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A HARDY-LITTLEWOOD TYPE THEOREM FOR HARMONIC BERGMAN-ORLICZ SPACES AND APPLICATIONS

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Abstract. It is well known that a harmonic function is in a Bergman space if and only if it satisfies some Hardy-Littlewood type integral estimates. In this paper, we extend this result to harmonic Bergman-Orlicz spaces. As an application, Lipschitz-type characterizes of harmonic Bergman-Orlicz spaces on the unit ball with respect to pseudo-hyperbolic, hyperbolic and Euclidean metrics are established. In addition, the boundedness of a mapping defined by a difference quotient of harmonic function is discussed.

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1. Introduction and main results

Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$ be two vectors in the n-dimensional real vector space $\mathbb{R}^n$. We write

$$\langle x, y\rangle = x_1y_1 + ... + x_ny_n \quad \text{and} \quad |x| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + ... + x_n^2}.$$

For $a \in \mathbb{R}^n$, let $\mathbb{B}(a, r) = \{x : |x - a| < r\}$, $\mathbb{S}(a, r) = \partial\mathbb{B}(a, r)$ and $\overline{\mathbb{B}(a, r)} = \mathbb{B}(a, r) \cup \mathbb{S}(a, r)$. In particular, we use the notations $\mathbb{B} = \mathbb{B}(0, 1)$, $\mathbb{S} = \partial\mathbb{B}(0, 1)$ and $\overline{\mathbb{B}} = \mathbb{B} \cup \mathbb{S}$ the closure of $\mathbb{B}$. We denote by dv the normalized volume measure on $\mathbb{B}$ and $d\sigma$ the normalized surface measure on $\mathbb{S}$. As usual, the class of all harmonic functions on the unit ball $\mathbb{B}$ will be denoted by $h(\mathbb{B})$.

Given a function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, we say that Φ is a *growth function* if it is continuous and non-decreasing.

For $\alpha > -1$ and a growth function Φ, the *Orlicz space* $L^\Phi_\alpha(\mathbb{B})$ is the set of all functions f such that

$$\int_{\mathbb{B}} \Phi(|f(x)|)dv_\alpha(x) < \infty,$$

where $dv_\alpha(x) = c_\alpha(1 - |x|^2)^\alpha dv(x)$ and c_α is a positive constant so that $v_\alpha(\mathbb{B}) = 1$. Denote by $L^p_\alpha(\mathbb{B})$ the subspace of $L^\Phi_\alpha(\mathbb{B})$ for $\Phi(t) = t^p$ and $0 < p < \infty$.

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The harmonic *Bergman-Orlicz* space $\mathcal{B}_\alpha^\Phi$ is the subspace of $L_\alpha^\Phi(\mathbb{B})$ consisting of all $f \in h(\mathbb{B})$ such that

$$\|f\|_{\alpha,\Phi} = \inf\{\mu > 0 : \int_{\mathbb{B}} \Phi\Big(\frac{|f(x)|}{\mu}\Big) dv_\alpha(x) \leq 1\} < \infty.$$

In particular, if $0 < p < \infty$ and $\Phi(t) = t^p$, then the associated harmonic Bergman-Orlicz space is the weighted harmonic Bergman space $\mathcal{B}_\alpha^p$ (cf. [4, 11]).

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$. For $0 < p < \infty$ and $\alpha > -1$, the standard weighted Bergman space $\mathcal{A}_\alpha^p(\mathbb{D})$ consists of all analytic functions g on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |g(z)|^p dA_\alpha(z) < \infty$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ and $dA(z)$ is the area measure on $\mathbb{C}$ normalized so that $A(\mathbb{D}) = 1$. The famous Hardy-Littlewood theorem for weighted Bergman space $\mathcal{A}_\alpha^p(\mathbb{D})$ asserts that

$$\int_{\mathbb{D}} |g(z)|^p dA_\alpha(z) \approx |g(0)|^p + \int_{\mathbb{D}} |g'(z)|^p (1 - |z|^2)^p dA_\alpha(z). \tag{1.1}$$

It is well-known that integral estimate (1.1) plays an important role in the theory of analytic functions. For the generalizations and applications of (1.1) to the spaces of holomorphic functions, harmonic functions, and solutions to certain PDEs, see [4], [5], [7] - [11, 16] and the references therein.

In [14], B. Sehba considered the analogue of (1.1) in the setting of holomorphic functions spaces on the complex unit ball. By adding some suitable restrictions on the growth function Φ, he generalized the integral estimate (1.1) to the Bergman-Orlicz spaces of holomorphic functions. As applications, characterizations of the Gustavsson-Peetre interpolate and boundedness of Cesàro-type operators on Bergman-Orlicz spaces are discussed.

Motivated by the results in [7, 9, 14], our aim in this paper is to extend (1.1) to the setting of harmonic Bergman-Orlicz space $\mathcal{B}_\alpha^\Phi$. In order to state the our result, we need some more definitions on the growth function Φ.

We say that a growth function Φ is of upper type $q \geq 1$ if there exists $C > 0$ such that, for $s > 0$ and $t \geq 1$,

$$\Phi(st) \leq Ct^q \Phi(s). \tag{1.2}$$

Denote by $\mathcal{U}^q$ the set of growth functions Φ of upper type q, (for some $q \geq 1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-decreasing.

