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ON A GENERALIZATION OF AN OPERATOR PRESERVING TURÁN-TYPE INEQUALITY FOR COMPLEX POLYNOMIALS

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Abstract. Let $W(\zeta) = (a_0 + a_1\zeta + ... + a_n\zeta^n)$ be a polynomial of degree n having all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1 + k + k^n$, Govil and Mctume [7] showed that the following inequality holds

$$\max_{\zeta\in\mathbb{T}_1} |D_\alpha W(\zeta)| \geq n\left(\frac{|\alpha| - k}{1 + k^n}\right)\|W\| + n\left(\frac{|\alpha| - (1 + k + k^n)}{1 + k^n}\right)\min_{\zeta\in\mathbb{T}_k} |W(\zeta)|.$$

In this paper, we have obtained a generalization of this inequality involving sequence of operators known as polar derivatives. In addition, the problem for the limiting case is also considered.

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1. Introduction and statement of results

Let $\mathcal{L}$ be the space of all complex polynomials $W(\zeta) = \sum_{v=1}^{n} b_v\zeta^v$ of degree n. For each positive real number k let $\mathbb{T}_k = \{\zeta : |\zeta| = k\}$, $\mathbb{E}_k^- = \{\zeta \in \mathbb{C} : |\zeta| < k\}$ and $\mathbb{E}_k^+ = \{\zeta \in \mathbb{C} : |\zeta| > k\}$ respectively. For any holomorphic function f defined on $\mathbb{T}_1$, we write $\|f\| = \sup_{z\in\mathbb{T}_1} |f(z)|$, the supremum norm of f on $\mathbb{T}_1$. The Bernstein's classical inequality states that

$$\max\{|W'(\zeta)| : \zeta \in \mathbb{T}_1\} \leq n \max\{|W(\zeta)| : \zeta \in \mathbb{T}_1\} \tag{1.1}$$

holds for all polynomials $W \in \mathcal{L}$. This result is best possible and the extremal polynomial for (1.1) is $W(\zeta) = \alpha\zeta^n$, $\alpha \neq 0$. The relationships between the bounds, their refinements and extensions, and the distribution of zeros of W in a certain region of $\mathbb{C}$ have been studied extensively and has deeply influenced the sequel of such type of inequalities throughout the decades. Since the equality in the Bernstein's inequality (1.1) holds for polynomials which have all their zeros at the origin, improvement in (1.1) is not possible if we consider polynomials having all their zeros inside the unit circle. For this reason, in this case, it may be interesting to obtain inequality in the reverse direction, and in this connection, we have the inequality ascribed to Turán[10], which asserts that

$$\max\{|W'(\zeta)| : \zeta \in \mathbb{T}_1\} \geq \frac{n}{2} \max\{|W(\zeta)| : \zeta \in \mathbb{T}_1\} \tag{1.2}$$

holds for all polynomials $W \in \mathcal{L}$ having all zeros in $\mathbb{T}_1 \cup \mathbb{E}_1^-$. This result is best possible and the extremal polynomial for (1.2) is $W(\zeta) = \zeta^n + 1$. One would expect, the refinement of the lower bound estimate in (1.2) under the condition when W is free of zeros on $\mathbb{T}_1$. This assertion was observed in [1] in which the authors proved the following inequality under the same hypothesis as of (1.2)

$$\|W'\| \geq \frac{n}{2}\{\|W\| + \min_{\zeta\in\mathbb{T}_1} |W(\zeta)|\}. \tag{1.3}$$

Inequalities (1.2) and (1.3) are very useful in proving some well known classical polynomial inequalities.

