(z)$ defined by (1.9) lie in the half plane (1.7).

Remark 2.3. *Taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$ in* (2.5) *and noting that $N_v[P](z) = \lambda_0 P(z)$, we get the following result which is of independent interest, because besides giving generalizations and refinements of* (1.4) *and* (1.5) *it also provides generalizations and refinements of some results of Zireh* [17], *Dewan and Hans* [5] *and Jain (*[6], [7]*).*

Corollary 2.1. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq k,\ k \leq 1$ then for $|\beta| \leq 1,\ R > r \geq k$, we have*

$$\begin{aligned} \left|P(Rz)+\beta\left(\frac{R+k}{r+k}\right)^n P(rz)\right| \leq \frac{1}{2}\Bigg[&\Bigg(\frac{1}{k^n}\left|R^n+r^n\beta\left(\frac{R+k}{r+k}\right)^n\right||z|^n \\ &+\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right|\Bigg)M_k-\Bigg(\frac{1}{k^n}\left|R^n+r^n\beta\left(\frac{R+k}{r+k}\right)^n\right||z|^n \\ &-\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right|\Bigg)m_k\Bigg] \quad \text{for } |z| \geq 1. \end{aligned} \tag{2.6}$$

Equality in (2.6) *holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$.*

Self-inversive polynomial: A polynomial $P \in P_n$ is said to be self-inversive if $P(z) = \zeta Q(z),\ |\zeta| = 1$. Finally, we prove the following result for self-inversive polynomials.

Theorem 2.5. *If $P \in P_n$ is self-inversive, then for $|\beta| \leq 1$ and $R > r \geq 1$, we have for $|z| \geq 1$,*

$$\begin{aligned} \left|N_v[P](Rz)+\beta\left(\frac{R+1}{r+1}\right)^n N_v[P](rz)\right| \leq \frac{1}{2}\Bigg[&\left|R^n+r^n\beta\left(\frac{R+1}{r+1}\right)^n\right|\left|N_v[\rho_n](z)\right| \\ &+|\lambda_0|\left|1+\beta\left(\frac{R+1}{r+1}\right)^n\right|\Bigg]M_1. \end{aligned} \tag{2.7}$$

Equality in (2.7) *holds for $P(z) = z^n + 1$.*

Remark 2.4. *For $\beta = 0$, the above result in particular reduces to a result of Rather et al. (*[15], *Theorem* 1.4*). Taking $\lambda_v = 0$ for $v = 1, 2, 3, ..., m$ in* (2.7) *and noting that $N_v[P](z) = \lambda_0 P(z)$, we get the following result for self-inversive polynomials.*

Corollary 2.2. *If $P \in P_n$ is self-inversive, then for $|\beta| \leq 1$ and $R > r \geq 1$, we have for $|z| \geq 1$,*

$$\begin{aligned} \left|P(Rz)+\beta\left(\frac{R+1}{r+1}\right)^n P(rz)\right| \leq \frac{1}{2}\Bigg[&\left|R^n+r^n\beta\left(\frac{R+1}{r+1}\right)^n\right||z|^n \\ &+\left|1+\beta\left(\frac{R+1}{r+1}\right)^n\right|\Bigg]M_1. \end{aligned} \tag{2.8}$$

For $\beta = 0$, the inequality (2.8) *shows that the inequality* (1.5) *also holds for self-inversive polynomials.*

3. Auxiliary results

In order to prove our main results, we need the following lemmas.

Lemma 3.1. *([3]) If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 0$, then for every $R \geq r$ and $rR \geq k^2$,*

$$|P(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |P(rz)| \quad \text{for} \quad |z| = 1.$$

If we take $r = s = 1$ and $\sigma = \frac{n}{2}$ in Theorem 1.1 of Rather et al. [15], we get the following:

Lemma 3.2. *If all the zeros of polynomial $P \in P_n$ lie in $|z| \leq 1$, then all the zeros of $N_v[P(z)]$ defined by (1.8) also lie in $|z| \leq 1$.*

We now prove the following lemma from which we can obtain Theorem C as a special case.

Lemma 3.3. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k > 0$, and $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = k$, then for every $|\beta| \leq 1$, $R > r \geq k$ and for $|z| \geq 1$,*

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq \left|N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right|.$$

Proof of Lemma 3.3. By hypothesis $|P(z)| \leq |f(z)|$ for $|z| = k$, therefore any zero of $f(z)$ that lies on $|z| = k$ is also a zero of $P(z)$. On the other hand, for every $\zeta \in \mathbb{C}$ with $|\zeta| > 1$, we have $|P(z)| < |\zeta f(z)|$, for $|z| = k$, when all the zeros of $f(z)$ lie in $|z| < k$, it follows by Rouché's theorem that all the zeros of the polynomial $g(z) = P(z) - \zeta f(z)$ lie in $|z| \leq k$. On applying Lemma 3.1 to the polynomial $g(z)$, we have

$$|g(Rz)| > \left(\frac{R+k}{r+k}\right)^n |g(rz)| \quad \text{for} \quad |z| = k.$$

