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OPERATORS ON MIXED-NORM AMALGAM SPACES VIA EXTRAPOLATION

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Abstract. Let $t \in (0, \infty)$, $\vec{p} \in (1, \infty)^n$ and $\vec{q} \in [1, \infty)^n$. We establish versions of the Rubio de Francia extrapolation theorem, and further obtain the bounds for some classical operators and the commutators in harmonic analysis on the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. As an application, a characterization of the mixed-norm amalgam spaces is given.

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1. Introduction

In 1926, the first appearance of amalgam spaces can be traced to Wiener [36]. But the first systematic study of these spaces was undertaken by Holland [20] in 1975. Feichtinger initially called these spaces Wiener-type spaces in the early 1980's in a series of papers [14, 15, 16], and then, following a suggestion of Benedetto, adopted the name Wiener amalgam spaces. That is, for p, $q \in (0, \infty)$, the amalgam space $(L^p, L^q)(\mathbb{R})$ is defined by

$$(L^p, \ell^q)(\mathbb{R}) := \left\{ f \in L^p_{\rm loc}(\mathbb{R}) : \left[\sum_{n \in \mathbb{Z}} \left(\int_n^{n+1} |f(x)|^p \, dx \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty \right\}.$$

Wiener amalgam spaces are a central object of the time-frequency analysis, another area with links to several mathematical subjects as well as its applications. The mixed amalgam spaces provide for a basic tool for harmonic analysis. And that makes these spaces extremely prominent to us. Very recently, lots of vital work has been done in the study of amalgam spaces. In 2011, Ruzhansky, Sugimoto, Toft and Tomita [29] established various properties of global and local changes of variables as well as properties of canonical transforms on Wiener amalgam spaces. In 2016, Delgado, Ruzhansky and Wang proved the metric approximation property for Wiener amalgam spaces in [8] and [9]. In 2022, Wang [35] obtained global

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regularity estimates for solutions of non-divergence elliptic equations on amalgam spaces spaces if the coefficient matrix is symmetric. For some historical notes and for an introduction about Wiener amalgam spaces on the real line can also be refered to [18].
Recently, to study the weak solutions of boundary value problems for a t-independent elliptic systems in the upper half plane, Auscher and Mourgoglou [2] introduced a particular amalgam space, the slice space $E_t^p(\mathbb{R}^n)$. Moreover, Auscher and Prisuelos-Arribas [3] introduced a more general slice space $(E_r^q)_t(\mathbb{R}^n)$ for $r \in (1,\infty)$, $t \in (0,\infty)$ and $q \in [1,\infty)$, and studied the boundedness of some classical operators on these spaces. For further study and a deeper account of developments on the slice spaces we may consult [19, 40] and the references therein. In 2022, Zhang and Zhou [39] first introduced the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$, as natural generalizations of the amalgam space $(L^p, L^q)_t(\mathbb{R}^n)$.

For $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty)^n$, the main purpose of this paper is to establish a version of the Rubio de Francia extrapolation theorem on mixed-norm amalgam spaces, and obtain the boundedness of some classical operators and the linear commutators by this theorem over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. Moreover, we study characterizations of mixed-norm amalgam spaces via the Littlewood–Paley functions. The bounds for the commutators and the characterizations of the mixed-norm amalgam spaces are new results even for the classical amalgam spaces.

This paper is organized as follows. Main definitions and necessary lemmas will be given in Section 2. In Section 3, we establish versions of the Rubio de Francia extrapolation theorem over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. In Section 4, the boundedness of some operators and the commutators are given on mixed-norm amalgam spaces by the results of Section 3. In the final section, characterizations of mixed-norm amalgam spaces via the Littlewood–Paley functions is given.

Finally, we make some convention on notation. Let $\vec{p} = (p_1, ..., p_n)$ be n-tuples and $1 \le p_1, ..., p_n \le \infty$. $\vec{p} < \vec{q}$ means that $p_i < q_i$ holds, $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$ means that $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ holds, and $\vec{p}/p_0$ means that p_i/p_0 holds, where $p_0 \in [1,\infty)$, $i = 1, ..., n$. For $\alpha > 0$ and a cube $Q \subset \mathbb{R}^n$. $A \sim B$ means that A is equivalent to B, that is, $A \lesssim B$ $(A \le CB)$ and $B \lesssim A$ $(B \le CA)$, where C is a positive constant. Throughout this paper, the letter C will be used for positive constants independent of relevant variables that may change from one occurrence to another.

2. Definitions and main lemmas

To state our main definitions, we begin with the definition of the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ which introduced by Benedek and Panzone [4] in 1961.

Let $\vec{p} \in [1,\infty]^n$. The mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined by the set of all measurable functions f on $\mathbb{R}^n$, such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |f(x_1,...,x_n)|^{p_1}\, dx_1\right)^{\frac{p_2}{p_1}} \cdots dx_n\right)^{\frac{1}{p_n}} < \infty,$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1,...,n\}$.

Definition 2.1. *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$. The mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is defined as the space of all measurable functions f on $\mathbb{R}^n$ satisfying*

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} := \left\| \frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} < \infty,$$

with the usual modification for $q_i = \infty$ for each $i = 1,...,n$.

Remark 2.1. *If $p_1 = \cdots = p_n = p$ and $q_1 = \cdots = q_n = q$, then $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is the slice space $(E_p^q)_t(\mathbb{R}^n)$ and the amalgam space $(L^p, L^q)_t(\mathbb{R}^n)$ (see* [3, 2]*).*

Definition 2.2. *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$. The weak mixed-norm amalgam space $W(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is defined as the space of all measurable functions f on $\mathbb{R}^n$ satisfying*

$$\|f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda \left\|\chi_{\{x\in\mathbb{R}^n : |f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} < \infty,$$

with the usual modification for $q_i = \infty$, $i = 1,...,n$.

Note that if $p_1 = \cdots = p_n = p$ and $q_1 = \cdots = q_n = p$, then $W(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ is the weak Lebesgue space $L^{p,\infty}(\mathbb{R}^n)$, where

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}} < \infty.$$

We still recall the definition of Muckenhoupt's weights $A_p (1 \le p \le \infty)$. These weights introduced in [26] were used to characterize the boundedness of the Hardy–Littlewood maximal operator on weighted Lebesgue spaces. For a locllay integrable function f and for every $x \in \mathbb{R}^n$, the centered Hardy–Littlewood maximal operator is defined by,

$$Mf(x) := \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|\, dy,$$

and the uncentered Hardy–Littlewood maximal operator is defined by,

$$M_u f(x) := \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Definition 2.3. *Let $1 < p < \infty$. A weight w is said to be of class A_p if*

$$[w]_{A_p} := \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{\frac{1}{1-p}}\,dx\right)^{p-1} < \infty.$$

A weight w is said to be of class A_1 if

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty, \quad \textit{for almost all } x \in \mathbb{R}^n.$$

For $p = \infty$, $A_\infty := \cup_{p\geq 1} A_p$.