For polynomials of a complex variable, we also have the following more general result, due to Govil [4], which is one of the most known polynomial inequality in this direction and will be useful for our results. More precisely, the inequality

$$\|W'\| \geq \frac{n}{1+k^n}\|W\| \tag{1.4}$$

holds for all polynomials $W \in \mathcal{L}$ having all zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, where $k \geq 1$. As is easy to see that (1.4) becomes equality when $W(\zeta) = \zeta^n + k^n$. Again, excluding the class of polynomials having all zeros on $\mathbb{T}_k$, then one may expect that the bound (1.4) could be amended. In this direction, under the same hypothesis as of (1.4), it was shown by Govil [3] that the following inequality holds good

$$\|W'\| \geq \frac{n}{1+k^n}\{\|W\| + \min_{\zeta\in\mathbb{T}_1} |W(\zeta)|\}. \tag{1.5}$$

The research on mathematical objects associated with Turán type inequalities has been active over a period; there are many research papers published in a variety of journals each year and different approaches have been taken for different targets. The present article is concerned with Turán type inequalities for the polar derivative of a polynomial with restricted zeros. Before moving on to our main results, we will take a moment to introduce the concept of the polar derivative being involved.

Definition 1.1. *Let $W \in \mathcal{L}$, and α is any complex number, then*

$$\begin{aligned} D_\alpha W(\zeta) &= -\left[\frac{W(\zeta)}{(\zeta-\alpha)^n}\right]'(\zeta-\alpha)^{n+1} \\ &= nW(\zeta) + (\alpha-\zeta)W'(\zeta), \end{aligned} \tag{1.6}$$

is called the polar derivative of $W(\zeta)$. Note that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it generalizes the concept of "ordinary derivative" is evident and convincing from the fact that

$$\lim_{\alpha\to\infty} \frac{D_\alpha W(\zeta)}{\alpha} = W'(\zeta) \tag{1.7}$$

uniformly with respect to ζ for $\mathbb{T}_R \cup \mathbb{E}_R^-$, $R > 0$.

In the polar derivative milieu, all the above inequalities have been widely investigated, the research in this field has taken many different directions and resulting in slew of publications see, e.g.,([5], [8], [11], [6]). In this paper our interest is mainly motivated upon the study of various versions of inequalities (1.4) and (1.5), their refinements, strengthening and generalizations in the polar derivative setting by introducing constraints on the zeros of $W \in \mathcal{L}$, the modulus of largest root of W or restrictions on coefficients etc. In this contexture, the inequality

$$\max_{\zeta\in\mathbb{T}_1}|D_\alpha W(\zeta)| \geq \frac{n(|\alpha|-k)}{1+k^n}\|W\| \tag{1.8}$$

holds for all polynomials $W \in \mathcal{L}$ which has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$. This result is ascribed to Aziz and Rather [2]. Another result in this direction ascribed to Govil and Mctume [7] acts as a refinement of (1.7) and states that the inequality

$$\max_{\zeta\in\mathbb{T}_1}|D_\alpha W(\zeta)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\|W\| + n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\min_{\zeta\in\mathbb{T}_k}|W(\zeta)| \tag{1.9}$$

holds for all polynomials $W \in \mathcal{L}$ which has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1+k+k^n$.

Definition 1.2. *Given a polynomial $W \in \mathcal{L}$, we can construct a sequence of polar derivatives or so-called higher order derivatives with respect to finitely many poles as given below*

$$\begin{aligned}
D_{\alpha_1}W(\zeta) &= nW(\zeta) + (\alpha_1-\zeta)W'(\zeta)\\
D_{\alpha_2}D_{\alpha_1}W(\zeta) &= (n-1)D_{\alpha_1}W(\zeta) + (\alpha_2-\zeta)(D_{\alpha_1}W(\zeta))'\\
&\dots\ \dots\ \dots\\
D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta) &= (n-t+1)D_{\alpha_{t-1}}...D_{\alpha_1}W(\zeta) + (\alpha_t-\zeta)(D_{\alpha_{t-1}}...D_{\alpha_1}W(\zeta))',\\
&\textit{for}\ \ 2\leq t\leq n.
\end{aligned}$$

Like the t^{th} ordinary derivative $W^{(t)}(\zeta)$ of $W(\zeta)$, the t^{th} polar derivative $D_{\alpha_t}...D_{\alpha_2}$ $D_{\alpha_1}W(\zeta)$ of $W(\zeta)$ is a polynomial of degree at most $n-t$. For the sake of simplicity, we use the following notations:

$$A^k_{\alpha_t} = (|\alpha_1|-k)(|\alpha_2|-k)...(|\alpha_t|-k),$$

$$N_t = n(n-1)(n-2)...(n-t+1).$$

In this paper we obtain a generalization of inequalities (1.8) and (1.9), and besides our theorem includes many quality inequalities in this connection as special cases. To be more precise, we prove

