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ON THE UNIQUENESS OF L-FUNCTIONS AND MEROMORPHIC FUNCTIONS SHARING A SET

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Abstract. The paper presents general criterions for the uniqueness of a non-constant meromorphic function having finitely many poles and a non-constant L-function in the Selberg class when they share a set. Our results significantly improve all the existing results in this direction [22, 17, 16, 4] with extent to the most general setting. As a consequence, we have incorporated a large number of examples in the application section showing the far reaching applications of our results.

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1. Introduction and main results

At the outset, we assume that by an L-function we always mean an L-function $\mathcal{L}$ in the Selberg class which includes the Riemann zeta function $\zeta(s)=\sum\limits_{n=1}^{\infty} n^{-s}$ and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined [18, 19] to be a Dirichelet series

$$\mathcal{L}(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^{-s}} \tag{1.1}$$

satisfying the following axioms:

- (i) *Ramanujan hypothesis :* $a(n) \ll n^{\varepsilon}$ for every $\varepsilon>0$;
- (ii) *Analytic continuation :* There is a non-negative integer m such that $(s-1)^m\mathcal{L}(s)$ is an entire function of finite order;
- (iii) *Functional equation:* $\mathcal{L}$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s)=\omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})}, \tag{1.2}$$

where

$$\Lambda_{\mathcal{L}}(s)=\mathcal{L}(s)Q^s\prod_{j=1}^{k}\Gamma(\lambda_j s+\nu_j), \tag{1.3}$$

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with positive real numbers Q, λ_j and complex numbers ν_j, ω with $\mathrm{Re}(\nu_j) \geq 0$ and $|\omega| = 1$;

- (iv) *Euler product hypothesis :* $\log \mathcal{L}(s) = \sum\limits_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Also, throughout the paper by any meromorphic function we always mean a meromorphic function defined in $\mathbb{C}$. We denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By $\mathbb{N}$ we mean the set of all natural numbers. Though for standard definitions used in this paper we refer our readers to follow [9], yet for the sake of our convenience we denote the order of f by $\rho(f)$, where

$$\rho(f) = \frac{\log(T(r, f))}{\log r}. \tag{1.4}$$

By $S(r, f)$ we mean any quantity satisfying $S(r, f) = O(\log(rT(r, f)))$ for all r possibly outside a set of finite linear measure. If f is a function of finite order, then $S(r, f) = O(\log r)$ for all r.

The importance of L-functions in number theory is needless to say and an L-function can be analytically continued to a meromorphic function in $\mathbb{C}$. Hence like the value distribution of meromorphic functions, the value distribution of L-functions is a natural consequence. In this respect, during the last few years an extensive study for the distribution of zeros of L-functions have been done by various researchers [11, 14, 22, 10, 18, 19]. In due course of time, the study have been confined to the direction of uniquely determining an L-function via the shared values or sets. Hence let us recall these basic definitions of value and set sharing.

Definition 1.1. [6] *For a non-constant meromorphic function f and $a \in \mathbb{C}$, let* $E_f(a) = \{(z, p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \text{ with multiplicity } p\}$

$$\left(\overline{E}_f(a) = \{(z, 1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\right),$$

then we say f, g share the value a CM(IM) if $E_f(a) = E_g(a) \left(\overline{E}_f(a) = \overline{E}_g(a)\right)$. *For $a = \infty$, we define* $E_f(\infty) := E_{1/f}(0)$ $\left(\overline{E}_f(\infty) := \overline{E}_{1/f}(0)\right)$.

Definition 1.2. [6] *For a non-constant meromorphic function f and $S \subset \overline{\mathbb{C}}$, let* $E_f(S) = \bigcup_{a \in S}\{(z, p) \in \mathbb{C} \times \mathbb{N} : f(z) = a \text{ with multiplicity } p\}$

$$\left(\overline{E}_f(S) = \bigcup_{a \in S}\{(z, 1) \in \mathbb{C} \times \mathbb{N} : f(z) = a\}\right),$$

then we say f, g share the set S CM(IM) if $E_f(S) = E_g(S)$ $\left(\overline{E}_f(S) = \overline{E}_g(S)\right)$.

Definition 1.3. [12, 13] *Let k be a non-negative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is* *counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.4. [12] *For $S \subset \overline{\mathbb{C}}$ we define $E_f(S,k) = \cup_{a \in S} E_k(a;f)$, where k is a non-negative integer $a \in S$ or infinity. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$. If $E_f(S,k) = E_g(S,k)$, then we say that f and g share the set S with weight k.*

Obviously Definition 1.3 and Definition 1.4 are the refined notions of Definition 1.1 and Definition 1.2 respectively. However, now we recall the first result in this direction due to Steuding.

Theorem A. [19] *If two L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Since every L-function have meromorphic continuation in $\mathbb{C}$, so natural quest for the uniqueness of a meromorphic function and an L-function enters into the course of uniqueness theory vis-a-vis value distribution theory. Since an L-function can have at most one pole in $\mathbb{C}$, so it is reasonable to study the uniqueness of L-functions with meromorphic functions having finitely many poles. Pertinent to that, in 2010 Li proved the following uniqueness theorem.

