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$(P, Q) - \varepsilon$-PSEUDO CONDITION SPECTRUM FOR 2×2 MATRICES. LINEAR OPERATOR AND APPLICATION

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Abstract. We define a new type of spectrum, called the $(P, Q) - \varepsilon$-pseudo condition spectra

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \sigma^{(2)}_{(P,Q)}(T) \bigcup \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| > \frac{1}{\varepsilon} \right\}.$$

This $(P, Q) - \varepsilon$-pseudo condition spectrum shares some properties of the usual spectrum such as non emptiness. Our aim in this paper is to show some properties of $(P, Q) - \varepsilon$-pseudo condition spectra of a linear operator T in Banach spaces and reveal the relation between their $(P, Q) - \varepsilon$-pseudo condition spectra. Additionally, we investigate the $(P, Q) - \varepsilon$-pseudo condition spectrum of a block matrix in a Banach space.

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1. Introduction

For the past ten years, there has been in the field of mathematics digital technology has a keen interest in the study of the notion of pseudo-spectrum and pseudo condition spectra. The development of this notion is explained by the fact that in a certain number of mathematical engineering problems were natural non-self-employed operators. This original observation suggests that in some cases, knowledge of the spectrum of an operator alone does not sufficiently understand his action. It is as well as to make up for this apparent lack of information contained in the spectrum, new subsets of the complex plane called pseudo-spectra have been introduced. There are several generalizations of the concept of the spectrum in literature such as Ransford spectrum [8], pseudo spectrum [12], condition spectrum [1, 2, 5, 10], pseudo spectra of multivalued linear operator [3]. Unlike the spectrum, which is a purely algebraic concept, both the pseudo spectrum and condition spectrum depend on the norm. Also, both these sets contain the spectrum as a subset.

Consider two idempotent elements $P, Q \in \mathcal{B}(X)$ i.e. $P^2 = P$ and $Q^2 = Q$.

Definition 1.1. *Let $T \in \mathcal{B}(X)$. An operator $S \in \mathcal{B}(X)$ satisfying,*

$$STS = S, \quad ST = P \quad \text{and} \quad I - TS = Q$$

will be called a (P,Q)-outer generalized inverse of T and it is denoted by $T^{(2)}_{P,Q}$.

The detailed treatment of outer generalized inverses of operators on Banach and Hilbert spaces can be found in [4, 7].

Definition 1.2. *For an element $T \in \mathcal{B}(X)$, the (P,Q)-resolvent set is defined as*

$$\rho^{(2)}_{(P,Q)}(T) := \Big\{\lambda \in \mathbb{C} : (\lambda - T)^{(2)}_{P,Q} \text{ exist}\Big\}.$$

The complement of the set $\rho^{(2)}_{(P,Q)}(T)$ is called (P,Q)-spectrum and it is denoted by $\sigma^{(2)}_{(P,Q)}(T)$.

From now onwards, we consider the idempotent $P \neq 0$ and $P \neq I$ and we fix the operator $Q = I - P$. If $\lambda \in \rho^{(2)}_{(P,Q)}(T)$, then we denote $(\lambda - T)^{(2)}_{P,Q}$ by $R_T(\lambda)$. For given $T \in \mathcal{B}(X)$, if $R_T(\lambda)$ exists for some $\lambda \in \mathbb{C}$, then from Definition 1.1,

$$R_T(\lambda)(\lambda - T) = P \quad \text{and} \quad (\lambda - T)R_T(\lambda) = P \tag{1.1}$$

By (Eq. 1.1), $TP = PT$. Consequently, if $TP \neq PT$ then $\sigma^{(2)}_{(P,Q)}(T) = \mathbb{C}$. Because of this reason, in the rest of the paper we assume $TP = PT$.

In this note, we dedicate to research the $(P,Q) - \varepsilon$-pseudo condition spectra of a linear operator and its properties. The remainder of this paper is organized as follows. In Section 2, we first suggest a characterize for the $(P,Q) - \varepsilon$-pseudo condition spectra of a linear operator. Then, in Section 3, we investigate the $(P,Q)-$ ε-pseudo condition spectra, of a block matrix in a Banach space.