Some vital lemmas over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ will be given in the following.

(b) For any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_n(\cdots(M_1(f)\cdots))(x) = \sup_{I_n\in\mathbb{I}_{x_n}} \left\{\frac{1}{|I_n|}\int_{I_n} \cdots \sup_{I_1\in\mathbb{I}_{x_1}} \left[\frac{1}{|I_1|}\int_{I_1} |f(y_1,...,y_n)|\,dy_1\right]\cdots dy_n\right\},$$

where, for any $k \in \{1,...,n\}$, $\mathbb{I}_{x_k}$ denotes the set of all intervals in $\mathbb{R}$ containing x_k. Then, it is easy to see that, for any $x \in \mathbb{R}^n$,

$$M(f)(x) \leq M_n(\cdots(M_1(f)\cdots))(x).$$

Lemma 2.1. [39] *Let $t \in (0,\infty)$. Given $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$,*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}, \;\; f \in (L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n) \;\textit{and}\; g \in (L^{\vec{p}'}, L^{\vec{q}'})_t(\mathbb{R}^n).$$

where $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = \frac{1}{\vec{q}} + \frac{1}{\vec{q}'} = 1$.

Lemma 2.2. [39] *Let $t \in (0,\infty)$. Given $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$,*

$$[(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)]' = (L^{\vec{p}'}, L^{\vec{q}'})_t(\mathbb{R}^n),$$

where $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = \frac{1}{\vec{q}} + \frac{1}{\vec{q}'} = 1$, and as for dual space of mixed-norm amalgam spaces, then

$$[(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)]' := \left\{f : \|f\|_{[(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)]'} := \sup_{\|g\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\leq 1} \int_{\mathbb{R}^n} f(x)g(x)dx\right\}.$$

Lemma 2.3. [39] *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $\vec{q} \in [1,\infty]^n$. For any constant $\rho \in [1,\infty)$, we have*

$$C_1 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_{\rho t}(\mathbb{R}^n)} \leq \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq C_2 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_{\rho t}(\mathbb{R}^n)},$$

where the positive constants C_1, C_2 are independent of f and t.

Lemma 2.4. [5, 21]*If $\vec{q} = (q_1,...,q_n)$ satisfies one of the following conditions:*

(a) $1 < q_1,...,q_n < \infty$;

(b) $q_1 = \cdots = q_n = \infty$.

Then for any $f \in L^{\vec{q}}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator M is bounded on $L^{\vec{q}}(\mathbb{R}^n)$.

The boundedness of the Hardy–Littlewood maximal operator on mixed Lebesgue spaces $L^{\vec{q}}(\mathbb{R}^n)$ for the case of (a) is just [21, Lemma 3.5]. And the case of (c) holds by a similar argument to the bounds for M on $L^{\infty}(\mathbb{R})$ (see [5, p.14]).

Lemma 2.5. *Let $t \in (0,\infty)$ and $\vec{p} \in (1,\infty)^n$. Assume that $\vec{q}$ satisfies the conditions of Lemma 2.4, then the Hardy–Littlewood maximal operator is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.*

Proof. Fix $x \in \mathbb{R}^n$ and $t > 0$, and split the sumprem into $0 < r \leq t$ and $t < r$, and then

$$Mf(y) \leq \sup_{0<r\leq t} \frac{1}{|Q(y,r)|} \int_{Q(y,r)} |f(z)|dz + \sup_{r>t} \frac{1}{|Q(y,\tau)|} \int_{Q(y,r)} |f(z)|dz := I + II.$$

For I, since $y \in Q(x,t)$, $Q(y,r) \subset Q(x,2t)$. Then

$$I \lesssim \sup_{0<r\leq t} \frac{1}{|Q(y,r)|} \int_{Q(y,r)} |f(z)|\,\chi_{Q(x,2t)}(z)dz \leq M(f\chi_{Q(\cdot,2t)})(y).$$

For II, for any $z, \xi \in \mathbb{R}^n$, $\xi \in Q(z,t)$ is equivalent to $z \in Q(\xi,t)$. If $z \in Q(y,r)$, $\xi \in Q(z,t)$, then $\xi \in Q(y,2r)$. Besides, owing to $x \in Q(y,t)$, then $x \in Q(y,2r)$. Applying the Fubini's theorem and the Hölder inequality, then we get

$$\begin{aligned}
II &= \sup_{r>t} \frac{1}{|Q(y,r)|} \int_{Q(y,r)} |f(z)|_{Q(z,t)} d\xi dz \\
&\lesssim \sup_{r>t} \frac{1}{|Q(y,2r)|} \int_{Q(y,2r)} \frac{1}{|Q(\xi,t)|} \int_{Q(\xi,t)} |f(z)|dzd\xi \\
&\leq M_u\left(\frac{1}{|Q(\cdot,t)|} \int_{Q(\cdot,t)} |f(z)|dz\right)(x) \leq M_u\left(\frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}\right)(x).
\end{aligned}$$

Therefore, by Lemmas 2.3 and 2.4, we write

$$\begin{aligned}
\|Mf\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &\lesssim \left\| \frac{\|M(f\chi_{Q(\cdot,2t)})\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\quad + \left\| \frac{\left\| M_u\left(\frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}}}\right)\chi_{Q(\cdot,t)} \right\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\lesssim \left\| \frac{\|f\chi_{Q(\cdot,2t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\quad + \left\| \frac{\|f\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_{Q(\cdot,t)}\|_{L^{\vec{p}}(\mathbb{R}^n)}} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof of the Lemma 2.5. □

3. Extrapolation

In this section, we establish a new version of extrapolation theorem on mixed-norm amalgam spaces via the algorithm of Rubio de Francia for generating A_1

weights with certain properties (see [17]). To state our results, we begin with some necessary definitions.