Theorem B. [14] *Let a and b be two distinct finite values, and let f be a meromorphic function in the complex plane such that f has finitely many poles in the complex plane. If f and a non-constant L-function $\mathcal{L}$ share a CM and b IM, then $\mathcal{L} = f$.*

After that in 2018, taking the famous Gross Problem [8] into account, Yuan, Li and Yi [22] proposed an analogous version of the same for L-functions as follows.

Question 1.1. [22] *What can be said about the relationship between a meromorphic function f and an L-function $\mathcal{L}$ if f and $\mathcal{L}$ share one or two sets?*

Apropos of *Question 1.1*, in the same paper Yuan, Li and Yi provided the following result.

Theorem D. [22] *Let $Q(z) = z^n + az^m + b$, where a, b are non-zero constants with $\gcd(m,n) = 1$ and $n \geq 2m+5$. Further suppose f is a non-constant meromorphic* *function having finitely many poles and* $\mathcal{L}$ *is a non-constant L-function such that* $E_f(S,\infty) = E_{\mathcal{L}}(S,\infty)$*, where* $S = \{z : Q(z) = 0\}$*. Then* $f = \mathcal{L}$*.*

Later on with the aid of weighted sharing Sahoo-Sarkar [17] improved *Theorem D* as follows.

Theorem E. [17] *Let* S *be defined same as in Theorem D and* $n \geq 2m+5$*. Suppose* f *is a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *is a non-constant L-function. If* f *and* $\mathcal{L}$ *share* $(S,2)$*, then* $f = \mathcal{L}$*.*

Considering the ignoring multiplicities of the shared set Sahoo-Halder proved the following theorem.

Theorem F. [16] *Let* S *be defined same as in Theorem D and* $n \geq \max\{2m+5, 4q+9\}$*, where* $q = n - m \geq 1$*. Let* f *be a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *be a non-constant L-function. If* f *and* $\mathcal{L}$ *share* $(S,0)$*, then* $f = \mathcal{L}$*.*

Pertinent to *Theorem E* and *Theorem F*, Banerjee-Kundu [4] found out some gaps in these theorems and they provided the following theorem rectifying these gaps.

Theorem G. [4] *Let* S *be defined as in* Theorem D*,* f *be a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *be a non-constant L-function such that* $E_f(S,t) = E_{\mathcal{L}}(S,t)$*. If*

(i) $t \geq 2$ *and* $n \geq 2m+5$*, or*
(ii) $t = 1$ *and* $n \geq 2m+6$*, or*
(iii) $t = 0$ *and* $n \geq 2m+11$*,*

then $f = \mathcal{L}$*.*

In the same paper Banerjee-Kundu proved another result analogous to *Theorem G* which is as follows.

Theorem H. [4] *Let* $S = \{z : z^n + az^{n-m} + b = 0\}$*, where* a, b *are non-zero constants and* $\gcd(n,m) = 1$*. Let* f *be a non-constant meromorphic function having finitely many poles in* $\mathbb{C}$ *and* $\mathcal{L}$ *be a non-constant L-function such that* $E_f(S,t) = E_{\mathcal{L}}(S,t)$*. If*

(i) $t \geq 2$ *and* $n \geq 2m+5$*, or*
(ii) $t = 1$ *and* $n \geq 2m+6$*, or*
(iii) $t = 0$ *and* $n \geq 2m+11$*,*

then $f = \mathcal{L}$*.*

Note that the set S used in *Theorem D-H* are generated from the zeros of the polynomial

$$P(z) = z^n + az^m + b \ \ or \ \ P(z) = z^n + az^{n-m} + b, \tag{1.5}$$

where a, b are non-zero constants and $\gcd(n, m) = 1$. In [4, see Lemma 4], authors proved that these polynomials are critically injective and they may have multiple zero but that must be one in number. On this occasion let us invoke the definition of critically injective polynomial.

Definition 1.5. *Let $P(z)$ be a polynomial such that $P^{'}(z)$ has mutually r distinct zeros given by $d_1, d_2, \ldots, d_r$ with multiplicities $q_1, q_2, \ldots, q_r$ respectively. Then $P(z)$ is said to be a critically injective polynomial if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \cdot\cdot\cdot, r\}$.*

Any polynomial which is not critically injective is called a non-critically injective polynomial.

Observe that the following points come out of the above discussions.

(i) All the authors always used one of the polynomials given by (1.5).
(ii) The authors always used the set of zeros of critically injective polynomials to show the uniqueness of f and $\mathcal{L}$.
(iii) In the above theorems authors have improved the previous results by relaxing the nature of sharing of the sets.
(iv) The authors also considered the set of zeros of the polynomials having multiple zeros.

Apropos of observation (i) and (ii), One would naturally raise the following questions.

Question 1.2. *Does there exist any other polynomial except the polynomials given by (1.5) whose set of zeros provide uniqueness of f and $\mathcal{L}$?*

Question 1.3. *Does there exist any non-critically injective polynomial whose set of zeros provide the uniqueness of f and $\mathcal{L}$?*

Pertinent to observation (iii) the following questions become inevitable.

Question 1.4. *Can we have the answer of* Question 1.1 *under more relaxed sharing hypothesis than that obtained in the latest results* Theorem G-H*?*

Question 1.5. *Can we have a set with lesser cardinality than that obtained in the latest results* Theorem G-H *for the uniqueness of f and $\mathcal{L}$?*

Finally with respect to observation (iv), we have the following note.