We say that Φ is of lower type $p > 0$ if there exists $C > 0$ such that, for $s > 0$ and $0 < t \leq 1$,

$$\Phi(st) \leq Ct^p \Phi(s). \tag{1.3}$$

Denote by $\mathcal{L}_p$ the set of growth functions Φ of lower type p, (for some $p < 1$), such that the function $t \to \frac{\Phi(t)}{t}$ is non-increasing.

From the above definitions on Φ, we may always suppose that any $\Phi \in \mathcal{L}_p$ (resp. $\mathscr{U}^q$), is concave (resp. convex) and that Φ is a $\mathcal{C}^1$ function with derivative $\Phi'(t) \approx \frac{\Phi(t)}{t}$ (cf. [14, 15]).

For $f \in h(\mathbb{B})$, recall that the radial derivative $\mathcal{R}$ of f is given by

$$\mathcal{R}f(x) = x \cdot \nabla f(x) = \frac{\partial}{\partial t}(f(tx))_{t=1} = \sum_{m=1}^{\infty} m f_m(x),$$

where ∇ is the usual gradient and the last form is the homogeneous expansion of f. The fundamental theorem of calculus shows that

$$f(x) - f(0) = \int_0^1 (\mathcal{R}f)(tx)\frac{dt}{t}.$$

Theorem 1.1. *Let $\alpha > -1$, $f \in h(\mathbb{B})$ and $\Phi \in \mathscr{U}^q \cup \mathcal{L}_p$. Then the following statements are equivalent.*
(a) $f \in \mathcal{B}_\alpha^\Phi$;
(b) $(1 - |x|^2)|\nabla f(x)| \in L_\alpha^\Phi(\mathbb{B})$;
(c) $(1 - |x|^2)|\mathcal{R}f(x)| \in L_\alpha^\Phi(\mathbb{B})$.

As applications of Theorem 1.1, we establish several characterizations of harmonic Bergman-Orlicz spaces in terms of Lipschitz-type conditions with respect to pseudo-hyperbolic, hyperbolic and Euclidean metrics, which can be viewed as extensions of [16, Theorem 1.1] into the general setting.

Theorem 1.2. *Let $\alpha > -1$, $\Phi \in \mathscr{U}^q \cup \mathcal{L}_p$ and $f \in h(\mathbb{B})$. Then the following statements are equivalent.*
(a) $f \in \mathcal{B}_\alpha^\Phi$;
(b) There exists a positive continuous function $g \in L_\alpha^\Phi(\mathbb{B})$ such that

$$|f(x) - f(y)| \leq \rho(x, y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$;
(c) There exists a positive continuous function $g \in L_\alpha^\Phi(\mathbb{B})$ such that

$$|f(x) - f(y)| \leq \varrho(x, y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$.

Theorem 1.3. *Let $\alpha > -1$, Φ be a given growth function and $f \in \mathcal{B}_\alpha^\Phi$.*
(1) If $\Phi \in \mathscr{U}^q$, then there exists a positive continuous function $g \in L_{\alpha+k}^\Phi(\mathbb{B})$ ($k \in [q, \infty)$ such that $|f(x) - f(y)| \leq |x - y|\big(g(x) + g(y)\big)$ for all $x, y \in \mathbb{B}$;

(2) *If* $\Phi \in \mathcal{L}_p$, *then there exists a positive continuous function* $g \in L^{\Phi}_{\alpha+k}(\mathbb{B})$ $(k \in [1,\infty)$ *such that* $|f(x) - f(y)| \le |x-y|\big(g(x)+g(y)\big)$ *for all* $x, y \in \mathbb{B}$.

The organization of this paper is as follows. In Section 2, some necessary terminologies and notations will be introduced. The proof of Theorems 1.1 $\sim$ 1.3 will be presented in Section 3. The Section 4 is devoted to discussing some applications of the main results. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$.

2. Preliminaries

In this section, we introduce notations and collect some preliminary results that we need later.

2.1. Pseudo-hyperbolic metric. For $a \in \mathbb{B}$, let

$$\varphi_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{[x,a]^2}, \;\; x \in \mathbb{B},$$

where $[a,x] = \sqrt{1 - 2\langle a, x\rangle + |a|^2|x|^2}$. Then φ_a is a Möbius transformation from $\mathbb{B}$ onto $\mathbb{B}$ with $\varphi_a(0) = a$ and $\varphi_a(a) = 0$.

We denote by $\mathcal{M}(\mathbb{B})$ the set of all Möbius transformation on $\mathbb{B}$. It is known that if $\varphi \in \mathcal{M}(\mathbb{B})$, then there exist $a \in \mathbb{B}$ and an orthogonal transformation A such that

$$\varphi(x) = A\varphi_a(x), \quad x \in \mathbb{B}.$$

In terms of φ_a, the *pseudo-hyperbolic metric* ρ and the *hyperbolic metric* ϱ in $\mathbb{B}$ are given by

$$\rho(a,b) = |\varphi_a(b)| = \frac{|a-b|}{[a,b]}, \quad a, b \in \mathbb{B}$$

and

$$\varrho(a,b) = \ln \frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|},$$

respectively.