A weight w is a positive and locally integrable function on $\mathbb{R}^n$. For $p \in (0, \infty)$, the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right]^{\frac{1}{p}} < \infty.$$

The weak weighted Lebesgue space $L^{p,\infty}_w(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\infty}_w(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})^{\frac{1}{p}} < \infty.$$

For $p = \infty$,

$$\|f\|_{L^\infty_w(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| < \infty.$$

Theorem 3.1. *Given a family of extrapolation pairs $\mathcal{F}$. Assume that for some $1 \le p_0 \le q_0 < \infty$, and for all $w \in A_1$,*

$$\left[\int_{\mathbb{R}^n} f(x)^{q_0} w(x) dx\right]^{\frac{1}{q_0}} \le C_{w,p_0} \left[\int_{\mathbb{R}^n} g(x)^{p_0} w(x)^{p_0/q_0} dx\right]^{\frac{1}{p_0}}, \quad \forall (f,g) \in \mathcal{F}. \tag{3.1}$$

Let $t \in (0,\infty)$, $\vec{r}$, $\vec{s} \in (p_0,\infty)^n$ and $\vec{p}$, $\vec{q} \in (q_0,\infty)^n$ with $1/r_i - 1/p_i = 1/s_i - 1/q_i = 1/p_0 - 1/q_0 > 0$ for each $i = 1, ..., n$. Then

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \le C\|g\|_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)}. \tag{3.2}$$

The positive constant C is independent of f and t.

Proof. Let $m := \|M\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)\to(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}$. By Lemma 2.5, we conclude that the Hardy–Littlewood maximal operator M is bounded on $(L^{(\vec{p}/q_0)'}, L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)$. We begin the proof by using the Rubio de Francia iteration algorithm. The algorithm $\mathcal{R} : L^0(\mathbb{R}^n) \to [0,\infty]$ is defined by

$$\mathcal{R}h(x) := \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k m^k},$$

where for $k \ge 1$, $M^k = M \circ \cdots \circ M$ is k iterations of M, and $M^0 h := |h|$. We show the following properties for all $h \in (L^{(\vec{p}/q_0)'}, L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)$:

(A) $|h| \le \mathcal{R}h$,

(B) $\|\mathcal{R}h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \le 2\|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}$,

(C) $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \le 2m$.

Property (A) holds since $\mathcal{R}h \geq M^0(h) = |h|$. Property (B) holds by the fact that $m < \infty$, since

$$\|\mathcal{R}h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty} \frac{\left\|M^k h\right\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}}{2^k m^k} \leq 2\|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)}.$$

Let us then prove (C). We may assume that $h \neq 0$, since the claim is trivial otherwise. It is equivalent to prove that $M(\mathcal{R}h)(x) \leq 2m\mathcal{R}h(x)$. By the definition of $\mathcal{R}$ and the sublinearity of the maximal operator, we obtain

$$M(\mathcal{R}h)(x) = M\left(\sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k m^k}\right) \leq \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^k m^k} = 2m \sum_{k=0}^{\infty} \frac{M^{k+1}h(x)}{2^{k+1}m^{k+1}} \leq 2m\mathcal{R}h(x).$$

Fix $(f,g) \in \mathcal{F}$ and define $\mathcal{H} := \{h : \|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \leq 1\}$. Note that $\|f\|^{q_0}_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \|f^{q_0}\|_{(L^{\vec{p}/q_0},L^{\vec{q}/q_0})_t(\mathbb{R}^n)}$. By Lemma 2.2 and (A), we see

$$\|f\|^{q_0}_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \sup_{h\in\mathcal{H}} \int_{\mathbb{R}^n} f(x)^{q_0} h(x)dx \leq \sup_{h\in\mathcal{H}} \int_{\mathbb{R}^n} f(x)^{q_0}\mathcal{R}h(x)dx. \tag{3.3}$$

To apply our hypothesis, by our convention on families of extrapolation pairs we need to show that the left-hand side in (3.1) is finite. This follows from Hölder's inequality and property (B): for all $h \in \mathcal{H}$,

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{q_0}\mathcal{R}h(x)dx &\lesssim \|f^{q_0}\|_{(L^{\vec{p}/q_0},L^{\vec{q}/q_0})_t(\mathbb{R}^n)} \|\mathcal{R}h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \\ &\lesssim \|f\|^{q_0}_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \|h\|_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} < \infty.\end{aligned}$$

Given this and (C), we can apply our hypothesis (3.1) in (3.3) to get that

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq \sup_{h\in\mathcal{H}} \left(\int_{\mathbb{R}^n} f(x)^{q_0}\mathcal{R}h(x)dx\right)^{\frac{1}{q_0}} \leq \sup_{h\in\mathcal{H}} \left(\int_{\mathbb{R}^n} g(x)^{p_0}(\mathcal{R}h(x))^{p_0/q_0}dx\right)^{\frac{1}{p_0}}. \tag{3.4}$$

Then for any $h \in \mathcal{H}$, by Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)^{p_0}(\mathcal{R}h(x))^{p_0/q_0}dx &\lesssim \|g^{p_0}\|_{(L^{\vec{r}/p_0},L^{\vec{s}/p_0})_t(\mathbb{R}^n)} \left\|(\mathcal{R}h)^{p_0/q_0}\right\|_{(L^{(\vec{r}/p_0)'},L^{(\vec{s}/p_0)'})_t(\mathbb{R}^n)} \\ &\leq \|g\|^{p_0}_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)} \|\mathcal{R}h\|^{p_0/q_0}_{(L^{p_0(\vec{r}/p_0)'/q_0},L^{p_0(\vec{s}/p_0)'/q_0})_t(\mathbb{R}^n)}.\end{aligned}$$

Notice that

$$\frac{p_0}{q_0}\left(\frac{\vec{r}}{p_0}\right)' = \left(\frac{\vec{p}}{q_0}\right)' \quad \text{and} \quad \frac{p_0}{q_0}\left(\frac{\vec{s}}{p_0}\right)' = \left(\frac{\vec{q}}{q_0}\right)'.$$

It follows from the property (B) that

$$\int_{\mathbb{R}^n} g(x)^{p_0}\left(\mathcal{R}h(x)\right)^{p_0/q_0}dx \leq \|g\|^{p_0}_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)} \|\mathcal{R}h\|^{p_0/q_0}_{(L^{(\vec{p}/q_0)'},L^{(\vec{q}/q_0)'})_t(\mathbb{R}^n)} \lesssim \|g\|^{p_0}_{(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)}. \tag{3.5}$$

Combined with (3.4) and (3.5), the desired result is concluded. $\square$

Corollary 3.1. *Given a family of extrapolation pairs $\mathcal{F}$, assume that for some $1 \le p_0 < \infty$, and for all $w \in A_1$,*

$$\left[\int_{\mathbb{R}^n} f(x)^{p_0} w(x)dx\right]^{\frac{1}{p_0}} \le C_{w,p_0} \left[\int_{\mathbb{R}^n} g(x)^{p_0} w(x)dx\right]^{\frac{1}{p_0}}, \quad \forall (f,g) \in \mathcal{F}.$$

Let $t \in (0,\infty)$ and $\vec{p}$, $\vec{q} \in (p_0,\infty)^n$. Then

$$\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \le C\|g\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

The positive constant C is independent of f and t.