Note 1.1. *Recently in* [5, see paragraph between Theorem H and Theorem I] *Banerjee-Kundu have clarified the fact that all the results obtained till date in this direction of shared set problems for the uniqueness of f and $\mathcal{L}$; i.e.,* Theorem D-H *(except Theorem F) have an analytical gap while considering multiple zero of the polynomials and the sharing of the sets with some non-zero weight. Thus conclusion of* Theorem D-H *(except Theorem F) become false when the multiple zero of the generating polynomials are taken into account and the sharing of the sets with some non-zero weight. But in the same scenario, the results obtained with IM sharing of the sets are correct; i.e., conclusion (iii) of* Theorem G-H *and* Theorem F*. Though* Theorem F *has a different flaw contradicting their own conclusion of cardinality $n \geq \max\{2m+5, 4q+9\}$ which is analysed in* [4, Remark 3]*. Another point is that all these results are true when the polynomial has only simple zeros.*

Hence in this paper, we shall solely concentrate on the polynomials having only simple zeros and answer all the above questions from *Question 1.2-1.5* affirmatively which improve all the existing results from *Theorem D-H.* Moreover, we present general criterions for any general polynomial so that the set of zeros of the same would provide the uniqueness of f and $\mathcal{L}$ when shared by these functions. In a nutshell, our results bring all the existing results under a single umbrella in a more improved version with extent to the most general setting.

In the 4th section of this paper, that is in the **"Application"** section we have proved all our claims to be true by exhibiting a number of examples showing the wide-ranging applications of our results.

Before going to our main results, we make a short discussion on the structure of a general polynomial as this will play an important role throughout the rest of this paper.

Let us consider the following general polynomial $P(z)$ of degree n having only simple zeros.

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \tag{1.6}$$

where $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n, a_0 \neq 0$, a_i being the first non-vanishing coefficient from $a_{n-1}, a_{n-2}, \ldots, a_1$. Let

$$S = \{z : P(z) = 0\}. \tag{1.7}$$

Observe that (1.6) can be written in the form

$$P(z) = a_n \prod_{i=1}^{p} (z - \alpha_i)^{m_i} + a_0, \tag{1.8}$$

where p denotes the number of distinct zeros of $P(z)-a_0$. Let us also denote by s the number of distinct zeros of $P^{'}(z)$. Hence we would have

$$P^{'}(z)=na_n\prod_{i=1}^{s}(z-\eta_i)^{r_i}, \tag{1.9}$$

where r_i denotes the multiplicities of distinct zeros of $P^{'}(z)$.

Set

$$R(z)=-\frac{a_nz^n}{a_{n-1}z^{n-1}+\ldots+a_1z+a_0}=-\frac{a_nz^n}{a_i\prod\limits_{j=1}^{k}(z-\beta_j)^{m_j}}=-\frac{a_nz^n}{\phi(z)}, \tag{1.10}$$

where $a_0,a_1,\ldots,a_n$ are as defined in (1.6) and $\beta_1,\beta_2,\ldots,\beta_k$ are the roots of the equation

$$\phi(z)=a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\ldots+a_1z+a_0=0,$$

with multiplicities $m_1,m_2,\ldots,m_k$. Clearly

$$R(z)-1=-\frac{P(z)}{\phi(z)}, \tag{1.11}$$

where $P(z)$ is defined by (1.6) and obviously $P(z)$ and $\phi(z)$ do not share any common zero. Hence S as defined in (1.7) can be treated as

$$S=\{z:P(z)=0\}=\{z:R(z)-1=0\}. \tag{1.12}$$

Let $R^{'}(z)$ has l distinct zeros say $\delta_1,\delta_2,\ldots,\ldots,\delta_l$ with multiplicities $q_1,q_2,\ldots,q_l$ respectively. Then From (1.10) we would have

$$R'(z) \quad = \quad \frac{\gamma\prod\limits_{j=1}^{l}(z-\delta_j)^{q_j}}{\prod\limits_{j=1}^{k}(z-\beta_j)^{p_j}}, \tag{1.13}$$

where $\gamma\in\mathbb{C}-\{0\}$ and $p_j\in\mathbb{N}$ for all $j\in\{1,2,\ldots,k\}$.

Remark 1.1. *Observe that in the definition (1.6) of the general polynomial $P(z)$, the condition $a_i\neq 0$ for $i=\{1,2,\ldots,n-1\}$ is necessary. Because otherwise we would find a non-constant L-function $\mathcal{L}$ and a non-constant meromorphic function f which share the set $S=\{z:P(z)=0\}$ CM but $f\not\equiv\mathcal{L}$.*

For example, let $a_i=0$ for $i=\{1,2,\ldots,n-1\}$. Then $S=\{z:a_nz^n+a_0=0\}$. Consider a non-constant L-function $\mathcal{L}$ and a non-constant meromorphic function f such that $f=\zeta\mathcal{L}$, where ζ is the nth root of unity. Then clearly,

$$a_nf^n+a_0=a_n\mathcal{L}^n+a_0;$$

$$i.e.,\ \prod_{i=1}^{n}(f-\sigma_i)=\prod_{i=1}^{n}(\mathcal{L}-\sigma_i),$$

where $\sigma_i \in S$ for $i = \{1, 2, \ldots, n\}$. That is f and $\mathcal{L}$ share S CM but $f \neq \mathcal{L}$.

Now we provide the following two theorems as the main results of this paper.