Let $a \in \mathbb{B}$ and $r \in (0,1)$, the *pseudo-hyperbolic ball* with center a and radius r is denoted by

$$E(a,r) = \{x \in \mathbb{B} : \rho(a,x) = |\varphi_a(x)| < r\}.$$

A straightforward calculation shows that $E(a,r)$ is actually a Euclidean ball with center c_a and radius r_a given by

$$c_a = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad r_a = \frac{r(1-|a|^2)}{1-|a|^2r^2}, \tag{2.1}$$

respectively (cf. [1, 14]).

Lemma 2.1. [13] *Let $a \in \mathbb{B}$, $r \in (0,1)$ and $x \in E(a,r)$. Then*

$$1-|a|^2 \approx 1-|x|^2 \approx [a,x] \quad \text{and} \quad |E(a,r)| \approx (1-|a|^2)^n,$$

where $|E(a,r)|$ denotes the Euclidean volume of $E(a,r)$.

The following standard estimate will be needed in the sequel.

Lemma 2.2. [13] *Let $\alpha > -1$ and $\beta \in \mathbb{R}$. Then for any $x \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|y|^2)^\alpha}{[x,y]^{n+\alpha+\beta}} dv(y) \approx \begin{cases} (1-|x|^2)^{-\beta}, & \beta > 0, \\ \log \dfrac{1}{1-|x|^2}, & \beta = 0, \\ 1, & \beta < 0. \end{cases}$$

2.2. Operators on Orlicz spaces. Let Φ be a growth function. Recall that the lower and the upper indices of Φ are respectively defined by

$$a_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \text{ and } b_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

It is known that when Φ is convex, then $1 \le a_\Phi \le b_\Phi < \infty$ and, if Φ is concave, then $0 \le a_\Phi \le b_\Phi \le 1$ (cf. [6, 14]).

Definition 2.1. *Let Φ be a growth function. A linear operator T defined on $L^\Phi_\alpha(\mathbb{B})$ is said to be of mean strong type $(\Phi,\Phi)_\alpha$ if*

$$\int_{\mathbb{B}} \Phi(|Tf|) dv_\alpha(x) \le C \int_{\mathbb{B}} \Phi(|f|) dv_\alpha(x)$$

for any $f \in L^\Phi_\alpha(\mathbb{B})$, and T is said to be mean weak type $(\Phi,\Phi)_\alpha$ if

$$\sup_{t>0} \Phi(t) v_\alpha(\{x \in \mathbb{B} : |Tf(x)| > t\}) \le C \int_{\mathbb{B}} \Phi(|f|) dv_\alpha(x)$$

for any $f \in L^\Phi_\alpha(\mathbb{B})$, where C is independent of f.

We remark that if $\Phi(t) = t^p$, then the mean strong type $(t^p, t^p)_\alpha$ is the usual strong type (p,p) coincide. The following result comes from [4, Theorem 4.3].

Lemma 2.3. *Let Let Φ_0, Φ_1 and Φ_2 be three convex growth functions. Sup- pose that their upper and lower indices satisfy the following condition*

$$1 \le a_{\Phi_0} \le b_{\Phi_0} < a_{\Phi_2} \le b_{\Phi_2} < a_{\Phi_1} \le b_{\Phi_1} < \infty.$$

If T is of mean weak types $(\Phi_0,\Phi_0)_\alpha$ and $(\Phi_1,\Phi_1)_\alpha$, then it is of mean strong type $(\Phi_2,\Phi_2)_\alpha$.

Let $\beta \in \mathbb{R}$ and consider the operator E_β defined for functions f on $\mathbb{B}$ by

$$E_\beta f(x) = \int_{\mathbb{B}} f(y) \frac{(1-|y|^2)^\beta}{[x,y]^{n+\beta}} dv(y).$$

We refer to [2] and [3] for more details on Bergman type projection E_β. For a proof of the following lemma see, for example [9, Theorem 1.6].

Lemma 2.4. *Let $1 \le p < \infty$ and $\alpha, \beta > -1$. The operator $E_\beta : L^p_\alpha(\mathbb{B}) \to L^p_\alpha(\mathbb{B})$ is bounded if and only if $\alpha + 1 < p(\beta + 1)$.*

Combing Lemmas 2.3 and 2.4, the following result can be easily derived.

Lemma 2.5. *Let $\alpha, \beta > -1$ and Φ be a convex growth function with its lower indice a_Φ. If $1 < p < a_\Phi$ and $\alpha + 1 < p(\beta + 1)$, then E_β is of mean strong type $(\Phi, \Phi)_\alpha$.*

2.3. Harmonic functions. It is well-known that the weighted harmonic Bergman spaces $\mathcal{B}^2_\alpha$ for $\alpha > -1$ is a reproducing kernel Hilbert space with reproducing kernel $R_\alpha(x, y)$:

$$f(x) = \int_{\mathbb{B}} f(y) R_\alpha(x, y) dv_\alpha(y), \ \ f \in \mathcal{B}^2_\alpha.$$

The reproducing kernels $R_\alpha(x, y)$ can be expressed in terms of zonal harmonics as

$$R_\alpha(x, y) = \sum_{k=0}^{\infty} \frac{(1 + \frac{n}{2} + \alpha)_k}{(\frac{n}{2})_k} Z_k(x, y) = \sum_{k=0}^{\infty} \gamma_k(\alpha) Z_k(x, y),$$

where the series absolutely and uniformly converges on $K \times \mathbb{B}$, for any compact subset K of $\mathbb{B}$. $R_\alpha(x, y)$ is real-valued, symmetric in the variables x and y and harmonic with respect to each variable since the same is true for all $Z_k(x, y)$. For the extension of reproducing kernels $R_\alpha(x, y)$ to all $\alpha \in \mathbb{R}$, see [7, 9].