For a linear operator $\mathcal{T}$ and a locally integrable function b, the commutators of $\mathcal{T}$ is defined for smooth functions f by

$$[b,\mathcal{T}]f(x) = b(x)\mathcal{T}f(x) - \mathcal{T}(bf)(x).$$

Now, we recall the definition of $BMO(\mathbb{R}^n)$. $BMO(\mathbb{R}^n)$ is the Banach function space modulo constants with the norm $\|\cdot\|_{BMO}$ defined by

$$\|b\|_{BMO} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q |b(y) - b_Q|dy < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ and b_Q implies the mean value of b over Q; that is, $b_Q := \frac{1}{|Q|}\int_Q b(y)dy$.

Theorem 3.2. *Let $t \in (0,\infty)$, $\vec{p} \in (1,\infty)^n$ and $w \in A_1$. Let $\mathcal{T}$ be a sublinear operator.*

(a) For $\vec{q} = (1,...,1)$, suppose that the operator $\mathcal{T}$ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$. Then

$$\|\mathcal{T}f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

(b) For $\vec{q} = (1,...,1)$, suppose that the commutators $[b,\mathcal{T}]$ with $b \in BMO(\mathbb{R}^n)$ satisfies

(3.6)

$$w\left(\{y \in \mathbb{R}^n : \ |[b,\mathcal{T}]\, f(y)| > \lambda\}\right) \lesssim \|b\|_{BMO} \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\left(1 + \log^+\left(\frac{|f(y)|}{\lambda}\right)\right) w(y)dy.$$

Then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1 + \log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Proof. By Lemma 2.2, there exists $g \in (L^{\vec{p}'}, L^{\vec{q}'})_t(\mathbb{R}^n)$ such that

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |\mathcal{T}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n:|\mathcal{T}f(x)|>\lambda\}}(x)g(x)dx.$$

Let $w(x) := \left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)$ with $\gamma > 1$. Then $w \in A_1$. Since $g(x) \le \left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)$, by Lemma 2.1, the hypothesis that $\mathcal{T}$ is bounded $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$ and Lemma

2.5, then we can obtain that

$$\begin{aligned}\lambda\left\|\chi_{\{x\in\mathbb{R}^n:\ |\mathcal{T}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &\le \lambda\int_{\mathbb{R}^n}\chi_{\{x\in\mathbb{R}^n:\ |\mathcal{T}f(x)|>\lambda|\}}(x)\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim \int_{\mathbb{R}^n}|f(x)|\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\left\|\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}\right\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\\ &\lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}.\end{aligned}$$

By taking the supremum over all $\lambda>0$, then we get

$$\|\mathcal{T}f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}\lesssim\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

For the part of (b). Argue similarly for the weight $w(x):=\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)$ with $\gamma>1$. There exists $g\in(L^{\vec{p}'},L^{\vec{1}'})_t(\mathbb{R}^n)$ such that $g(x)\le w(x)$. Lemma 2.1, the hypothesis of $[b,\mathcal{T}]$ and Lemma 2.5 yield

$$\begin{aligned}&\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\le\int_{\mathbb{R}^n}\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f(x)|>\lambda|\}}(x)\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim\|b\|_{BMO}\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\left(1+\log^+\left(\frac{|f(x)|}{\lambda}\right)\right)\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}(x)dx\\ &\lesssim\left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\left\|\left[M\left(|g|^{\frac{1}{\gamma}}\right)\right]^{\gamma}\right\|_{(L^{\vec{p}'},L^{\vec{1}'})_t(\mathbb{R}^n)}\\ &\lesssim\left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\|g\|_{(L^{\vec{p}'},L^{\vec{1}'})_t(\mathbb{R}^n)}.\end{aligned}$$

Hence

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mathcal{T}]f|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}\lesssim\left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

This completes the proof of Theorem 3.2. □

4. Some estimates on mixed-norm amalgam spaces

In this section, we apply our extrapolation theorem to prove norm inequalities over mixed-norm amalgam spaces.

We apply the results of Section 3 to the singular integral operators, and establish the mapping properties of these operators and the commutators in this subsection.

Let $\delta>0$. The *Calderón–Zygmund singular integral operator of non-convolution type* is a bounded linear operator $T:\ L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ satisfying that, for all $f\in C_c^\infty(\mathbb{R}^n)$ and $x\notin\operatorname{supp}(f)$,

$$T(f)(x):=\int_{\mathbb{R}^n}K(x,y)f(y)dy,$$

where the distributional kernel coincides with a locally integrable function K defined away from the diagonal on $\mathbb{R}^n\times\mathbb{R}^n$. When K also satisfies that, for $x,\ y\in\mathbb{R}^n$ with

$x \neq y$,

$$|K(x,y)| \leq \frac{C_0}{|x-y|^n}, \tag{4.1}$$

$$|K(x,y) - K(x,y+h)| + |K(x,y) - K(x+h,y)| \leq \frac{C_1|h|^\delta}{|x-y|^{n+\delta}}, \tag{4.2}$$

whenever $|x-y| \geq 2|h|$, and we call K the *standard kernel.*

In [13], it is proved that for the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2), if $1 < p < \infty$ and $w \in A_p$, then T is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$ and $w \in A_1$, then T is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$. In [31], the commutator $[b,T]$ are bounded in the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ whenever $1 < q < \infty$ and $w \in A_p$, and in [28], if $p = 1$ and $w \in A_1$, then

$$w\left(\{y \in \mathbb{R}^n : |[b,T]f(y)| > \lambda\}\right) \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\left(1 + \log^+\left(\frac{|f(y)|}{\lambda}\right)\right) w(y)dy.$$

Thus, by Theorems 3.1 and 3.2, we can easily get the boundedness of the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2) and the linear commutators $[b,T]$ over the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ in the following.