Since $g(Rz)$ has all its zeros in $|z| \leq \frac{k}{R} \leq 1$. Therefore, if β is any complex number such that $|\beta| \leq 1$, it follows that all the zeros of the polynomial $g(Rz) + \beta\left(\frac{R+k}{r+k}\right)^n g(rz)$ also lie in $|z| \leq 1$. Applying Lemma 3.2 and noting that N_v is a linear operator, we conclude that all the zeros of the polynomial

$$J(z) := N_v[g](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[g](rz)$$

lie in $|z| \leq 1$, for every $|\beta| \leq 1$ and $R > r \geq k$. Replacing $g(z)$ by $P(z) - \zeta f(z)$, we conclude that all the zeros of the polynomial

$$\begin{aligned} J(z) := {} & N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz) \\ & - \zeta\left[N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right] \end{aligned}$$

lie in $|z| \leq 1$ for all real or complex number β with $|\beta| \leq 1$ and $R > r \geq k$. This implies,

$$\begin{aligned} & \left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ & \leq \left|N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right| \quad \text{for } |z| \geq 1. \end{aligned} \tag{3.1}$$

If inequality (3.1) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\begin{aligned} & \left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ & \qquad > \left|N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)\right|. \end{aligned}$$

Taking

$$\zeta = \frac{N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)}{N_v[f](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[f](rz)},$$

so that $|\zeta| > 1$ and with this choice of ζ, we have $J(z_0) = 0$ for $|z_0| \geq 1$, which is a clear contradiction to the fact that $J(z) \neq 0$ for $|z| \geq 1$. Thus for every complex number β with $|\beta| \leq 1$ and $R > r \geq k$, we have (3.1) holds. This proves Lemma 3.3 completely.

Remark 3.1. *On applying Lemma 3.3 with $f(z) = M_k z^n / k^n$, giving us the following inequality:*

$$\begin{aligned} & \left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ & \leq \frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right| \big|N_v[\rho_n](z)\big| M_k \quad \textit{for } |z| \geq 1. \end{aligned} \tag{3.2}$$

Lemma 3.4. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k$, $k > 0$, then for every $|\beta| \leq 1$ and $R > r \geq k$, we have*

$$\begin{aligned} & \left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ & \leq k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right| \quad \textit{for } |z| \geq 1. \end{aligned}$$

Proof of Lemma 3.4. Since $P(z) \neq 0$ in $|z| < k$, therefore, all the zeros of polynomial $Q\big(z/k^2\big)$ lie in $|z| < k$. Also $|k^n Q(z/k^2)| = |P(z)|$ for $|z| = k$. Applying Lemma 3.3 to $P(z)$ with $f(z)$ replaced by $k^n Q(z/k^2)$, we get for every $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$\begin{aligned}&\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \\ &\qquad \leq k^n \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|.\end{aligned}$$

4. Proofs of main results

Proof of Theorem 2.1. Let $M_k = \max_{|z|=k} |P(z)|$, then by Rouché's theorem the polynomial $U(z) = P(z) - \zeta M_k$ has no zeros in $|z| < k$ for every $\zeta \in \mathbb{C}$ with $|\zeta| > 1$. On using Lemma 3.4 to $U(z)$, we have for $|\beta| \leq 1$ and $R > r \geq k$,

$$\begin{aligned}&\left|N_v[U](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[U](rz)\right| \\ &\qquad \leq k^n \left|N_v[L](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[L](rz/k^2)\right|,\end{aligned}$$

where $L(z) = z^n \overline{U(\frac{1}{\bar{z}})} = Q(z) - \bar{\zeta} z^n M_k$. Using $U(z) = P(z) - \zeta M_k$, $L(z) = Q(z) - \bar{\zeta} z^n M_k$, and the fact that N_v is linear and $N_v[1] = \lambda_0$, we get from above inequality for $|\beta| \leq 1$, $|z| \geq 1$ and $R > r \geq k$,

$$\begin{aligned}&\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz) - \lambda_0 \zeta M_k\left[1 + \beta\left(\frac{R+k}{r+k}\right)^n\right]\right| \\ &\leq k^n \Bigg|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2) \\ &\qquad - \frac{\bar{\zeta} M_k}{k^{2n}}\left[R^n + r^n \beta\left(\frac{R+k}{r+k}\right)^n\right] N_v[\rho_n](z)\Bigg|, \end{aligned} \tag{4.1}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Now choosing the argument of ζ suitably on the right hand side of (4.1) such that

$$\begin{aligned}&k^n \Bigg|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2) \\ &\qquad\qquad - \frac{\bar{\zeta} M_k}{k^{2n}}\left[R^n + r^n \beta\left(\frac{R+k}{r+k}\right)^n\right] N_v[\rho_n](z)\Bigg| \\ &= \frac{|\bar{\zeta}| M_k}{k^n}\left|R^n + r^n \beta\left(\frac{R+k}{r+k}\right)^n\right| \big|N_v[\rho_n](z)\big| \\ &\qquad\qquad - k^n \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|,\end{aligned}$$

which is possible by applying inequality (3.2) to the polynomial $Q(z/k^2)$ and using the fact that $\max_{|z|=k}\left|Q(z/k^2)\right| = M_k/k^n$, we get for $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| - |\lambda_0||\zeta|M_k\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right|$$
$$\leq \frac{|\bar{\zeta}|M_k}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right|$$
$$- k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right|.$$

The required result follows on making $|\zeta| \to 1$.