We recall some useful inequalities concerning harmonic functions which are useful for our investigations.

Lemma 2.6. [4, 13] *Let $0 < p < \infty$, $0 < r < 1$ and $f \in h(\mathbb{B})$. Then there exists some positive constant C such that*

(1) $|f(x)|^p \le C \int_{E(x,r)} |f(y)|^p d\tau(y)$;

(2) $|\nabla f(x)|^p \le \dfrac{C}{(1 - |x|^2)^p} \int_{E(x,r)} |f(y)|^p d\tau(y)$,

where $d\tau(x) = (1 - |x|^2)^{-n} dv(x)$ is the invariant measure on $\mathbb{B}$.

The above lemma leads to the following integral inequality (cf. [7, Lemma 5.1]).

Lemma 2.7. *Let $0 < p < 1$ and $\alpha > -1$. Then*

$$\int_{\mathbb{B}} |f(x)g(x)|(1 - |x|^2)^{(n+\alpha)/p - n} dv_\alpha(x) \lesssim \|f(x)g(x)\|_{L^p_\alpha}$$

for all $f, g \in h(\mathbb{B})$.

3. Proofs of main results

The purpose of this section is to prove our main results. Before the proofs, we need the following lemmas.

Lemma 3.1. *Let $\Phi \in \mathcal{L}_p$. Then the growth function Φ_p, defined by $\Phi_p(t) = \Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$ for some $q \geq 1$. Moreover, for $s > 0$ and $t \geq 1$,*

$$\Phi_p(ts) \leq t^{\frac{1}{p}} \Phi_p(s). \tag{3.1}$$

Proof. We only need to prove (3.1) since the assertion $\Phi_p \in \mathcal{U}^q$ for some $q \geq 1$ can be found in [14, Lemma 2.1]. Let $s > 0$ and $t \geq 1$. By the monotonicity of $\frac{\Phi(t)}{t}$, it deduces that

$$\Phi_p(ts) = \Phi(t^{\frac{1}{p}} s^{\frac{1}{p}}) \leq t^{\frac{1}{p}} \Phi(s^{\frac{1}{p}}) = t^{\frac{1}{p}} \Phi_p(s).$$

This gives (3.1).

Lemma 3.2. *Let $\alpha > -1$ and $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$. Then there exists a constant $C > 0$ such that for any $f \in \mathcal{B}_\alpha^\Phi$,*

$$\int_{\mathbb{B}} \Phi((1-|x|^2)|\nabla f(x)|) dv_\alpha(x) \leq C \int_{\mathbb{B}} \Phi(|f(x)|) dv_\alpha(x). \tag{3.2}$$

Proof. Let

$$p_\Phi = \begin{cases} 1, & \text{if} \quad \Phi \in \mathcal{U}^q, \\ p, & \text{if} \quad \Phi \in \mathcal{L}_p. \end{cases} \tag{3.3}$$

By Lemma 2.6, for each $x \in \mathbb{B}$, there exist $C > 0$ such that

$$\big((1-|x|^2)|\nabla f(x)|\big)^{p_\Phi} \leq C \int_{E(x,r)} |f(y)|^{p_\Phi} d\tau(y).$$

Set

$$\Phi_p(t) = \begin{cases} \Phi(t), & \text{if} \quad \Phi \in \mathcal{U}^q, \\ \Phi(t^{\frac{1}{p}}), & \text{if} \quad \Phi \in \mathcal{L}_p. \end{cases} \tag{3.4}$$

It follows from Lemma 3.1 and the convexity of $\Phi_p(t)$ that

$$\Phi((1-|x|^2)|\nabla f(x)|) \leq C \int_{E(x,r)} \Phi(|f(y)|) d\tau(y).$$

Integrating both sides of the above inequality over $\mathbb{B}$ with respect to $dv_\alpha(x)$ and applying Fubini's theorem and Lemma 2.1, (3.2) follows.

Lemma 3.3. *Let $\alpha > -1$, $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$ and $f \in h(\mathbb{B})$. If $(1-|x|^2)|\mathcal{R}f(x)| \in L_\alpha^\Phi(\mathbb{B})$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{B}} \Phi(|f(x) - f(0)|) dv_\alpha(x) \leq C \int_{\mathbb{B}} \Phi((1-|x|^2)|\mathcal{R}f(x)|) dv_\alpha(x). \tag{3.5}$$

Proof. Assume that $f \in h(\mathbb{B})$ and $(1-|x|^2)|\mathcal{R}f(x)| \in L^{\Phi}_{\alpha}(\mathbb{B})$. In view of [9, Theorem 1.4], we see that for large enough s,

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y) R_s(x,y) dv_s(y).$$

Since $\int_{\mathbb{B}} \mathcal{R}f(y) dv_s(y) = 0$, subtracting this from the previous equation yields

$$\mathcal{R}f(x) = \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(x,y) - 1) dv_s(y).$$

Consequently,

$$\begin{aligned} |f(x) - f(0)| &= \Big| \int_0^1 \int_{\mathbb{B}} \mathcal{R}f(y)(R_s(tx,y) - 1) dv_s(y) \frac{dt}{t} \Big| \\ &= \Big| \int_{\mathbb{B}} \mathcal{R}f(y) \int_0^1 \frac{R_s(tx,y) - 1}{t} dt dv_s(y) \Big|. \end{aligned}$$