Theorem 1.1. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex numbers $\alpha_1, \alpha_2, ..., \alpha_t$ with $|\alpha_i| \geq 1 + k + k^n$, $i = 1, 2, ..., t$,*

$$\max_{\zeta \in \mathbb{T}_1} |D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{N_t}{K_t} \left[A^k_{\alpha_t} \|W\| + \{A^k_{\alpha_t} - K_t\} \min_{\zeta \in \mathbb{T}_k} |W(\zeta)| \right], \tag{1.10}$$

where $K_t = (1 + k^n)(1 + k^{n-1})...(1 + k^{n-t+1})$.

Remark 1.1. *For $t = 1$, one easily gets inequality (1.9) from Theorem 1.1 and when there is no information about minimum of a polynomial $W(\zeta)$ we get inequality (1.8) as a special case.*

If we choose $\alpha_1 = \alpha_2 = ... = \alpha_t = \alpha$, then by dividing both sides of inequality (1.10) by $|\alpha|^t$ and letting $|\alpha| \to \infty$, therefore taking (1.7) into consideration, we obtain the following result.

Corollary 1.1. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then*

$$\max_{\zeta \in \mathbb{T}_1} |W^{(t)}(\zeta)| \geq \frac{N_t}{K_t} \left\{ \|W\| + \min_{\zeta \in \mathbb{T}_k} |W(\zeta)| \right\}, \tag{1.11}$$

where $K_t = (1 + k^n)(1 + k^{n-1})...(1 + k^{n-t+1})$.

Inequality (1.5) can easily be obtained from above Corollary 1.1 for $t = 1$.

2. Lemmas

Lemma 2.1. *If all the zeros of an nth degree polynomial W lie in a circular region C and if none of the points $\alpha_t, \alpha_{t-1}, ..., \alpha_1$ lie in the region C, then each of the polar derivatives*

$$D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} W(\zeta), \ t = 1, 2, .., n - 1$$

has all of its zeros in C.

This lemma follows by repeated applications of Laguerre's theorem [9].

Lemma 2.2. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex numbers α with $|\alpha| \geq k$,*

$$|D_\alpha W(\zeta)| \geq \frac{n(|\alpha| - k)}{1 + k^n} |W(\zeta)|.$$

This lemma is due to Aziz and Rather [2].

Lemma 2.3. *Let $W \in \mathcal{L}$, and $W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$, $k \geq 1$, then for every real or complex numbers $\alpha_1, \alpha_2, ..., \alpha_t$ with $|\alpha_i| \geq k$, $i = 1, 2, ..., t$,*

$$|D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{N_t}{(1 + k^n)(1 + k^{n-1})...(1 + k^{n-t+1})} A^k_{\alpha_t} |W(\zeta)|. \tag{2.1}$$

Proof of Lemma 2.3. Well, Lemma follows trivially if $|\alpha_i| = k$ for at least one i, $1 \leq i \leq t \leq n$. Therefore from now on we will assume that $|\alpha_i| > k$. We will prove this lemma by mathematical induction. Lemma is true for $t = 1$ by Lemma 2.2 i.e., if $|\alpha_1| > k$ then

$$|D_{\alpha_1} W(\zeta)| \geq \frac{n(|\alpha_1| - k)}{1 + k^n} |W(\zeta)|. \tag{2.2}$$

Now for $t = 2$. Except for one value (say a) of α_1, $D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-1)$. Let us take any $\alpha_1 (\alpha_1 \neq a$ if $|a| > k)$ with $|\alpha_1| > k$ and fix it up. Thus $D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-1)$ and by Lemma 2.1 it has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$. Therefore on applying ?Lemma 2.2 to $D_{\alpha_1} W(\zeta)$ with $\alpha = \alpha_2$, $|\alpha_2| > k$ we get,

$$|D_{\alpha_2}(D_{\alpha_1} W(\zeta))| \geq \frac{n-1}{1+k^{n-1}} (|\alpha_2| - k) |D_{\alpha_1} W(\zeta)|.$$