2. $(P,Q) - \varepsilon$-Pseudo condition spectra of linear operator

The $(P,Q) - \varepsilon$-pseudo spectrum were studied in [6, 11]. Let $\varepsilon > 0$ and $T \in \mathcal{B}(X)$. The $(P,Q) - \varepsilon$-pseudo spectrum is defined as

$$\sigma^{(2)}_{(P,Q)-\varepsilon}(T) := \Big\{\lambda \in \mathbb{C} : (\lambda - T)^{(2)}_{P,Q} \text{ does not exist or } \Big\|(\lambda - T)^{(2)}_{P,Q}\Big\| > \varepsilon\Big\}.$$

By convention, we write $\|R_T(\lambda)\| = \infty$ if $R_T(\lambda)$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma^{(2)}_{(P,Q)}(T)$. It is well known that $\rho^{(2)}_{(P,Q)-\varepsilon}(T)$ for any $T \in \mathcal{B}(X)$ is a nonempty open subset, the following remark prove the same for $(P,Q) - \varepsilon$-pseudo resolvent set. In this section, we define the pseudo spectra of linear relation and study some properties.

Definition 2.1. *($(P,Q)-\varepsilon$-pseudo spectra of T)*
Let $T \in \mathcal{B}(X)$ where X is a normed space and $\varepsilon > 0$ we define the $(P,Q)-\varepsilon$-pseudo spectra of T by

$$\sigma^{(2)}_{(P,Q)-\varepsilon}(T) \quad = \quad \sigma^{(2)}_{(P,Q)}(T) \bigcup \left\{ \lambda \in \mathbb{C} : \quad \|R_T(\lambda)\| > \frac{1}{\varepsilon} \right\}.$$

We denote the $(P,Q)-\varepsilon$-pseudo resolvent set of T

$$\rho^{(2)}_{(P,Q)-\varepsilon}(T) = \mathbb{C} \backslash \sigma^{(2)}_{(P,Q)-\varepsilon}(T) = \rho^{(2)}_{(P,Q)}(T) \bigcap \left\{ \lambda \in \mathbb{C} : \quad \|R_T(\lambda)\| \leq \frac{1}{\varepsilon} \right\}.$$

Definition 2.2. *($(P,Q)-\varepsilon$-pseudo condition spectra of T)*
Let $T \in \mathcal{B}(X)$ where X is a normed space and $\varepsilon > 0$ we define the $(P,Q)-\varepsilon$-pseudo condition spectra of T by

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T) \quad = \quad \sigma^{(2)}_{(P,Q)}(T) \bigcup \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| > \frac{1}{\varepsilon} \right\}$$

with the convention that $\|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| = \infty$, if $(\lambda - T)^{(2)}_{(P,Q)}$ is not exists. Notice that the uniqueness of $\Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ allows us to consider the $(P,Q)-\varepsilon$-pseudo condition spectrum and $(P,Q)-\varepsilon$-pseudo spectrum.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$ and $0 < \varepsilon < 1$. Then,*

(1) $\sigma^{(2)}_{(P,Q)}(T) = \bigcap\limits_{0<\varepsilon<1} \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.

(2) *If $0 < \varepsilon_1 < \varepsilon_2 < 1$, then*

$$\sigma^{(2)}_{(P,Q)}(T) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon_1}(T) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon_2}(T).$$

(3) *If $\alpha \in \mathbb{C}$, then*

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T + \alpha I) = \alpha + \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Proof. (1) It is clear that $\sigma^{(2)}_{(P,Q)}(T) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ for all $0 < \varepsilon < 1$. Then,

$$\sigma^{(2)}_{(P,Q)}(T) \subset \bigcap_{0<\varepsilon<1} \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Conversely, if $\lambda \in \bigcap\limits_{0<\varepsilon<1} \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then for all $0 < \varepsilon < 1$, we get $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. We will discuss these two cases:

<u>1^{st} case</u> : If $\lambda \in \sigma^{(2)}_{(P,Q)}(T)$, then we get the desired result.