Let

$$G(x,y) = \int_0^1 \frac{R_s(tx,y) - 1}{t} dt.$$

By an argument similar to the one in the proof of [9, Lemma12.1], we have

$$|G(x,y)| \leq \int_0^1 \Big| \frac{R_s(tx,y) - 1}{t} \Big| dt \lesssim \int_0^1 \frac{dt}{[tx,y]^{n+s}} \lesssim \frac{1}{[x,y]^{n+s-1}}.$$

Therefore,

$$|f(x) - f(0)| \lesssim \int_{\mathbb{B}} (1 - |y|^2)|\mathcal{R}f(y)| \frac{1}{[x,y]^{n+s-1}} dv_{s-1}(y).$$

We first consider the case $\Phi \in \mathcal{U}^q$. Fix p so that $1 < p < a_{\Phi}$. Since $(1-|x|^2)|\mathcal{R}f(x)| \in L^{\Phi}_{\alpha}(\mathbb{B})$, by taking s large enough so that $\alpha + 1 < ps$, we obtain from Lemma 2.5 that

$$\int_{\mathbb{B}} \Phi(|f(x) - f(0)|) dv_{\alpha}(x) \leq C \int_{\mathbb{B}} \Phi((1 - |x|^2)|\mathcal{R}f(x)|) dv_{\alpha}(x).$$

We now consider the case of $\Phi \in \mathcal{L}_p$. Set $s = (n + \alpha')/p - n$ and $\alpha' > \alpha + p$. By Lemma 2.7, it deduces that

$$\begin{aligned} |f(x) - f(0)|^p &\lesssim \int_{\mathbb{B}} |\mathcal{R}f(y)|^p |G(x,y)|^p dv_{\alpha'}(y) \\ &\lesssim \int_{\mathbb{B}} \frac{|\mathcal{R}f(y)|^p}{[x,y]^{p(n+s-1)}} dv_{\alpha'}(y) \\ &\lesssim \int_{\mathbb{B}} \frac{|(1 - |y|^2)\mathcal{R}f(y)|^p}{[x,y]^{n+\alpha'-p}} dv_{\alpha'-p}(y). \end{aligned}$$

As the growth function $t \to \Phi_p(t) = \Phi(t^{\frac{1}{p}})$ is in $\mathcal{U}^q$, proceeding as in the first part of this proof yields that

$$\begin{aligned} &\int_{\mathbb{B}} \Phi(|f(x) - f(0)|) dv_{\alpha}(x) = \int_{\mathbb{B}} \Phi_p(|f(x) - f(0)|^p) dv_{\alpha}(x) \\ \lesssim &\int_{\mathbb{B}} \Phi_p((1 - |x|^2)|\mathcal{R}f(x)|)^p) dv_{\alpha}(x) \lesssim \int_{\mathbb{B}} \Phi((1 - |x|^2)|\mathcal{R}f(x)|) dv_{\alpha}(x). \end{aligned}$$

The proof of this lemma is complete.

Proof of Theorem 1.1. $(a) \Rightarrow (b)$ follows from Lemma 3.2. $(b) \Rightarrow (c)$ is trivial since $|\mathcal{R}f(x) \leq |\nabla f(x)|$ for $x \in \mathbb{B}$. Lemma 3.3 implies $(c) \Rightarrow (a)$.

Proof of Theorem 1.2. We first prove $(b) \Rightarrow (a)$. Assume that (b) holds. Then for each fixed $x \in \mathbb{B}$ and all y sufficiently close to x

$$\frac{|f(x) - f(y)|}{\rho(x,y)} \leq g(x) + g(y), \quad x \neq y.$$

Letting y approach x in the direction of each real coordinate axis, we conclude

$$(1 - |x|^2)|\nabla f(x)| \leq Cg(x).$$

It follows from the assumption $g \in L^{\Phi}_{\alpha}(\mathbb{B})$ that

$$\int_{\mathbb{B}} \Phi(|(1 - |x|^2)|\nabla f(x)|) dv_{\alpha}(x) < \infty.$$

Hence $f \in \mathcal{B}^{\Phi}_{\alpha}$ by Theorem 1.1.

$(b) \Rightarrow (a)$. Assume that $f \in \mathcal{B}^{\Phi}_{\alpha}$. Fix a small positive r and consider any two points $x, y \in \mathbb{B}$ with $y \in E(x,r)$. Since $E(x,r)$ is a Euclidean ball, by Lemma 2.1, it is given that

$$\begin{aligned} |f(x) - f(y)| &\leq C|x - y| \int_0^1 |\nabla f(sy + (1-s)x)| ds \\ &\leq C\rho(x,y) \sup\{(1 - |x|^2)|\nabla f(\xi)| : \xi \in E(x,r)\} \\ &= \rho(x,y)h(x), \end{aligned}$$

where

$$h(x) = C_r \sup\{(1 - |\xi|^2)|\nabla f(\xi)| : \xi \in E(x,r)\}.$$

If $\rho(x,y) \geq r$, then the triangle inequality implies

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x)| + |f(y)| \\ &\leq \rho(x,y)\Big(\frac{|f(x)|}{r} + \frac{|f(y)|}{r}\Big). \end{aligned}$$