Theorem 1.1. *Let $P(z)$ be given by (1.8) with $p \geq 2$ and S, s be defined by (1.7), (1.9) respectively. Suppose f is a non-constant meromorphic function having finitely many poles and $\mathcal{L}$ is a non-constant L-function sharing (S, t). Then for $n \geq \max\{2p+1, 2s+3\}$, when $t \geq 1$; and for $n \geq \max\{2p+1, 2s+6\}$, when $t = 0$; the following are equivalent :*

(i) $P(f) = P(\mathcal{L}) \implies f = \mathcal{L}$;
(ii) $E_f(S, t) = E_{\mathcal{L}}(S, t) \implies f = \mathcal{L}$.

Theorem 1.2. *Let $R(z)$ be defined by (1.10) with $k \geq 2$ or $k = 1$ with $n > 2m_1$ and S, l be defined by (1.12), (1.13) respectively. Let f be a non-constant meromorphic function having finitely many poles and $\mathcal{L}$ be a non-constant L-function sharing (S, t). Then for $n \geq \max\{2k+3, 2l+3\}$, when $t \geq 1$; and for $n \geq \max\{2k+3, 2l+6\}$, when $t = 0$; the following are equivalent :*

(i) $R(f) = R(\mathcal{L}) \implies f = \mathcal{L}$;
(ii) $E_f(S, t) = E_{\mathcal{L}}(S, t) \implies f = \mathcal{L}$.

Remark 1.2. *Obviously* Theorem G-H *are the latest results in this direction for simple zeros of the polynomials given by* (1.5). *In the application section* (Example 4.5 *and* Example 4.2)*, we shall show that the conclusions of* Theorem G-H *are true for $n \geq 7$ when $E_f(S, 1) = E_{\mathcal{L}}(S, 1)$, whereas the same is true in* Theorem G-H *for $n \geq 8$. Thus our result directly improves* Theorem G-H *by reducing the cardinality of the set S when shared by the functions with weight* 1*. We also find that weight* 2 *in* Theorem G-H *can be relaxed to weight* 1 *keeping the carinality of the set fixed as an application of our result. Hence the answer of* Question 1.4 *is also obtained with improvement. Moreover, in* Theorem G-H *the least cardinality of the sets when shared IM is 13 whereas the same result can be obtained when the cardinalities of the sets are 10, which is a significant improvement of* Theorem G-H*. Thus we obtain a threefold improvement of* Theorem G-H *by the application of our main results and obtain the answer of* Question 1.4-1.5*.*

We shall also obtain similar results in the application section for other polynomials including critically injective polynomials, non-critically injective polynomials and even those polynomials which are still uncertain to be critically injective or non-critically injective (*see* Example 4.1*,* Example 4.3 *and* Example 4.4)*. These results* *provide us the answers of* Question 1.2-1.3 *with improvements in the nature of sharing of the sets as well as the least cardinalities of the sets.*

Remark 1.3. *The reason behind proving two similar but different theorems are clarified in the first two paragraphs of section 5 named* **"Conclusion and an Open Question"**.

For standard definitions and notations we have already suggested our readers to follow [9]. Furthermore, we explain the following notations which will be used throughout the paper for the proof of the *Theorem 1.1* and *Theorem 1.2*.

Definition 1.6. [21] *Let f and g be two non-constant meromorphic functions such that f and g share $(1,0)$. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those 1-points of f and g where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p=q=1$. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a \in \overline{\mathbb{C}}$.*

Definition 1.7. [12, 13] *Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Definition 1.8. [13] *For $a \in \mathbb{C}\cup\{\infty\}$we denote by $N(r,a;f\mid=1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f\mid\leq m)(N(r,a;f\mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.*

$\overline{N}(r,a;f\mid\leq m)$ $(\overline{N}(r,a;f\mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid<m)$, $N(r,a;f\mid>m)$, $\overline{N}(r,a;f\mid<m)$ and $\overline{N}(r,a;f\mid>m)$ are defined analogously.

Definition 1.9. [2] *Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid\ g\neq b_1,b_2,\ldots,b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i=1,2,\ldots,q$.*

2. Lemmas

For two non-constant meromorphic functions F and G, set

$$H=\left(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1}\right)-\left(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [21] *Let F, G share $(1,0)$ and $H\not\equiv 0$. Then*

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. [2] *Let F, G share $(1,t)$, where $t\in\mathbb{N}\cup\{0\}$. Then*

$$\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_E^{1)}(r,1;F)+\left(t-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\leq\frac{1}{2}[N(r,1;F)+N(r,1;G)].$$

Lemma 2.3. *Let f be a non-constant meromorphic function having finite number of poles and $\mathcal{L}$ be an non-constant L-function sharing a set S IM, where $|S|\geq 3$. Then $\rho(f)=\rho(\mathcal{L})=1$. Furthermore, $\overline{N}(r,\infty;f)=O(\log r)=\overline{N}(r,\infty;\mathcal{L})$ and $S(r,f)=O(\log r)=S(r,\mathcal{L})$.*

Proof. Proceeding in a similar method as done in the proof of Theorem 5, [16, p. 6] we can obtain $\rho(f)=\rho(\mathcal{L})=1$. So we omit it.