<u>2^{nd} case</u> : If $\lambda \in \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| > \frac{1}{\varepsilon} \right\}$, then taking limits as $\varepsilon \longrightarrow 0^+$, we get

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| \|\lambda - T\| = \infty.$$

We deduce that $\lambda \in \sigma^{(2)}_{(P,Q)}(T)$.

(2) Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon_1}(T)$, so

$$\|(\lambda - T)^{(2)}_{(P,Q)}\|\|\lambda - T\| > \frac{1}{\varepsilon_1} > \frac{1}{\varepsilon_2}.$$

We conclude that $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon_2}(T)$. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T + \alpha I)$, hence

$$\|((\lambda - \alpha) - T)^{(2)}_{(P,Q)}\|\|(\lambda - \alpha) - T\| > \frac{1}{\varepsilon}.$$

Therefore, $\lambda - \alpha \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. This yields to

$$\lambda \in \alpha + \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Lemma 2.1. *Let $T \in \mathcal{B}(X)$, $0 < \varepsilon < 1$ and P is invertible. Then, $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash \sigma^{(2)}_{(P,Q)}(T)$ if and only if there exists x such that*

$$\|P^{-1}(\lambda - T)x\| < \varepsilon\|\lambda - T\|\|x\|.$$

Proof. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T)$, then

$$\|(\lambda - T)^{(2)}_{(P,Q)}\|\|\lambda - T\| > \frac{1}{\varepsilon}.$$

Thus

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Moreover

$$\sup_{y\neq 0} \frac{\|(\lambda - T)^{(2)}_{(P,Q)}y\|}{\|y\|} > \frac{1}{\varepsilon\|\lambda - T\|}.$$

Then, there exists a nonzero $y \in X$ such that

$$\|(\lambda - T)^{(2)}_{(P,Q)}y\| > \frac{\|y\|}{\varepsilon\|\lambda - T\|}.$$

Putting $x = (\lambda - T)^{(2)}_{(P,Q)}y$, then $(\lambda - T)x = (\lambda - T)(\lambda - T)^{(2)}_{(P,Q)}y = Py$. Hence,

$$\varepsilon\|\lambda - T\|\|x\| > \|P^{-1}(\lambda - T)x\|.$$

Conversely, we assume that there exists $x \in X$ such that

$$\varepsilon\|\lambda - T\|\|x\| > \|P^{-1}(\lambda - T)x\|.$$

Let $\lambda \notin \sigma^{(2)}_{(P,Q)}(T)$ and $x = (\lambda - T)^{(2)}_{(P,Q)}y$, then

$$\|x\| \le \|(\lambda - T)^{(2)}_{(P,Q)}\|\|y\|.$$

Moreover,

$$\varepsilon\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\|\|y\| > \|P^{-1}(\lambda - T)x\| = \|y\|.$$

It follows that $1 < \varepsilon\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\|$. We conclude that,

$$\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T).$$

Suppose X is a Banach space with the following property: For all generalized invertible operator $T \in \mathcal{B}(X)$ there exist $B \in \mathcal{B}(X)$ such that B is not generalized invertible and

$$\|T - B\| = \frac{1}{\|T^{(2)}_{(P,Q)}\|}.$$

Theorem 2.2. *Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. Then, there exists $D \in \mathcal{B}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \Sigma^{(2)}_{(P,Q)}(T + D)$.*

Proof. Suppose $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. We will discuss these two cases:

$\underline{1^{st}\ case}$: If $\lambda \in \sigma^{(2)}_{(P,Q)}(T)$, then it is sufficient to take $D = 0$.

$\underline{2^{nd}\ case}$: If $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T)$. Hence, by assumption, there exists an element $B \in \mathcal{B}(X)$ such that

$$\|\lambda - T - B\| = \frac{1}{\|(\lambda - T)^{(2)}_{(P,Q)}\|}.$$

Let $D = \lambda - T - B$. Then

$$\|D\| = \frac{1}{\|(\lambda - T)^{(2)}_{(P,Q)}\|} \leq \varepsilon\|\lambda - T\|.$$

Also $B = \lambda - (T + D)$, is not generalized invertible. So, $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$.