Using (2.2) we have

$$|D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)}{(1+k^n)(1+k^{n-1})} (|\alpha_1| - k)(|\alpha_2| - k) |W(\zeta)|.$$

It follows Lemma is true for $t = 2$. We assume that Lemma is true for $t = s$ i.e., for $\zeta \in \mathbb{T}_1$ and $|\alpha_i| > k, \ i = 1, 2, ..., s$

$$|D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{N_s}{(1+k^n)(1+k^{n-1})...(1+k^{n-s+1})} A_{\alpha_s}^k |W(\zeta)|, \tag{2.3}$$

and we will prove that Lemma is true for $t = s+1$, $(< n)$. Again except for one value (say $\alpha_1^{'}$) of α_1, $D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-1)$. Let us take any $\alpha_1 (\alpha_1 \neq \alpha_1^{'}$ if $|\alpha_1^{'}| > k)$ with $|\alpha_1| > k$ and fix it up. Thus $D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-1)$. Now $D_{\alpha_2} D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-2)$ for every α_2, except for one value $\alpha_2^{'}$, (say), of α_2. Let us take any $\alpha_2 (\alpha_2 \neq \alpha_2^{'}$ if $|\alpha_2^{'}| > k)$ with $|\alpha_2| > k$ and fix it up. Therefore, $D_{\alpha_2} D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-2)$. Likewise one can continue and say that $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ will be a polynomial of degree $(n-s)$ for every for every α_s, except for one value $\alpha_s^{'}$, (say), of α_s. Let us take any $\alpha_s (\alpha_s \neq \alpha_s^{'}$ if $|\alpha_s^{'}| > k)$ with $|\alpha_s| > k$ and fix it up. Thus $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ is a polynomial of degree $(n-s)$ with fixed $\alpha_s (|\alpha_s| > k)$, fixed $\alpha_{s-1} (|\alpha_{s-1}| > k)$,...,fixed $\alpha_1 (|\alpha_1| > k)$ and by Lemma 2.1 $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ has all its zeros in $\mathbb{T}_k \cup \mathbb{E}_k^-$. Therefore on applying Lemma 2.2 to $D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)$ with $\alpha = \alpha_{s+1}, \quad |\alpha_{s+1}| > k$ we get,

$$|D_{\alpha_{s+1}}(D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta))| \geq \frac{n-s}{1+k^{n-s}} (|\alpha_{s+1}| - k) |D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)|,$$

which on being combined with (2.3) gives for $\zeta \in \mathbb{T}_1$ and $|\alpha_{s+1}| > k$

$$|D_{\alpha_{s+1}} D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})} A_{\alpha_{s+1}}^k |W(\zeta)|. \tag{2.4}$$

Further $\alpha_s(\alpha_s \neq \alpha_s^{'}$ if $|\alpha_s^{'}| > k)$ with $|\alpha_s| > k$ was a fixed element but was chosen arbitrarily. Accordingly (2.4) is true for every $\alpha_{s+1}(|\alpha_{s+1}| > k)$ and every $\alpha_s(\alpha_s \neq \alpha_s^{'}$ if $|\alpha_s^{'}| > k)$ with $|\alpha_s| > k$, i.e.

$$|D_{\alpha_{s+1}} D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})} A^k_{\alpha_{s+1}} |W(\zeta)|, \quad \text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k \ \& \ |\alpha_s| > k (\alpha_s \neq \alpha_s^{'} \text{ if } |\alpha_s^{'}| > k). \tag{2.5}$$

Now if $|\alpha_s^{'}| > k$ then for a fixed $\alpha_{s+1}(|\alpha_{s+1}| > k)$, $D_{\alpha_{s+1}} D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1}$ is continuous function of α_s and hence by continuity we can say (2.5) will be true for a fixed α_{s+1} and $\alpha_s^{'}$ and accordingly (2.5) will be true for every $\alpha_{s+1}(|\alpha_{s+1}| > k)$ and $\alpha_s^{'}$. Thus (2.5) is true for every $\alpha_{s+1}(|\alpha_{s+1}| > k)$ and every $\alpha_s(|\alpha_s| > k)$. That is

$$|D_{\alpha_{s+1}} D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})} A^k_{\alpha_{s+1}} |W(\zeta)|, \quad \text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k \ \& \ |\alpha_s| > k. \tag{2.6}$$