By letting $g(x) = h(x) + \frac{|f(x)|}{r}$, we have

$$|f(x) - f(y)| \leq \rho(x,y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$. It is easy to see that $g(x) = h(x) + \frac{|f(x)|}{r}$ is the desired function provided that $h \in L^{\Phi}_{\alpha}(\mathbb{B})$. From (2.1), we can find r' such that $0 < r < r' < 1$ and $E(\xi, r) \subset E(x, r')$ for every $\xi \in E(x,r)$. It follows from Lemma 2.6 and the proof of Lemma 3.2 that

$$|h(x)|^{p_\Phi} \quad \leq \quad C \int_{E(x,r')} |f(y)|^{p_\Phi} d\tau(y)$$

and

$$\Phi(|h(x)|) \quad \leq \quad C\int_{E(x,r')} \Phi(|f(y)|)d\tau(y),$$

where p_Φ is given in (3.3).

Hence by Fubini's theorem and Lemma 2.1,

$$\begin{aligned}\int_{\mathbb{B}} \Phi(|h(x)|)dv_\alpha(x) \quad &\leq \quad C\int_{\mathbb{B}}(1-|x|^2)^\alpha \int_{E(x,r')} \Phi(|f(y)|)d\tau(y)dv(x)\\ &\leq \quad C\int_{\mathbb{B}} \Phi(|f(y)|)d\tau(y)\int_{E(y,r')}(1-|x|^2)^\alpha d\tau(x)\\ &\leq \quad C\int_{\mathbb{B}} \Phi(|f(y)|)dv_\alpha(y).\end{aligned}$$

This proves $(a) \Leftrightarrow (b)$.

As $\rho \leq \varrho$, so the implication $(a) \Rightarrow (c)$ is trivial. For the converse, suppose that there exists a positive continuous function $g \in L^\Phi_\alpha(\mathbb{B})$ such that

$$|f(x)-f(y)| \leq \varrho(x,y)\big(g(x)+g(y)\big)$$

for all $x, y \in \mathbb{B}$. By a discussion similar to the proof of $(b) \Rightarrow (a)$, it concludes that

$$(1-|x|^2)|\nabla f(x)| \leq Cg(x),$$

which implies that $f \in \mathcal{B}^\Phi_\alpha$. The proof of Theorem 1.2 is finished.

Proof of Theorem 1.3. Since $f \in \mathcal{B}^\Phi_\alpha$, we know that there exists a positive continuous function $g_1 \in L^\Phi_\alpha(\mathbb{B})$ such that

$$|f(x)-f(y)| \leq \rho(x,y)\big(g_1(x)+g_1(y)\big), \quad x, y \in \mathbb{B},$$

from Theorem 1.2. As for $x, y \in \mathbb{B}$,

$$[x,y] \geq 1-|x|, \quad [x,y] \geq 1-|y|,$$

we deduce that

$$|f(x)-f(y)| \leq |x-y|\Big(\frac{g_1(x)}{[x,y]}+\frac{g_1(y)}{[x,y]}\Big) \leq |x-y|\big(g(x)+g(y)\big),$$

where

$$g(x) = \frac{g_1(x)}{1-|x|} \leq \frac{2g_1(x)}{1-|x|^2}.$$

This means that $(1-|\cdot|^2)g(\cdot) \in L^\Phi_\alpha(\mathbb{B})$.

(1) If $\Phi \in \mathcal{U}^q$ and $k \in [q, \infty)$, by (1.2) we have

$$\begin{aligned}\int_{\mathbb{B}} \Phi(|g(x)|)dv_{\alpha+k}(x) \quad &\leq \quad C\int_{\mathbb{B}} \Phi((1-|x|^2)|g(x)|)(1-|x|^2)^{k-q}dv_\alpha(x)\\ &\leq \quad C\int_{\mathbb{B}} \Phi((1-|x|^2)|g(x)|)dv_\alpha(x),\end{aligned}$$

which implies that $g \in L^\Phi_{\alpha+k}(\mathbb{B})$.

(2) If $\Phi \in \mathcal{L}_p$ and $k \in [1, \infty)$, then by Lemma 3.1, $\Phi_p(t) = \Phi(t^{\frac{1}{p}}) \in \mathcal{U}^{1/p}$. It follows from (1.2) again that

$$\begin{aligned} \int_{\mathbb{B}} \Phi(|g(x)|) dv_{\alpha+k}(x) &= \int_{\mathbb{B}} \Phi_p(|g(x)|^p) dv_{\alpha+k}(x) \\ &\leq C \int_{\mathbb{B}} \Phi_p((1-|x|^2)^p |g(x)|^p)(1-|x|^2)^{k-1} dv_\alpha(x) \\ &\leq C \int_{\mathbb{B}} \Phi_p((1-|x|^2)^p |g(x)|^p) dv_\alpha(x) \\ &\leq C \int_{\mathbb{B}} \Phi((1-|x|^2)|g(x)|) dv_\alpha(x), \end{aligned}$$

as desired. The proof is complete.

4. A difference quotient of harmonic function on $\mathbb{B}$

In this section we present an application of our main results.