Corollary 4.1. *Let $0 < t < \infty$, $\vec{p} \in (1,\infty)^n$.*

(a) If $\vec{q} \in (1,\infty)^n$, then the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2) is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then the Calderón–Zygmund singular integral operator T with the kernel satisfying (4.1) and (4.2) is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Corollary 4.2. *Let $0 < t < \infty$, $\vec{p} \in (1,\infty)^n$ and $b \in BMO(\mathbb{R}^n)$. Let T be the Calderón–Zygmund singular integral operator with the kernel satisfying (4.1) and (4.2),*

(a) If $\vec{q} \in (1,\infty)^n$, then the operator $[b,T]$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,T]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1 + \log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Let $\mathbb{S}^{n-1}(n \geq 2)$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$, $\Omega(x)$ is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta \leq \infty$ and such that

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{4.3}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$, the homogeneous singular integral operator T_Ω can be defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)dy,$$

and the Marcinkiewicz integral of higher dimension μ_Ω can be defined by

$$\mu_\Omega f(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

The commutators of Marcinkiewicz operator μ_Ω and a locally integrable function b can be defined by

$$[b, \mu_\Omega]f(x) = \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left[b(x) - b(y) \right] f(y)dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}.$$

Lemma 4.1. [12] *For $\Omega \in L^\theta(\mathbb{S}^{n-1})$ and $1 < \theta < \infty$. If $\theta' \leq p < \infty$ and $w \in A_{p/\theta'}$, then T_Ω is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$ and $w \in A_1$, then T_Ω is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

From Theorems 3.1, 3.2 and Lemma 4.1, we can easily get the results as follows.

Corollary 4.3. *Let $0 < t < \infty$, $\vec{p} \in (1, \infty)^n$.*

(a) If $\vec{q} \in (1, \infty)^n$, then T_Ω is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then T_Ω is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Lemma 4.2. [10] *For $\Omega \in L^\theta(\mathbb{S}^{n-1})$ and $1 < \theta \leq \infty$, if $1 < p < \infty$, and $w \in A_p$. Then μ_Ω is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, then μ_Ω is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

Applying Theorems 3.1, 3.2 and Lemma 4.3, we have the following results.

Corollary 4.4. *Let $0 < t < \infty$, $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta \leq \infty$, and $\vec{p} \in (\theta', \infty)^n$.*

(a) If $\vec{q} \in (\theta', \infty)^n$, then μ_Ω is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then μ_Ω is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Lemma 4.3. [10, 11] *Let $\Omega \in L^\theta(\mathbb{S}^{n-1})$, $1 < \theta \leq \infty$, $b \in BMO(\mathbb{R}^n)$. If $\theta' < p < \infty$, and $w \in A_{p/\theta'}$, then $[b, \mu_\Omega]$ is bounded on $L^p_w(\mathbb{R}^n)$. If $w \in A_1$, then there exists a constant $C > 0$ such that*

$$w\left(\{y \in \mathbb{R}^n : |[b, \mu_\Omega] f(y)| > \lambda\}\right) \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(y)|}{\lambda}\right)\right) w(y)dy.$$

Therefore we have

Corollary 4.5. *Let $b \in BMO(\mathbb{R}^n)$, $0 < t < \infty$, $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta \le \infty$, and $\vec{p} \in (\theta', \infty)^n$.*

(a) If $\vec{q} \in (\theta', \infty)^n$, then the operator $[b, \mu_\Omega]$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,\mu_\Omega]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

The Bochner–Riesz operators of order $\delta > 0$ in terms of the Fourier transforms is defined by

$$\left(T_R^\delta f\right)^\wedge(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\delta \hat{f}(\xi),$$

where $\hat{f}$ denote the Fourier transform of f. These operators can be defined by

$$T_R^\delta f(x) = \left(f * \phi_{1/R}\right)(x),$$

where $\phi(x) = [(1 - |\cdot|^2)_+^\delta]^\vee(x)$, and $f^\vee$ is the inverse Fourier transform of f.

The associated maximal operators is defined by

$$T_*^\delta f(x) = \sup_{R>0} |T_R^\delta f(x)|.$$

Lemma 4.4. [32, 33, 34] *Let $n \ge 2$. If $1 < p < \infty$ and $w \in A_p$, then $T_*^{(n-1)/2}$ is bounded on $L^p_w(\mathbb{R}^n)$. For a fixed $R > 0$, if $p = 1$, $w \in A_1$, then $T_R^{(n-1)/2}$ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

Corollary 4.6. *Let $0 < t < \infty$, and $\vec{p} \in (1, \infty)^n$.*

(a) If $\vec{q} \in (1, \infty)^n$, then $T_*^{(n-1)/2}$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, then $T_R^{(n-1)/2}$ is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

Lemma 4.5. [1, 24] *Let $n \ge 2$, and $b \in BMO(\mathbb{R}^n)$. If $1 < p < \infty$, $w \in A_p$, and $\delta \ge \frac{n-1}{2}$, then then $[b, T_R^\delta]$ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, and $\delta > \frac{n-1}{2}$, then*

$$w\left(\left\{y \in \mathbb{R}^n : \left|\left[b, T_*^\delta\right] f(y)\right| > \lambda\right\}\right) \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda}\left(1+\log^+\left(\frac{|f(y)|}{\lambda}\right)\right) w(y)dy.$$

Corollary 4.7. *Let $b \in BMO(\mathbb{R}^n)$, $0 < t < \infty$, and $\vec{p} \in (1, \infty)^n$.*

(a) If $\vec{q} \in (1, \infty)^n$, and $\delta \ge \frac{n-1}{2}$, then the operator $[b, T_R^\delta]$ is bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{q} = (1, ..., 1)$, and $\delta > \frac{n-1}{2}$, then

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |[b,T_*^\delta]f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Let $0 < \alpha < n$, the fractional integral operator I_α is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(\xi)}{|x-\xi|^{n-\alpha}}\, d\xi,$$

And the associated fractional maximal operator M_α is defined by

$$M_\alpha f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy.$$

We note that the fractional maximal functions enjoys the same boundedness as that of the fractional integrals since the pointwise inequality $M_\alpha f(x) \lesssim I_\alpha f(x)$.

We also recall the definition of $A_{p,q}$ weights which are closely related to the weighted boundedness of the fractional integrals in [27].