Corollary 2.1. *Let X be a Banach space satisfying the hypothesis of Theorem 2.3. Then, $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ if, and only if, there exists $D \in \mathcal{B}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$.*

Theorem 2.3. *Let $T \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$, and $0 < \varepsilon < 1$. If there is $D \in \mathcal{B}(X)$ such that $\|D\| \leq \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$. Then, $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.*

Proof. We assume that there exists D such that $\|D\| < \varepsilon\|\lambda - T\|$ and $\lambda \in \sigma^{(2)}_{(P,Q)}(T + D)$. Let $\lambda \notin \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then for all $(\lambda - T)^{(2)}_{(P,Q)}$ a generalized inverse of $\lambda - T$ we have

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \leq \frac{1}{\varepsilon}.$$

Now, we define the operator $S : X \longrightarrow X$ by

$$S := \sum_{n=0}^{\infty}(\lambda - T)^{(2)}_{(P,Q)}\left(D(\lambda - T)^{(2)}_{(P,Q)}\right)^n.$$

Since,

$$\|D(\lambda - T)^{(2)}_{(P,Q)}\| < 1,$$

we can write

$$S = (\lambda - T)^{(2)}_{(P,Q)}\left(I - D(\lambda - T)^{(2)}_{(P,Q)}\right)^{-1}.$$

Then, there exists $y \in X$ such that

$$S\Big(I - D(\lambda - T)^{(2)}_{(P,Q)}\Big)y = (\lambda - T)^{(2)}_{(P,Q)}y.$$

Let $y = P(\lambda - T)x$. Then,

$$S(\lambda - T - D)Px = Px$$

for every $x \in X$. Hence, $\lambda - T - D$ is generalized invertible, so $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.

Theorem 2.4. *Let $T \in \mathcal{B}(X)$, $k = \|T\|\|T^{(2)}_{(P,Q)}\|$ and $0 < \varepsilon < 1$. Then,*

(i)$\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ if, and only if, $\overline{\lambda} \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T^{'})$.

(ii) If $\lambda_n \notin \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)(T)$ such that $\lambda_n \to \lambda$ for all $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)(T)$, then $\|(\lambda - T)^{(2)}_{(P,Q)}\| = \infty$.

Proof. (i) Using the identity

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| = \|\overline{\lambda} - T^{'}\|\|(\overline{\lambda} - T^{'})^{(2)}_{(P,Q)}\|,$$

it is easy to see that the $(P,Q) - \varepsilon$-pseudo condition spectrum of $T^{'}$ is given by the mirror image of $\Sigma_{\varepsilon}(T)$ with respect to the real axis.

(ii) Suppose $\|(\lambda - T)^{(2)}_{(P,Q)}\| \leq \dfrac{1}{\delta}$ for some $\delta \in \mathbb{R}$ and since $\lambda_n \to \lambda$ for all $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then there exists $n_0 \in \mathbb{N}$ such that

$$|\lambda_n - \lambda| < \delta - 1 < \delta \leq \frac{1}{\|(\lambda - T)^{(2)}_{(P,Q)}\|} \quad \text{for all } n \geq n_0.$$

Hence, $\lambda \notin \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$. This is a contradiction.

Theorem 2.5. *Let $T, E \in \mathcal{B}(X)$ such that $\|E\| < \dfrac{\varepsilon}{2}\|\lambda - T\|$ and $0 < \varepsilon < 1$. Then,*

$$\Sigma^{(2)}_{(P,Q)-(\frac{\varepsilon}{2}-\|E\|)}(T) \subseteq \Sigma^{(2)}_{(P,Q)-\varepsilon}(T + E) \subseteq \Sigma^{(2)}_{(P,Q)-\tau_{\varepsilon}}(T)$$

where, $0 < \tau_{\varepsilon} = \dfrac{\varepsilon^2}{2} + \varepsilon < 1$ and $0 < \frac{\varepsilon}{2} - \|E\| < 1$.