Since f has finitely many poles and $\mathcal{L}$ has at most one pole in $\mathbb{C}$, so obviously

$$\overline{N}(r,\infty;f)=O(\log r)=\overline{N}(r,\infty;\mathcal{L}). \tag{2.2}$$

Since $\rho(f)=\rho(\mathcal{L})=1$, so from the definition of $S(r,f)$ we get $S(r,f)=O(\log r)=S(r,\mathcal{L})$. □

Lemma 2.4. *Let $F^*-1=\dfrac{a_n\prod_{i=1}^{n}(f-w_i)}{\psi(f)}$ and $G^*-1=\dfrac{a_n\prod_{i=1}^{n}(\mathcal{L}-w_i)}{\psi(\mathcal{L})}$, where f be a non-constant meromorphic function having finite number of poles, $\mathcal{L}$ be an non-constant L-function, $a_n,w_i\in\mathbb{C}-\{0\}$; $\forall i\in\{1,2,\ldots,n\}$ and $\psi(z)$ be a polynomial of degree less than n with $\psi(w_i)\neq 0$; $\forall i\in\{1,2,\ldots,n\}$. Further suppose that F^* and G^* share $(1,t)$, where $t\in\mathbb{N}\cup\{0\}$. Then*

$$\overline{N}_L(r,1;F^*)\leq\frac{1}{t+1}\left[\overline{N}(r,0;f)-N_1(r,0;f')\right]+O(\log r), \tag{2.3}$$

where $N_1(r,0;f')=N(r,0;f'|f\neq 0,w_1,w_2,...,w_n)$. Similar expression also holds for $\overline{N}_L(r,1;G^*)$.

Proof. Since F^* and G^* share $(1,t)$, so in view of *Lemma 2.3* using the first fundamental theorem we find that

$$\begin{aligned}
\overline{N}_L(r,1;F^*) \quad & \leq \overline{N}(r,1;F^*| \geq t+2) \leq \frac{1}{t+1}\left[N(r,1;F^*) - \overline{N}(r,1;F^*)\right] \\
& \leq \frac{1}{t+1}\left[\sum_{i=1}^{n}\left(N(r,w_i;f) - \overline{N}(r,w_i;f)\right)\right] \\
& \leq \frac{1}{t+1}\left[N(r,0;f'|f\neq 0) - N_1(r,0;f')\right] \\
& \leq \frac{1}{t+1}\left[N(r,0;\frac{f'}{f}) - N_1(r,0;f')\right] \\
& \leq \frac{1}{t+1}\left[N(r,\infty;\frac{f}{f'}) - N_1(r,0;f')\right] + O(\log r) \\
& \leq \frac{1}{t+1}\left[N(r,\infty;\frac{f^{'}}{f}) - N_1(r,0;f')\right] + O(\log r) \\
& \leq \frac{1}{t+1}\left[\overline{N}(r,\infty;f) + \overline{N}(r,0;f) - N_1(r,0;f')\right] + O(\log r) \\
& \leq \frac{1}{t+1}\left[\overline{N}(r,0;f) - N_1(r,0;f')\right] + O(\log r).
\end{aligned}$$

This proves the lemma. □

Lemma 2.5. *Let $P(z)$, S and s as defined by* (1.6), (1.7) *and* (1.9) *respectively. Suppose that f, $\mathcal{L}$ share (S,t), where $t\in\mathbb{N}\cup\{0\}$ and f, $\mathcal{L}$ be a non-constant meromorphic function and an L-function respectively. Further suppose that*

(2.4)
$$\mathcal{F} = \frac{P(f)-a_0}{-a_0} = -\frac{a_n}{a_0}\prod_{i=1}^{p}(f-\alpha_i)^{m_i} \;\; \text{and} \;\; \mathcal{G} = \frac{P(\mathcal{L})-a_0}{-a_0} = -\frac{a_n}{a_0}\prod_{i=1}^{p}(\mathcal{L}-\alpha_i)^{m_i}.$$

Then for $n\geq 2s+3$, when $t\geq 1$ and for $n\geq 2s+6$, when $t=0$ we get the following.

$$\frac{1}{\mathcal{F}-1} = \frac{A}{\mathcal{G}-1} + B,$$

where $A(\neq 0), B\in\mathbb{C}$.

Proof. According to the assumptions of the lemma we clearly have $\mathcal{F}$, $\mathcal{G}$ share $(1,t)$ and

$$\mathcal{F}^{'} = -\frac{na_n}{a_0}\prod_{i=1}^{s}(f-\eta_i)^{r_i}f^{'}; \quad \mathcal{G}^{'} = -\frac{na_n}{a_0}\prod_{i=1}^{s}(\mathcal{L}-\eta_i)^{r_i}\mathcal{L}^{'},$$

where $\sum\limits_{i=1}^{s} r_i = n-1$. Now consider H as given by (2.1) for $\mathcal{F}$ and $\mathcal{G}$.

Case-I: Suppose $H\not\equiv 0$. Then, it can be easily verified that H has only simple poles and these poles come from the following points.

(i) α_i-points of f and $\mathcal{L}$.

(ii) Poles of f and $\mathcal{L}$.

(iii) 1-points of $\mathcal{F}$ and $\mathcal{G}$ having different multiplicities.

(iv) Those zeros of $f^{'}$ and $\mathcal{L}^{'}$ which are not zeros of $\prod\limits_{i=1}^{s}(f-\eta_i)(\mathcal{F}-1)$ and $\prod\limits_{i=1}^{s}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively.