Proof of Theorem 2.2. Let $m_k = \min_{|z|=k}|P(z)|$. In case $m_k = 0$, there is nothing to prove. Assume that $m_k > 0$, so that all the zeros of $P(z)$ lie in $|z| < k$ and we have, $m_k\big|z/k\big|^n \leq |P(z)|$ for $|z| = k$. Applying Lemma 3.3 with $f(z)$ replaced by $m_k\big(z/k\big)^n$, we obtain for every $|\beta| \leq 1$ and $R > r \geq k$,

$$\min_{|z|=1}\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right|$$
$$\geq \frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right|m_k,$$

which is inequality (2.2). This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. The desired result immediately follows by combining Lemma 3.4 and Theorem 2.1.

Proof of Theorem 2.4. The result follows obviously in case $P(z)$ has a zero on $|z| = k$ (by Theorem 2.3). Therefore, we assume that $P(z)$ has all its zeros in $|z| > k$, so that $m_k = \min_{|z|=k}|P(z)| > 0$. Now for every real or complex number ζ with $|\zeta| < 1$, it follows by Rouché's theorem, that the polynomial $U(z) = P(z) - \zeta m_k$ does not vanish in $|z| < k$. On applying Lemma 3.4 to the polynomial $U(z)$ and noting that N_v is a linear operator with $N_v[1] = \lambda_0$, we get for every $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$\left|N_v[U](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[U](rz)\right|$$
$$\leq k^n\left|N_v[L](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[L](rz/k^2)\right|,$$

where $L(z) = z^n\overline{W(\frac{1}{\bar z})}$. Equivalently,

$$\begin{aligned}
&\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz) - \zeta\lambda_0 m_k\left[1+\beta\left(\frac{R+k}{r+k}\right)^n\right]\right| \\
&\leq k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right. \\
&\qquad\left. - \frac{\overline{\zeta} m_k}{k^{2n}}\left[R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right]N_v[\rho_n](z)\right| \text{ for } |z|\geq 1, \qquad (4.2)
\end{aligned}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar z})}$.

Now choosing the argument of ζ on the right hand side of (4.2) such that

$$\begin{aligned}
&k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right. \\
&\qquad\left. - \frac{\overline{\zeta} m_k}{k^{2n}}\left[R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right]N_v[\rho_n](z)\right| \\
&= k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right| \\
&\qquad - \frac{|\overline{\zeta}| m_k}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right|,
\end{aligned}$$

which is possible by Theorem 2.2 applied to $Q(z/k^2)$, we get

$$\begin{aligned}
&\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| - |\zeta||\lambda_0| m_k\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right| \\
&\leq k^n\left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[Q](rz/k^2)\right| \\
&\qquad - \frac{|\zeta|}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right| m_k \quad \text{for } |z|\geq 1.
\end{aligned}$$

This gives by letting $|\zeta|\to 1$,

$$\begin{aligned}
&\left|N_v[P](Rz) + \beta\left(\frac{R+k}{r+k}\right)^n N_v[P](rz)\right| \leq \left|N_v[Q](Rz/k^2) + \beta\left(\frac{R+k}{r+k}\right)^n\right| \\
&- \left[\frac{1}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{r+k}\right)^n\right|\left|N_v[\rho_n](z)\right| - |\lambda_0|\left|1+\beta\left(\frac{R+k}{r+k}\right)^n\right|\right] m_k. \qquad (4.3)
\end{aligned}$$

Inequality (4.3) in conjunction with Theorem 2.1 yields (2.4). This completes the proof of Theorem 2.4.

Proof of Theorem 2.5. By hypothesis $P\in P_n$ is self-inversive, therefore $P(z) = \zeta Q(z),\ |\zeta| = 1$. It gives for every $|\beta|\leq 1$, $R > r \geq 1$ and for all z,

$$\left|N_v[P](Rz) + \beta\left(\frac{R+1}{r+1}\right)^n N_v[P](rz)\right| = \left|N_v[Q](Rz) + \beta\left(\frac{R+1}{r+1}\right)^n N_v[Q](rz)\right|.$$

The above equality when combined with Theorem 2.1 (for $k = 1$) yields (2.7). This completes the proof of Theorem 2.5.

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