Proof. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-(\frac{\varepsilon}{2}-\|E\|)}(T)$. Then, by Theorem 2.3, there exists a bounded operator $D \in \mathcal{B}(X)$ with

$$\|D\| < \Big(\frac{\varepsilon}{2} - \|E\|\Big)\|\lambda - T\|$$

such that

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T + D) = \sigma^{(2)}_{(P,Q)}\Big((T + E) + (D - E)\Big).$$

The fact that

$$\|D - E\| \leq \|D\| + \|E\| < \Big(\frac{\varepsilon}{2} - \|E\|\Big)\|\lambda - T\| + \|E\| < \varepsilon\|\lambda - T\|,$$

allows us to deduce that $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T+E)$. Now, let us prove the second inclusion. Suppose $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T+E)$, then there exists $D \in \mathcal{B}(X)$ verifying

$$\|D\| < \varepsilon\|\lambda - T - E\| \leq \varepsilon\|\lambda - T\| + \varepsilon\|E\|$$

and $\lambda \in \sigma^{(2)}_{(P,Q)}(T+E+D)$. The fact that $\|D+E\| \leq \tau_\varepsilon\|\lambda - T\|$ allows us to deduce that $\lambda \in \Sigma^{(2)}_{(P,Q)-\tau_\varepsilon}(T)$.

Theorem 2.6. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then, $\Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ has no isolated points.*

Proof. Suppose $\Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ has an isolated point μ. Then there exists an $\delta > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$ and there exists a generalized invertible $(\lambda - T)^{(2)}_{(P,Q)}$ such that

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| < \frac{1}{\varepsilon}.$$

Let $\mu \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)\backslash\sigma^{(2)}_{(P,Q)}(T)$. Then, using the Hahn-Banach Theorem, there exist $x' \in X'$ such that

$$x'\Big((\mu - T)^{(2)}_{(P,Q)}\Big) = \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \ \text{ with } \ \|x'\| = 1.$$

Now, we define

$$\begin{cases} \phi : \rho^{(2)}_{(P,Q)}(T) \longrightarrow \mathbb{R}, \\ \qquad\qquad \lambda \longrightarrow \phi(\lambda) = x'\Big((\lambda - T)^{(2)}_{(P,Q)}\Big). \end{cases}$$

Since ϕ is is well-defined and continuous; in $B(\mu, \delta)$ and for all $\lambda \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, we have

$$|\phi(\lambda)| \quad = \quad \Big|x'((\lambda - T)^{(2)}_{(P,Q)})\Big| = \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| < \frac{1}{\varepsilon}.$$

But, $\phi(\mu) = \|\mu - T\|\|(\mu - T)^{(2)}_{(P,Q)}\| \geq \frac{1}{\varepsilon}$. This contradicts the maximum modulus principle.

Definition 2.3. *We define $T \in \mathcal{B}(X)$ to be of d-class operator if*

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall\lambda \in \mathbb{C}\backslash\sigma^{(2)}_{(P,Q)}(T).$$

In fact, we have the following theorem

Theorem 2.7. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. If $T \in \mathcal{B}(X)$ is of $d-$class operator, then*

$$\Sigma^{(2)}_{(P,Q)-\varepsilon}(T) \subseteq \Big\{\lambda \in \mathbb{C} : d(\lambda, \sigma^{(2)}_{(P,Q)}(T)) \leq \varepsilon\|\lambda - T\|\Big\}.$$

Proof. Let $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then

$$\|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| > \frac{1}{\varepsilon}.$$

Now, if $T \in \mathcal{B}(X)$ is a d-class, we already have

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall \lambda \in \mathbb{C} \backslash \sigma^{(2)}_{(P,Q)}(T).$$

Hence,

$$\frac{1}{\varepsilon} < \|\lambda - T\| \|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{\|\lambda - T\|}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall \lambda \in \mathbb{C} \backslash \sigma^{(2)}_{(P,Q)}(T).$$

Therefore,

$$\lambda \in \Big\{ \lambda \in \mathbb{C} : d(\lambda, \sigma^{(2)}_{(P,Q)}(T)) \leq \varepsilon \|\lambda - T\| \Big\}.$$

Theorem 2.8. *Let $T \in \mathcal{B}(X)$ and $\varepsilon > 0$. Then*
(i) If $T = \mu I$ for some number μ, then T is of d-class operator and $\sigma^{(2)}_{(P,Q)}(T) = \{\mu\}$.
(ii) If T is of d-class operator, then $\alpha T + \beta$ is also of d-class operator for every number α, β.