Definition 4.1. *A weight w is said to be of class $A_{p,q}$, for $1 < p, q < \infty$, if*

$$[w]_{A_{p,q}} := \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty,$$

where p' is the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

And a weight w is said to be of class $A_{1,q}$ with $1 < q < \infty$, if

$$[w]_{A_{1,q}} := \sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{\frac{1}{q}} \left(\operatorname*{ess\,sup}_Q \frac{1}{w(x)} \right) < \infty.$$

Lemma 4.6. [27] *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$, and $w \in A_{p,q}$, then there exists a positive constant C such that*

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x) w(x)|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x) w(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = 1$, and $w \in A_{1,q}$ with $q = \frac{n}{n-\alpha}$, then for all $\lambda > 0$, then there exists a positive constant C such that

$$w\left(\{x \in \mathbb{R}^n : |I_\alpha(f)(x)| > \lambda\}\right) \leq C \left(\frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x)^{\frac{1}{q}} dx \right)^q.$$

The universal positive constant C is independent of f and λ.

Corollary 4.8. *Let $0 < t < \infty$, $0 < \alpha < n$. Suppose that $\vec{p}, \vec{r} \in (1, n/\alpha)^n$ such that $1/r_i - 1/p_i = 1/s_i - 1/q_i = \alpha/n$.*

(a) If $\vec{s} \in (1, \infty)^n$, then I_α is bounded from $(L^{\vec{r}}, L^{\vec{s}})_t(\mathbb{R}^n)$ to $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

(b) If $\vec{s} = (1, ..., 1)$, then I_α is bounded from $(L^{\vec{r}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

Proof. By Theorem 3.1 and Lemma 4.6, the case of (a) holds, we only prove the case of (b).

For $r_i > 1$ and $s_i = 1$, $i = 1, ..., n$, let

$$\frac{1}{p_i} = \frac{1}{r_i} - \frac{\alpha}{n}, \quad \frac{1}{q_i} = 1 - \frac{\alpha}{n}, \quad \text{for each } i = 1, ..., n.$$

Take $\theta = q_i = \frac{n}{n-\alpha}$. Then, for $g \in (L^{(\vec{p}/\theta)'}, L^{\vec{\infty}})_t(\mathbb{R}^n)$, by Lemma 2.2, we write

$$\begin{aligned}\lambda \left\|\chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &= \lambda \left\|\left|\chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}\right|^\theta\right\|^{\frac{1}{\theta}}_{(L^{\vec{p}/\theta},L^{\vec{q}/\theta})_t(\mathbb{R}^n)} \\ &= \lambda \left(\int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}(x) g(x)dx\right)^{\frac{1}{\theta}}.\end{aligned}$$

And letting $w := [M_\eta(|g|)(x)]^{\frac{1}{\theta}}$ with $0<\eta<1$, we have $w^\theta \in A_1$ and hence $w^\theta \in A_{\theta\frac{n-\alpha}{n}}$. Then $w \in A_{1,\theta}$. By Lemma 2.1 and Lemma 4.6, we can obtain that

$$\begin{aligned}\lambda \left\|\chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &= \lambda \left[\int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n:\ |I_\alpha f(x)|>\lambda\}}(x) M_\eta g(x)dx\right]^{\frac{1}{\theta}} \\ &\le C\int_{\mathbb{R}^n} |f(x)|w(x)dx \le C\|f\|_{(L^{\vec{r}},L^1)_t(\mathbb{R}^n)} \left\|[M_\eta(g)]^{\frac{1}{\theta}}\right\|_{(L^{\vec{r}'},L^{\vec{\infty}})_t(\mathbb{R}^n)}.\end{aligned}$$

From Lemma 2.5 we see

$$\begin{aligned}\left\|[M_\eta(g)]^{\frac{1}{\theta}}\right\|_{(L^{\vec{r}'},L^{\vec{\infty}})_t(\mathbb{R}^n)} &= \left\|[M(|g|^\eta)]^{\frac{1}{\eta\theta}}\right\|_{(L^{\vec{r}'},L^{\vec{\infty}})_t(\mathbb{R}^n)} = \|M(|g|^\eta)\|^{\frac{1}{\eta\theta}}_{(L^{\vec{r}'/\eta\theta},L^{\vec{\infty}})_t(\mathbb{R}^n)} \\ &\le C\,\||g|^\eta\|^{\frac{1}{\eta\theta}}_{(L^{\vec{r}'/\eta\theta},L^{\vec{\infty}})_t(\mathbb{R}^n)} = C\|g\|^{\frac{1}{\theta}}_{(L^{\vec{r}'/\theta},L^{\vec{\infty}})_t(\mathbb{R}^n)}.\end{aligned}$$

since $\frac{1}{r_i'/\theta} = \left(1-\frac{1}{r_i}\right)\theta = \left(1-\frac{1}{p_i}-\frac{\alpha}{n}\right)\theta = \left(\frac{1}{\theta}-\frac{1}{p_i}\right)\theta = 1-\frac{1}{p_i/\theta} = \frac{1}{(p_i/\theta)'}$ for $i=1,...,n$, we see

$$\|I_\alpha f\|_{W(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \le C\|f\|_{(L^{\vec{r}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Thus, the result holds. □

For the boundedness of the commutator for the Riesz potential, we

Lemma 4.7. [7] *Let $0<\alpha<n$, $1<p<n/\alpha$ and $1/p-1/q=\alpha/n$. Let $b\in BMO(\mathbb{R}^n)$ and $w\in A_{p,q}$, then $[b,I_\alpha]$ is bounded from $L^p_w(\mathbb{R}^n)$ to $L^q_w(\mathbb{R}^n)$.*

The estimate of the operator $[b,I_\alpha]$ over the mixed-norm amalgam space is immediate in view of Lemma 4.7 and Theorem 3.1 as follows.

Corollary 4.9. *Let $0<t<\infty$ and $0<\alpha<n$. Let $b\in BMO(\mathbb{R}^n)$. Suppose that $\vec{p}$, $\vec{r}\in(1,n/\alpha)^n$ such that $1/r_i-1/p_i=1/s_i-1/q_i=\alpha/n$. If $\vec{q}\in(1,\infty)^n$, then $[b,I_\alpha]$ is bounded from $(L^{\vec{r}},L^{\vec{s}})_t(\mathbb{R}^n)$ to $(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)$.*

5. The Littlewood–Paley functions

The Littlewood–Paley theory, originated in the 1930s and developed in the late 1950s, is a very effective replacement. It has played a very prominent role in harmonic analysis, Complex analysis and PDE (see [6, 22, 30]). Therefore, it is a very interesting problem to discuss the boundedness of the Littlewood–Paley operators. The main purpose of this section is to study the characterization of the mixed-norm amalgam space $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ via the Littlewood–Paley functions. We first state the associated definitions.