Therefore we obtain

$$(2.5)\quad N(r,H) \leq \sum_{i=1}^{s}\left[\overline{N}(r,\eta_i;f)+\overline{N}(r,\eta_i;\mathcal{L})\right]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) +\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;\mathcal{L}^{'}),$$

where $\overline{N}_0(r,0;f^{'})$ and $\overline{N}_0(r,0;\mathcal{L}^{'})$ denotes the reduced counting functions of those zeros of $f^{'}$ and $\mathcal{L}^{'}$ which are not zeros of $\prod\limits_{i=1}^{s}(f-\eta_i)(\mathcal{F}-1)$ and $\prod\limits_{i=1}^{s}(\mathcal{L}-\eta_i)(\mathcal{G}-1)$ respectively. Using the second fundamental theorem we get

(2.6)

$$(n+s-1)T(r,f) \leq \overline{N}(r,1;\mathcal{F})+\sum_{i=1}^{s}\overline{N}(r,\eta_i;f)+\overline{N}(r,\infty;f)-N_0(r,0;f^{'})+S(r,f),$$

(2.7)

$$(n+s-1)T(r,\mathcal{L}) \leq \overline{N}(r,1;\mathcal{G})+\sum_{i=1}^{s}\overline{N}(r,\eta_i;\mathcal{L})+\overline{N}(r,\infty;\mathcal{L})-N_0(r,0;\mathcal{L}^{'})+S(r,\mathcal{L}).$$

For the sake of our convenience let us denote by $T(r) = T(r,f)+T(r,\mathcal{L})$. Now combining (2.6) and (2.7) with the help of *Lemma 2.2*, *Lemma 2.1* and then (2.5) we get

$$\begin{aligned}(2.8)(n+s-1)T(r) \leq\ & \overline{N}(r,1;\mathcal{F})+\overline{N}(r,1;\mathcal{G})\\ +\ & \sum_{i=1}^{s}\left[\overline{N}(r,\eta_i;f)+\overline{N}(r,\eta_i;\mathcal{L})\right]\\ & +\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]-N_0(r,0;f^{'})-N_0(r,0;\mathcal{L}^{'})\\ & +S(r,f)+S(r,\mathcal{L})\\ \leq\ & \frac{n}{2}T(r)+2\sum_{i=1}^{s}\left[\overline{N}(r,\eta_i;f)+\overline{N}(r,\eta_i;\mathcal{L})\right]\\ & +2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]+\left(\frac{3}{2}-t\right)\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\\ & +S(r,f)+S(r,\mathcal{L}).\end{aligned}$$

Hence in view of *Lemma 2.3*, for $t \geq 2$; (2.8) reduces to

$$(\frac{n}{2}-s-1)T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2s+3$ as $\rho(f)=1=\rho(\mathcal{L})$.

We know that $\overline{N}_*(r,1;\mathcal{F},\mathcal{G}) = \overline{N}_L(r,1;\mathcal{F}) + \overline{N}_L(r,1;\mathcal{G})$. Hence for $0 \leq t \leq 1$; using *Lemma 2.4* and *Lemma 2.3* we get from (2.8) that

$$(2.9) \qquad (\frac{n}{2} - s - 1)T(r) \leq \frac{(\frac{3}{2} - t)}{t+1}\left[\overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L})\right] + O(\log r).$$

Now for $t = 1$; from (2.9) we get

$$(\frac{n}{2} - s - \frac{5}{4})T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2s + 3$.

For $t = 0$; from (2.9) we get

$$(\frac{n}{2} - s - \frac{5}{2})T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2s + 6$.

Case-II: Suppose $H \equiv 0$. Hence on integration, we obtain $\dfrac{1}{\mathcal{F}-1} = \dfrac{A}{\mathcal{G}-1} + B$, where $A(\neq 0), B \in \mathbb{C}$. □

Lemma 2.6. *Let $R(z)$, S and l as defined by* (1.10), (1.12) *and* (1.13) *respectively. Suppose that f, $\mathcal{L}$ share (S,t), where $t \in \mathbb{N} \cup \{0\}$ and f, $\mathcal{L}$ be a non-constant meromorphic function and an L-function respectively. Further suppose that*

$$(2.10) \qquad \mathbb{F} = R(f) \ \ and \ \ \mathbb{G} = R(\mathcal{L}).$$

Then for $n \geq 2l+3$, when $t \geq 1$ and for $n \geq 2l+6$, when $t = 0$ we get the following.

$$\frac{1}{\mathbb{F}-1} = \frac{A}{\mathbb{G}-1} + B,$$

where $A(\neq 0), B \in \mathbb{C}$.

Proof. Clearly $\mathbb{F}, \mathbb{G}$ share $(1,t)$ and in view of (1.13) we have

$$(2.11) \qquad \mathbb{F}' = \frac{\gamma \prod_{j=1}^{l}(f-\delta_j)^{q_j}}{\prod_{j=1}^{k}(f-\beta_j)^{p_j}} f', \qquad \mathbb{G}' = \frac{\gamma \prod_{j=1}^{l}(\mathcal{L}-\delta_j)^{q_j}}{\prod_{j=1}^{k}(\mathcal{L}-\beta_j)^{p_j}} \mathcal{L}'.$$

Now consider H as given by (2.1) for $\mathbb{F}$ and $\mathbb{G}$.