Proof. (i) Let $T = \mu$. for some number μ. Then clearly $\sigma^{(2)}_{(P,Q)}(T) = \{\mu\}$. Hence for all $\lambda \in \mathbb{C} \setminus \sigma^{(2)}_{(P,Q)}(T)$, we have $\lambda \neq \mu$. Thus

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{|\lambda - \mu|} = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))}.$$

This shows that a is of d-class operator.
(ii) Next suppose that T is of d-class operator and $B = \alpha T + \beta$ for some complex numbers α, β. We want to prove that B is of d-class operator. If $\alpha = 0$, then it follows from (i). So assume that $\alpha \neq 0$. Let $w \notin \sigma^{(2)}_{(P,Q)}(B) = \{\alpha\lambda + \beta : \lambda \in \sigma^{(2)}_{(P,Q)}(B)\}$. Then, $\lambda := \frac{w-\beta}{\alpha} \notin \sigma^{(2)}_{(P,Q)}(B)$ and since T is of d-class operator,

$$\|(\lambda - T)^{(2)}_{(P,Q)}\| = \frac{1}{d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \qquad \forall \lambda \in \mathbb{C} \backslash \sigma^{(2)}_{(P,Q)}(T).$$

Now

$$\|(w - B)^{(2)}_{(P,Q)}\| = \|(\alpha\lambda + \beta - (\alpha T + \beta))^{(2)}_{(P,Q)}\| = \frac{1}{|\alpha|} \|(\lambda - T)^{(2)}_{(P,Q)}\|.$$

Therefore,

$$\begin{aligned} \|(w - B)^{(2)}_{(P,Q)}\| &= \frac{1}{|\alpha| d(\lambda, \sigma^{(2)}_{(P,Q)}(T))} \\ &= \frac{1}{d(\lambda\alpha, \sigma^{(2)}_{(P,Q)}(\alpha T))} = \frac{1}{d(w, \sigma^{(2)}_{(P,Q)}(B))}. \end{aligned}$$

This shows that B is of d-class operator.

Remark 2.1. *Under what additional conditions can we conclude that, if T is of d-class operator and $\sigma^{(2)}_{(P,Q)}(T) = \{\mu\}$, then $T = \mu$.*

Theorem 2.9. *Let $T \in \mathcal{B}(X)$ and for every $0 < \varepsilon < 1$ such that $\varepsilon < \|\lambda - T\|$ we have*
(i) $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ *if, and only if,* $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon\|\lambda-T\|}(T)$.
(ii) $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$ *if, and only if,* $\lambda \in \Sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{\|\lambda-T\|}}(T)$.

Proof. *(i)* If $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \ \text{ and } \ \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{1}{\varepsilon}.$$

Hence,

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \ \text{ and } \ \|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{1}{\varepsilon\|\lambda - T\|},$$

which implies that $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon\|\lambda-T\|)}(T)$. The converse is similar.
(ii) Let $\lambda \in \sigma^{(2)}_{(P,Q)-\varepsilon}(T)$, then

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \ \text{ and } \ \|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{1}{\varepsilon}.$$

Hence it follows that

$$\lambda \in \sigma^{(2)}_{(P,Q)}(T) \ \text{ and } \ \|\lambda - T\|\|(\lambda - T)^{(2)}_{(P,Q)}\| \geq \frac{\|\lambda - T\|}{\varepsilon}.$$

This proves that

$$\lambda \in \Sigma^{(2)}_{(P,Q)-\frac{\varepsilon}{\|\lambda-T\|}}(T).$$

The converse is similar.