Let $f \in h(\mathbb{B})$, we define a difference quotient of f by

$$Lf(x, y) = \frac{f(x) - f(y)}{x - y}, \quad x, y \in \mathbb{B}, x \neq y.$$

It is known that in [16], Wulan and Zhu introduced this kind of operator L in the setting of analytic functions spaces and discussed the boundedness of L between standard weighted Bergman space $\mathcal{A}^p_\alpha(\mathbb{D})$. Especially, they obtained the following:

(I) If $\alpha > -1$, and $p \in (0, \alpha + 2)$, then L is bounded from $\mathcal{A}^p_\alpha(\mathbb{D})$ into $\mathcal{A}^p_\alpha(\mathbb{D} \times \mathbb{D})$;

(II) If $\alpha > -1$, and $p > \alpha + 2$ and $\beta = \frac{p+\alpha-2}{2}$, then L is bounded from $\mathcal{A}^p_\alpha(\mathbb{D})$ into $\mathcal{A}^p_\beta(\mathbb{D} \times \mathbb{D})$.

We now extend this result to the harmonic *Bergman-Orlicz* space $\mathcal{B}^\Phi_\alpha$ as follows.

Theorem 4.1. *Let $\alpha > -1$ and $\Phi \in \mathcal{U}^q \cup \mathcal{L}_p$. If $q \in [1, n + \alpha)$, then $L : \mathcal{B}^\Phi_\alpha \to L^\Phi_\alpha(\mathbb{B} \times \mathbb{B})$ is bounded.*

Proof. Let $f \in \mathcal{B}^\Phi_\alpha$. By Theorem 1.2, there exists a positive continuous function $g \in L^\Phi_\alpha(\mathbb{B})$ such that

$$|f(x) - f(y)| \leq \rho(x, y)\big(g(x) + g(y)\big),$$

which in turn gives

$$|Lf(x, y)| = \Big|\frac{f(x) - f(y)}{x - y}\Big| \leq \frac{g(x) + g(y)}{[x, y]}, \quad x \neq y.$$

Set

$$p_\Phi = \begin{cases} 1, & \text{if } \ \Phi \in \mathcal{U}^q, \\ p, & \text{if } \ \Phi \in \mathcal{L}_p. \end{cases}$$

Then

$$|Lf(x, y)|^{p_\Phi} \leq \Big(\frac{g(x)}{[x, y]}\Big)^{p_\Phi} + \Big(\frac{g(y)}{[x, y]}\Big)^{p_\Phi}.$$

In what follows, we divide the proof into two cases based on the types of Φ.

Case I. $\Phi \in \mathcal{U}^q$, $q \in [1, n+\alpha)$ and $p_\Phi = 1$.

Applying the convexity of Φ, we have

$$\Phi\Big(\frac{|Lf(x,y)|}{2}\Big) \leq \frac{1}{2}\Phi\Big(\frac{g(x)}{[x,y]}\Big) + \frac{1}{2}\Phi\Big(\frac{g(y)}{[x,y]}\Big).$$

It follows from the definition of $\mathcal{U}^q$ that

$$\begin{aligned}
&\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2}\Big) dv_\alpha(x) dv_\alpha(y)\\
\lesssim\ & \frac{1}{2}\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(x)}{[x,y]}\Big) dv_\alpha(x) dv_\alpha(y) + \frac{1}{2}\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(y)}{[x,y]}\Big) dv_\alpha(x) dv_\alpha(y)\\
\lesssim\ & \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(x)}{[x,y]}\Big) dv_\alpha(x) dv_\alpha(y) \leq \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{2g(x)}{[x,y]}\Big) dv_\alpha(x) dv_\alpha(y)\\
\lesssim\ & \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x) \int_{\mathbb{B}} \frac{2^q(1-|y|^2)^\alpha}{[x,y]^q} dv(y).
\end{aligned}$$

Since $q \in [1, n+\alpha)$, according to Lemma 2.2, we have

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2}\Big) dv_\alpha(x) dv_\alpha(y) \quad \lesssim \quad \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x).$$

Case II. $\Phi \in \mathcal{L}_p$ and $p_\Phi = p$.

In view of Lemma 3.1, $\Phi_p(t) = \Phi(t^{\frac{1}{p}}) \in \mathcal{U}^{1/p}$ and by the convexity of Φ_p,

$$\Phi_p\Big(\Big|\frac{Lf(x,y)}{2^{\frac{1}{p}}}\Big|^p\Big) \leq \frac{1}{2}\Phi_p\Big(\Big|\frac{g(x)}{[x,y]}\Big|^p\Big) + \frac{1}{2}\Phi_p\Big(\Big|\frac{g(y)}{[x,y]}\Big|^p\Big).$$

By an argument similar to that in the proof of Case I and Lemma 3.1, we have

$$\begin{aligned}
&\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2^{\frac{1}{p}}}\Big) dv_\alpha(x) dv_\alpha(y) = \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi_p\Big(\Big|\frac{Lf(x,y)}{2^{\frac{1}{p}}}\Big|^p\Big) dv_\alpha(x) dv_\alpha(y)\\
\lesssim\ & \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi_p\Big(\Big|\frac{2g(x)}{[x,y]}\Big|^p\Big) dv_\alpha(x) dv_\alpha(y) \lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x) \int_{\mathbb{B}} \frac{2(1-|y|^2)^\alpha}{[x,y]} dv(y)\\
\lesssim\ & \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x).
\end{aligned}$$

Combing these two cases and using a standard argument based on the closed graph theorem, it concludes that the operator $L : \mathcal{B}^\Phi_\alpha \to L^\Phi_\alpha(\mathbb{B} \times \mathbb{B})$ is bounded.