Case-I: Suppose $H \not\equiv 0$. Since H has only simple poles and in this case these poles come from the following points.

(i) δ_j -points of f and $\mathcal{L}$.
(ii) Poles of f and $\mathcal{L}$.
(iii) 1-points of $\mathbb{F}$ and $\mathbb{G}$ having different multiplicities.

(iv) Those zeros of $f^{'}$ and $\mathcal{L}^{'}$ which are not zeros of $\prod_{j=1}^{l}(f-\delta_j)(\mathbb{F}-1)$ and $\prod_{j=1}^{l}(\mathcal{L}-\delta_j)(\mathbb{G}-1)$ respectively.

Therefore we obtain

$$N(r,H)\leq \overline{N}(r,\infty;f)+\sum_{j=1}^{l}\overline{N}(r,\delta_j;f)+\overline{N}_0(r,0;f')+\overline{N}(r,\infty;\mathcal{L}) \tag{2.12}$$

$$+\quad\sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})+\overline{N}_0(r,0;\mathcal{L}')+\overline{N}_*(r,1;\mathbb{F},\mathbb{G})+S(r,f)+S(r,\mathcal{L}),$$

where we write $\overline{N}_0(r,0;f')$ for the reduced counting function of the zeros of f' that are not zeros of $(\mathbb{F}-1)\prod_{j=1}^{l}(f-\delta_j)^{q_j}$ and $\overline{N}_0(r,0;\mathcal{L}')$ is similarly defined. By using *Lemma 2.1* , *Lemma 2.2* and (2.12) we observe that

$$\overline{N}(r,1;\mathbb{F})+\overline{N}(r,1;\mathbb{G})\leq N(r,H)+\frac{1}{2}\left[N(r,1;\mathbb{F})+N(r,1;\mathbb{G})\right] \tag{2.13}$$

$$-(t-\frac{1}{2})\overline{N}_*(r,1;\mathbb{F},\mathbb{G})\leq \overline{N}(r,\infty;f)+\sum_{j=1}^{l}\overline{N}(r,\delta_j;f)+\overline{N}(r,\infty;\mathcal{L})+\sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})$$

$$+\frac{n}{2}\left\{T(r,f)+T(r,\mathcal{L})\right\}+\left(\frac{3}{2}-t\right)\overline{N}_*(r,1;\mathbb{F},\mathbb{G})+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;\mathcal{L}')+S(r,f)+S(r,\mathcal{L}).$$

Set $T(r,f)+T(r,\mathcal{L})=T(r)$. Hence in view of (2.13), using the second fundamental theorem we have

$$(n+l-1)T(r)\leq \overline{N}(r,\infty;f)+\overline{N}(r,1;\mathbb{F})+\sum_{j=1}^{l}\overline{N}(r,\delta_j;f) \tag{2.14}$$

$$+\overline{N}(r,\infty;\mathcal{L})+\overline{N}(r,1;\mathbb{G})+\sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})-N_0(r,0;f')-N_0(r,0;\mathcal{L}')$$

$$+S(r,f)+S(r,\mathcal{L})\leq 2\sum_{j=1}^{l}\overline{N}(r,\delta_j;f)+2\sum_{j=1}^{l}\overline{N}(r,\delta_j;\mathcal{L})$$

$$+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]+\frac{n}{2}T(r)+\left(\frac{3}{2}-t\right)\overline{N}_*(r,1;\mathbb{F},\mathbb{G})$$

$$+S(r,f)+S(r,\mathcal{L})\leq (2l+\frac{n}{2})T(r)+2\left[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L})\right]$$

$$+\left(\frac{3}{2}-t\right)\overline{N}_*(r,1;\mathbb{F},\mathbb{G})+S(r,f)+S(r,\mathcal{L}).$$

Hence in view of *Lemma 2.3*, for $t\geq 2$; (2.14) reduces to

$$\left(\frac{n}{2}-l-1\right)T(r)\leq O(\log r),$$

which is a contradiction for $n\geq 2l+3$.

For $0 \leq t \leq 1$; using $\overline{N}_*(r,1;\mathbb{F},\mathbb{G}) = \overline{N}_L(r,1;\mathbb{F}) + \overline{N}_L(r,1;\mathbb{G})$, *Lemma 2.4* and *Lemma 2.3* we get from (2.14) that

$$\left(\frac{n}{2} - l - 1\right) T(r) \quad \leq \quad \frac{\left(\frac{3}{2} - t\right)}{t+1} \left[\overline{N}(r,0;f) + \overline{N}(r,0;\mathcal{L})\right] + O(\log r). \tag{2.15}$$

Now for $t = 1$; from (2.15) we get

$$\left(\frac{n}{2} - l - \frac{5}{4}\right) T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2l + 3$.

For $t = 0$; from (2.15) we get

$$\left(\frac{n}{2} - l - \frac{5}{2}\right) T(r) \leq O(\log r),$$

which is a contradiction for $n \geq 2l + 6$.

Case-II: Suppose $H \equiv 0$. Now integrating (2.1), we find that

$$\frac{1}{\mathbb{F} - 1} = \frac{A}{\mathbb{G} - 1} + B, \quad \text{where} \;\; A(\neq 0), B \in \mathbb{C}. \tag{2.16}$$

Lemma 2.7. [20] *Let F and G be two non-constant meromorphic functions such that*

$$\frac{1}{F-1} = \frac{A}{G-1} + B,$$

where $A(\neq 0), B \in \mathbb{C}$. If

$$\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) < T(r),$$

where $T(r) = \max\{T(r,F), T(r,G)\}$. Then either $FG = 1$ or $F = G$.