As argued for α_s and $\alpha_s^{'}$, we can argue for α_{s-1} and $\alpha_{s-1}^{'}$ and say that

$$|D_{\alpha_{s+1}} D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})} A^k_{\alpha_{s+1}} |W(\zeta)|, \quad \text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k, |\alpha_s| > k \ \& \ |\alpha_{s-1}| > k.$$

One can continue similarly and obtain

$$|D_{\alpha_{s+1}} D_{\alpha_s} ... D_{\alpha_2} D_{\alpha_1} W(\zeta)| \geq \frac{n(n-1)...(n-s)}{(1+k^n)(1+k^{n-1})...(1+k^{n-s})} A^k_{\alpha_{s+1}} |W(\zeta)|, \quad \text{for } \zeta \in \mathbb{T}_1, |\alpha_{s+1}| > k, |\alpha_s| > k, |\alpha_{s-1}| > k, ..., |\alpha_2| > k, |\alpha_1| > k.$$

Hence Lemma is true for $t = s+1$. This completes the proof of Lemma.

3. Proof of Theorem 1.1

Let $m = \min_{\zeta \in \mathbb{T}_k} |W(\zeta)|$, then $|W(\zeta)| \geq m$ on $\mathbb{T}_k$. Therefore, for every λ with $|\lambda| < 1$, $|W(\zeta)| > |\lambda|$ on $\mathbb{T}_k$. If $W(\zeta)$ has a zero on $\mathbb{T}_k$ then $m = 0$ and the result follows from Lemma 2.3. Therefore from now onwards we will assume that $W(\zeta)$ has all its zeros in $\mathbb{E}_k^-$, where $k \geq 1$. By Rouche's theorem the polynomial

$$F(\zeta) = W(\zeta) + \lambda m$$

also has all its zeros in $\mathbb{E}_k^-$. Thus, on applying Lemma 2.3 to $F(\zeta)$ we obtain for $|\alpha_1| \geq k, |\alpha_2| \geq k, ..., |\alpha_t| \geq k$

$$|D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} F(\zeta)| \geq \frac{N_t}{(1+k^n)(1+k^{n-1})...(1+k^{n-t+1})} A^k_{\alpha_t} |F(\zeta)|, \ \zeta \in \mathbb{T}_1,$$

i.e.,

$$|D_{\alpha_t} ... D_{\alpha_2} D_{\alpha_1} W(\zeta) + m\lambda n(n-1)...(n-t+1)| \geq \frac{N_t}{K_t} A^k_{\alpha_t} |W(\zeta) + \lambda m|, \ \zeta \in \mathbb{T}_1, \tag{3.1}$$

where $K_t = (1+k^n)(1+k^{n-1})...(1+k^{n-t+1})$.

If we choose the argument of λ such that

$$|W(\zeta)+\lambda m|=|W(\zeta)|+|\lambda|m,$$

then from (3.1), we get

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta)+m\lambda n(n-1)...(n-t+1)|\geq\frac{N_t}{K_t}A^k_{\alpha_t}\{|W(\zeta)|+|\lambda|m\},$$

this gives

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta)|+mN_t|\lambda|\geq\frac{N_t}{K_t}A^k_{\alpha_t}\{|W(\zeta)|+|\lambda|m\},\ \zeta\in\mathbb{T}_1.$$

Equivalently

$$|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta)|\geq\frac{N_t}{K_t}\left[A^k_{\alpha_t}|W(\zeta)|+|\lambda|\{A^k_{\alpha_t}-K_t\}m\right]. \tag{3.2}$$

Now letting $|\lambda|\to 1$ in (3.2),we get

$$\max_{\zeta\in\mathbb{T}_1}|D_{\alpha_t}...D_{\alpha_2}D_{\alpha_1}W(\zeta)|\geq\frac{N_t}{K_t}\left[A^k_{\alpha_t}\|W\|+\{A^k_{\alpha_t}-K_t\}\min_{\zeta\in\mathbb{T}_k}|W(\zeta)|\right],$$

which is (1.10) and Theorem 1.1 is thus proved.

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