Suppose that $\varphi(x) \in L^1(\mathbb{R}^n)$ satisfies the following conditions:

$$\int_{\mathbb{R}^n} \varphi(x)dx = 0. \tag{5.1}$$

There exist constants C, $\delta > 0$, such that

$$|\varphi(x)| \leq \frac{C}{(1+|x|)^{n+\delta}}, \quad \forall x \in \mathbb{R}^n. \tag{5.2}$$

and when $2|y| < |x|$, there exist constants γ, $\delta > 0$, such that

$$|\varphi(x+y) - \varphi(x)|dx \leq \frac{C|y|^\delta}{(1+|x-y|)^{n+\delta+\gamma}}. \tag{5.3}$$

For $t > 0$, $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$. For all $x \in \mathbb{R}^n$, the Littlewood–Paley g function g_φ, the square function S_φ and the Littlewood–Paley $g^*_{\lambda,\varphi}$-function are defined by

$$g_\varphi(f)(x) = \left(\int_0^\infty |(\varphi_t * f)(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$S_\varphi(f)(x) = \left(\iint_{\Gamma_\alpha(x)} |(\varphi_t * f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$g^*_{\lambda,\varphi}(f)(x) = \left(\iint_{R^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |(\varphi_t * f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

For a locally integrable function b, the commutators of the Littlewood–Paley function g_φ, S_φ and $g^*_{\lambda,\varphi}$ are defined by

$$g_{\varphi,b}(f)(x) = \left(\int_0^\infty \left|\int_{\mathbb{R}^n} (\varphi_t(x-y)f(y)((b(x)-b(y))dy\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$S_{\varphi,b}(f)(x) = \left(\iint_{\Gamma_\alpha(x)} \left|\int_{\mathbb{R}^n} (\varphi_t(y-z)f(z)((b(y)-b(z))dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$g^*_{\lambda,,\varphi,b}(f)(x) = \left(\iint_{R^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left|\int_{\mathbb{R}^n} (\varphi_t(y-z)f(z)((b(y)-b(z))dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma_\alpha(x) = \big\{(y,t) \in \mathbb{R}^{n+1}_+ : \ |x-y| < \alpha t\big\}$ and $\mathbb{R}^{n+1}_+ = \{(y,t) \in \mathbb{R}^{n+1}_+ : \ y \in \mathbb{R}^n, t > 0\}$.

Lemma 5.1. [25] *Suppose that $\varphi \in L^1(\mathbb{R}^n)$ satisfies (5.1), (5.2) and (5.3). If $1 < p < \infty$, $w \in A_p$, then g_φ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$ and $w \in A_1$, then g_φ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

Lemma 5.2. [23] *Let $b \in BMO(\mathbb{R}^n)$. Suppose that $\varphi \in L^1(\mathbb{R}^n)$ satisfies (5.1), (5.2) and (5.3). If $1 < p < \infty$, $w \in A_p$, then $g_{\varphi,b}$ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$ and $w \in A_1$, then*

$$w\left(\{x \in \mathbb{R}^n : |g_{\varphi,b}f(x)| > \lambda\}\right) \lesssim \|b\|_{BMO} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

Lemma 5.3. [37] *Let $0 < \alpha \leq 1$, if $p \in (1,\infty)$ and $w \in A_p$, then S_φ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, then S_φ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$.*

Lemma 5.4. [23] *Let $\alpha \in (0,1]$ and $b \in BMO(\mathbb{R}^n)$. If $p \in (1,\infty)$ and $w \in A_p$, then $S_{\varphi,b}$ is bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, then there exists a constant $C > 0$ such that*

$$w\left(\{x \in \mathbb{R}^n : |S_{\varphi,b}f(x)| > \lambda\}\right) \lesssim \|b\|_{BMO} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

Lemma 5.5. [38] *Let $\lambda > 2$ and $0 < \gamma < \min\{n(\gamma-2)/2, \delta\}$. Let $b \in BMO(\mathbb{R}^n)$. If $p \in (1,\infty)$ and $w \in A_p$, then $g^*_{\lambda,\varphi}$ and $g^*_{\lambda,\varphi,b}$ are bounded on $L^p_w(\mathbb{R}^n)$. If $p = 1$, $w \in A_1$, then $g^*_{\lambda,\varphi}$ is bounded from $L^1_w(\mathbb{R}^n)$ to $L^{1,\infty}_w(\mathbb{R}^n)$, and*

$$w\left(\left\{x \in \mathbb{R}^n : \left|g^*_{\lambda,\varphi,b}(f)(x)\right| > \lambda\right\}\right) \lesssim \|b\|_{BMO} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda}\left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

Theorem 5.1. *Let $0 < t < \infty$, $\lambda > 2$ and $0 < \gamma < \min\{n(\gamma-2)/2, \delta\}$. Let $\vec{p} \in (1,\infty)^n$. If $\vec{q} \in (1,\infty)^n$, then*

(a) $C_1 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq \|g_\varphi(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq C_2 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(b) $C_1 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq \|S_\varphi(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \leq C_2 \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(c) $\left\|g^*_{\lambda,\varphi}(f)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(d) If $\vec{q} = (1,...,1)$, then the operators g_φ, S_φ, $g^*_{\lambda,\varphi}$ is bounded from $(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{1}})_t(\mathbb{R}^n)$.

The positive constants C_1 and C_2 are independent of f and t.

Proof. We only need to prove the left case of (a) and (b), since Lemmas 5.1, 5.3, 5.5 and Theorems 3.1, 3.2.