For the case of $q > n + \alpha$, we can also prove the following result by using an argument similar to the one in the proof of Theorem 4.1.

Theorem 4.2. *Let $\alpha > -1$. Suppose that one of the following two conditions is satisfied:*

(1) $\Phi \in \mathcal{U}^q$ with $q \in (n+\alpha, \infty)$ and $2\beta + n = q + \alpha$;

(2) $\Phi \in \mathcal{L}_p$ and $2\beta + n = pq + \alpha$ for some $pq > n + \alpha$.

Then $L : \mathcal{B}^\Phi_\alpha \to L^\Phi_\beta(\mathbb{B} \times \mathbb{B})$ is bounded.

Proof. Let $f \in \mathcal{B}_\alpha^\Phi$. (1) If $\Phi \in \mathcal{U}^q$ with $q \in (n+\alpha, \infty)$ and $2\beta + n = q + \alpha$, then by the same reasoning as in the proof of the above results, we have

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2}\Big) dv_\beta(x) dv_\beta(y) \lesssim \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{g(x)}{[x,y]}\Big) dv_\beta(x) dv_\beta(y)$$

$$\lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\beta(x) \int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^q} dv(y) \lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x),$$

where the last inequality follows from Lemma 2.2. Hence $L : \mathcal{B}_\alpha^\Phi \to L_\beta^\Phi(\mathbb{B}\times\mathbb{B})$ is bounded.

(2) If $\Phi \in \mathcal{L}_p$ and $2\beta + n = pq + \alpha$ for some $pq > n + \alpha$, then by Lemmas 2.2 and 3.1 we have

$$\int_{\mathbb{B}}\int_{\mathbb{B}} \Phi\Big(\frac{|Lf(x,y)|}{2^{\frac{1}{p}}}\Big) dv_\beta(x) dv_\beta(y) \lesssim \int_{\mathbb{B}}\int_{\mathbb{B}} \Phi_p\Big(\Big|\frac{2g(x)}{[x,y]}\Big|^p\Big) dv_\beta(x) dv_\beta(y)$$

$$\lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\beta(x) \int_{\mathbb{B}} \frac{(1-|y|^2)^\beta}{[x,y]^{pq}} dv(y) \lesssim \int_{\mathbb{B}} \Phi(g(x)) dv_\alpha(x).$$

The result follows.

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Список литературы

[1] L. V. Ahlfors, Möbius Transformations in Several Variables, Univ. of Minnesota (1981).

[2] B. R. Choe and K. Nam, "Positive Toeplitz operators of Schatten-Herz type", Nagoya Math. J. **165**, 31 – 62 (2007).

[3] B. R. Choe and K. Nam, "Optimal norm estimate of operators related to the harmonic Bergman projection on the ball", Tohoku Math. J., **62**, 357 – 374 (2010).

[4] B. R. Choe and K. Nam, "Double integral characterizations of harmonic Bergman spaces", J. Math. Anal. Appl. **379**, 889 – 909 (2011).

[5] H. R. Cho and S. Park, Lipschitz characterization for exponentially weighted Bergman spaces of the unit ball, Proc. Japan Acad. Math Ser. A, **95**, 70 – 74 (2019).

[6] Y. Deng, L. Huang, T. Zhao and D. Zheng, "Bergman projection and Bergman spaces", J. Oper. Theor. **46**, 3 – 24 (2001).

[7] Ö. Doğan, "Harmonic Besov spaces with small exponents", Complex Var. Elliptic Equ. **65**, 1051 – 1075 (2020).

[8] X. Fu and J. Qiao, "Bergman spaces, "Bloch spaces and integral means of p-harmonic functions", Bull. Korean. Math. Soc. **58**, 481 – 495 (2021).

[9] S. Gergün, H. Kaptanğlu and A. Üreyen, "Harmonic Besov spaces on the ball", Intern J. Math. **27**, 1650070, 59 pp. (2016).

[10] R. Ma and J. Xu, "Lipschitz characterizations for Bergman-Orlicz spaces and their applications", Acta Math Scientia, **40**, 1445 – 1458 (2020).

[11] K. Nam, "Lipschitz type characterizations of harmonic Bergman spaces", Bull. Korean. Math. Soc., **50**, 1277 – 1288 (2013).

[12] M. M. Rao and Z. D. Ren, Theory of Orlicz functions, Pure and Applied Mathematics 146, Marcel Dekker Inc, New York (1991).

[13] G. Ren and U. Kähler, "Weighted Lipschitz continuity and harmonic Bloch and Besov spaces in the real unit ball", Proc. Edinb. Math. Soc., **48**, 743 – 755 (2005).
[14] B. Sehba, "Derivatives characterization of Bergman-Orlicz spaces, boundedness and compactness of Ces'aro-type operator", Bull. Sci. Math., **180**, 103193 (2022),
[15] B. Sehba and S. Stevic, "On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces", Appl. Math. Comput., **233**, 565 – 581 (2014).
[16] H. Wulan and K. Zhu, Lipschitz type characterizations for Bergman spaces, Canad Math Bull. **52**, 613 – 626 (2009).
[17] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Springer-Verlag, New York (2005).

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