3. Application for matrix 2×2

In this article we will apply the results of the previous section to determine the $(P,Q)-\varepsilon$-pseudo condition spectrum of 2×2 matrix operators by mean of measure of non-strict-singularity. Let X and Y be tow Banach spaces and consider the 2×2 block operator matrix defined on $X \times Y$ by

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where, $T_1, T_2 \in \mathcal{B}(X)$. Defining the norm of the linear operator matrix T as

$$\|T\| = \max\Big\{\|T_1\|, \|T_2\|\Big\}.$$

Now, we state an auxiliary result.

Lemma 3.1. [9, Lemma 3.1] *Let* $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ *and* $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. *If* $T^{(2)}_{P,Q}$ *exist, then,*

$$T^{(2)}_{(P,Q)} = \begin{pmatrix} (T_1)^{(2)}_{(P_1,Q_1)} & 0 \\ 0 & (T_2)^{(2)}_{(P_2,Q_2)} \end{pmatrix}.$$

Theorem 3.1. *Let* $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ *and* $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$. *If* $T^{(2)}_{P,Q}$ *exist, then,*

$$\Sigma^{(2)}_{(P_1,Q_1)-\varepsilon}(T_1) \bigcup \Sigma^{(2)}_{(P_2,Q_2)-\varepsilon}(T_2) \subset \Sigma^{(2)}_{(P,Q)-\varepsilon}(T).$$

Proof. Let $\lambda \in \Sigma^{(2)}_{(P_1,Q_1)-\varepsilon}(T_1) \bigcup \Sigma^{(2)}_{(P_2,Q_2)-\varepsilon}(T_2)$. These imply

$$\lambda \notin \Sigma^{(2)}_{(P_1,Q_1)}(T_1) \quad \text{or} \quad \|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\| \|(\lambda - T_1)\| > \frac{1}{\varepsilon}$$

or

$$\lambda \notin \Sigma^{(2)}_{(P_2,Q_2)}(T_2) \quad \text{or} \quad \|(\lambda - T_2)^{(2)}_{(P_2,Q_2)}\| \|(\lambda - T_2)\| > \frac{1}{\varepsilon}.$$

If either $(\lambda - T_1)^{(2)}_{(P_1,Q_1)}$ *or* $(\lambda - T_2)^{(2)}_{(P_2,Q_2)}$ *does not exists*, by Lemma 3.1, it follows:

$$(\lambda - T)^{(2)}_{(P,Q)} = \begin{pmatrix} (\lambda - T_1)^{(2)}_{(P_1,Q_1)} & 0 \\ 0 & (\lambda - T_2)^{(2)}_{(P_2,Q_2)} \end{pmatrix}$$

does not exists, then we have $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.
On the other hand, if $(\lambda - T_1)^{(2)}_{(P_1,Q_1)}$ and $(\lambda - T_2)^{(2)}_{(P_2,Q_2)}$ exists, it holds either

$$\|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\| \|(\lambda - T_1)\| > \frac{1}{\varepsilon} \quad \textit{or} \quad \|(\lambda - T_2)^{(2)}_{(P_2,Q_2)}\| \|(\lambda - T_2)\| > \frac{1}{\varepsilon}.$$

Without loss of generality, assume that $\|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\| \|(\lambda - T_1)\| > \frac{1}{\varepsilon}$ holds. Therefore,

$$\begin{aligned} &\|(\lambda - T)^{(2)}_{(P,Q)}\| \|(\lambda - T)\| = \\ &= \max\left\{ \|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\|, \|(\lambda - T_2)^{(2)}_{(P_2,Q_2)}\| \right\} \max\{\|(\lambda - T_1)\|, \|(\lambda - T_2)\|\} \\ &\geqslant \|(\lambda - T_1)^{(2)}_{(P_1,Q_1)}\| \|(\lambda - T_1)\| > \frac{1}{\varepsilon}. \end{aligned}$$

This proves that $\lambda \in \Sigma^{(2)}_{(P,Q)-\varepsilon}(T)$.

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