By Lemma 2.1, the boundedness of g_φ over $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ for $\vec{p}$, $\vec{q} \in (1,\infty)^n$ and Hölder's inequality, we see

$$\begin{aligned}
\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &= \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} f(x)g(x)dx \\
&\le \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n}\int_0^\infty |\varphi_t * f(x)|\cdot|\varphi_t * g(x)|\frac{dt}{t}dx \\
&\le \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} g_\varphi(f)(x) g_\varphi(g)(x)dx \\
&\le \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \|g_\varphi(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \|g_\varphi(g)\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)} \\
&\lesssim \|g_\varphi f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)},
\end{aligned}$$

For the operator S_φ, using these facts, Lemma 2.1 and $\|S_\varphi f\|_{L^2_H(\mathbb{R}^n)} = A\|f\|_{L^2(\mathbb{R}^n)}$ with $A > 0$ and H is a Hilbert space, we conclude that

$$\begin{aligned}
\|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} &= \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} f(x)g(x)dx \\
&= \frac{1}{A^2} \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} S_\varphi f(x) S_\varphi g(x)dx \\
&\le \frac{1}{A^2} \sup_{\|g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)}\le 1} \|S_\varphi f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \|S_\varphi g\|_{(L^{\vec{p}'},L^{\vec{q}'})_t(\mathbb{R}^n)} \\
&\lesssim \frac{1}{A^2} \|S_\varphi f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof of Theorem 5.1. □

Using Lemmas 5.1, 5.3, 5.5 and Theorems 3.1, 3.2, we obtain the following estimate for the operator $g_{\varphi,b}$, $S_{\varphi,b}$, $g^*_{\lambda,\varphi,b}$ on mixed-norm amalgam spaces.

Theorem 5.2. *Let $0 < t < \infty$, $\lambda > 2$ and $0 < \gamma < \min\{n(\gamma-2)/2, \delta\}$. Let $\vec{p} \in (1,\infty)^n$ and $b \in BMO(\mathbb{R}^n)$, If $\vec{q} \in (1,\infty)^n$, then $g_{\varphi,b}$, $S_{\varphi,b}$ and $g^*_{\lambda,\varphi,b}$ are bounded on $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$. If $\vec{q} = (1,...,1)$, then*

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |g_{\varphi,b}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)},$$

$$\left\|\chi_{\{x\in\mathbb{R}^n:\ |S_{\varphi,b}f(x)|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}$$

and

$$\left\|\chi_{\left\{x\in\mathbb{R}^n:\ \left|g^*_{\lambda,\varphi,b}f(x)\right|>\lambda\right\}}\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{1}})_t(\mathbb{R}^n)}.$$

Remark 5.1. *For the Littlewood–Paley functions with the non-convolution type kernels and their commutators, a similar result also holds.*

5.0.1. *Nononvolution type.* A function $K(x,y)$ defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$, is said to be a non-convolution type kernel, if for all $y \in \mathbb{R}^n$, there exists a positive constant C, such that K satisfies the following conditions:

$$\int_{\mathbb{R}^n} K(x,y)dy = 0 \tag{5.4}$$

$$|K(x,y)| \leq \frac{C}{(1+|x-y|)^{n+\delta}} \tag{5.5}$$

$$|K(x+z,y) - K(x,y)| \leq \frac{C|z|^\gamma}{(1+|x-y|)^{n+\delta+\gamma}} \tag{5.6}$$

for some δ, $\gamma > 0$, and $2|z| \leq |x-y|$.

For any $f \in \mathscr{S}$, $t > 0$, and $z \ni$ supp f, we denote

$$G_t f(z) = \int_{\mathbb{R}^n} K_t(z,y) f(y)dy,$$

where $K_t(z,y) = \frac{1}{t^n} K(\frac{z}{t}, \frac{y}{t})$. Let b be a locally integrable function. Then the Littlewood-Paley g-function, Lusin's area integral and Littlewood-Paley g_λ^*-function with non-convolution type kernels and their commutators are defined by

$$g(f)(x) = \left(\int_0^\infty |G_t f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$g_b(f)(x) = \left(\int_0^\infty |G_t f(x)|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$S(f)(x) = \left(\iint_{\Gamma(x)} |G_t f(z)|^2 \frac{dzdt}{t^{n+1}}\right)^{\frac{1}{2}}$$

and

$$g_\lambda^*(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-z|}\right)^{n\lambda} |G_t f(z)|^2 \frac{dzdt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\lambda > 1$, $\Gamma(x) = \{(z,t) \in \mathbb{R}_+^{n+1} : |z-x| < t\}$ and $\mathbb{R}_+^{n+1} = \{(z,t) \in \mathbb{R}_+^{n+1} : z \in \mathbb{R}^n, t > 0\}$.

Theorem 5.3. *Let $0 < t < \infty$. For $1 < \vec{p} < \infty$. If $1 < \vec{q} < \infty$. Then*

(A) $\|g(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(B) $\|S(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(C) for $\lambda > 2$ and $0 < \gamma < \min\{n(\gamma-2)/2, \delta\}$, $\|g_\lambda^*(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \sim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

If $\vec{q} \in [1,\infty)$ and $\min\{q_1,...,q_m\} = 1$. Then the operators g_φ, S_φ, $g_{\lambda,\varphi}^*$ are bounded from $(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$ to $W(L^{\vec{p}}, L^{\vec{q}})_t(\mathbb{R}^n)$.

Using Lemmas 5.1,5.3,5.5 and Theorems 3.1,3.2, we obtain the following estimate for the operators $g_{\varphi,b}$, $S_{\varphi,b}$, $g_{\lambda,\varphi,b}^*$ on mixed-norm amalgam spaces.

Theorem 5.4. *Let* $0 < t < \infty$, $b \in BMO(\mathbb{R}^n)$. *For* $1 < \vec{p} < \infty$. *If* $1 < \vec{q} < \infty$. *Then*

(A) $\|g_b(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(B) $\|S_b(f)\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

(C) for $\lambda > 2$ *and* $0 < \gamma < \min\{n(\gamma-2)/2, \delta\}$, $\left\|g^*_{\lambda,b}(f)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \|f\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}$.

If $\vec{q} \in [1,\infty)$ *and* $\min\{q_1,...,q_m\} = 1$. *Then the operators* g_φ, S_φ, $g^*_{\lambda,\varphi}$ *satisfy*

$$\left\|\chi_{\{x\in\mathbb{R}^n:|g_bf|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)},$$

$$\left\|\chi_{\{x\in\mathbb{R}^n:|S_bf|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

and

$$\left\|\chi_{\{x\in\mathbb{R}^n:|g^*_{\lambda,b}f|>\lambda\}}\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)} \lesssim \left\|\frac{|f|}{\lambda}\left(1+\log^+\left(\frac{|f|}{\lambda}\right)\right)\right\|_{(L^{\vec{p}},L^{\vec{q}})_t(\mathbb{R}^n)}.$$

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