Lemma 2.8. *Let $\mathcal{F}$, $\mathcal{G}$ be defined by* (2.4) *with $p \geq 2$ and they share $(1,t)$ for $t \in \mathbb{N} \cup \{0\}$. Then $\mathcal{F}\mathcal{G} \neq a$, where a is non-zero complex constant.*

Proof. On the contrary, suppose that $\mathcal{F}\mathcal{G} = a$. Then

$$\prod_{i=1}^{p} (f - \alpha_i)^{m_i} \prod_{i=1}^{p} (\mathcal{L} - \alpha_i)^{m_i} = a \left(\frac{a_0}{a_n}\right)^2 = a_1(say). \tag{2.17}$$

It is clear from (2.17) that each α_i-point of f is a pole of $\mathcal{L}$ and vice-versa. Now let us consider the following cases.

<u>Case-1</u>: Let $p \geq 4$. Since an L- function has at most one pole, then in view of (2.17) we can say that f has at least three α_i-points which are picard exeptional values. That is, the meromorphic function f omits at least 3 values, so f must be constant. This contradicts our assumption.

Case-2: Let $p = 3$. Again like the arguments made above we can say that f omits two values say α_1, α_2. Hence using the second fundamental theorem in view of *Lemma 2.3*, we obtain

$$\begin{aligned} T(r,f) &\leq \sum_{i=1}^{2} \overline{N}(r,\alpha_i;f) + \overline{N}(r,\infty;f) + O(\log r) \\ &\leq O(\log r), \end{aligned}$$

which is a contradiction.

Case-3: Let $p = 2$. Note that applying similar argument as made in Case-1 we get f omits at-least one of the α_i's say α_1. On the other hand, f cannot omit both the α_i's. For if, f omits both the α_i's, then we again arrive at a contradiction like Case-2. Hence let us assume α_2 points of f are the poles of $\mathcal{L}$. Again as $z = 1$ is the only pole of $\mathcal{L}$, so let $z = 1$ be α_2 point of f of multiplicity r and the pole of $\mathcal{L}$ of multiplicity s. Then $m_2 r = ns$, which implies $m_2 r \geq n$; i.e., $\frac{1}{r} \leq \frac{m_2}{n}$. Now using the second fundamental theorem in view of *Lemma 2.3* we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,\alpha_1;f) + \overline{N}(r,\alpha_2;f) + \overline{N}(r,\infty;f) + O(\log r) \\ &\leq \frac{m_2}{n} T(r,f) + O(\log r), \end{aligned}$$

which is a contradiction as $n > m_2$. □

Lemma 2.9. *Let* $\mathbb{F}, \mathbb{G}$ *as defined by* (2.10)*. Then for*

(i) $k \geq 2$*; or*

(ii) $k = 1$ *with* $n > 2m_1$*;*

$\mathbb{F}\mathbb{G} \neq a$*, where* a *is non-zero complex constant.*

Proof. On the contrary suppose that $\mathbb{F}\mathbb{G} \equiv a$. Then

$$\frac{f^n}{\prod_{j=1}^{k}(f-\beta_j)^{m_j}} . \frac{\mathcal{L}^n}{\prod_{j=1}^{k}(\mathcal{L}-\beta_j)^{m_j}} \equiv a \left(\frac{a_i}{a_n}\right)^2 = a^{'}(say) \tag{2.18}$$

It is clear from (2.18) that β_j point of f is a zero of $\mathcal{L}$ and vice-versa and

$$T(r,f) = T(r,\mathcal{L}) + O(1). \tag{2.19}$$

Now we deal with the following cases.

Case I: Let $k \geq 2$. If z_0 be a zero of $f - \beta_j$ with multiplicity p, then z_0 is a zero of g with multiplicity q such that $m_j p = nq$ i.e., $p \geq \dfrac{n}{m_j}$. Therefore $\overline{N}(r,\beta_j;f) \leq \dfrac{m_j}{n} N(r,\beta_j;f)$.

So, in view of *Lemma 2.3* using the the second fundamental theorem, we get

$$\begin{aligned}(k-1)T(r,f) &\leq \sum_{j=1}^{k}\overline{N}(r,\beta_j;f)+\overline{N}(r,\infty;f)+O(\log r)\\ &\leq \sum_{j=1}^{k}\frac{m_j}{n}T(r,f)+O(\log r)\\ &\leq (1-\frac{1}{n})T(r,f)+O(\log r), \quad (2.20)\end{aligned}$$

which contradicts $k \geq 2$.

Case-II: For $k=1$, from (2.18) we have

$$\frac{\mathcal{L}^n}{(\mathcal{L}-\beta_1)^{m_1}} = \frac{a^{'}(f-\beta_1)^{m_1}}{f^n}. \quad (2.21)$$

From (2.21) we see that $\overline{N}(r,0;f) = \overline{N}(r,\beta_1;\mathcal{L})+\overline{N}(r,\infty;\mathcal{L}) = \overline{N}(r,\beta_1;\mathcal{L})+O(\log r)$. Also by similar calculation as in Case-I we have $\overline{N}(r,\beta_1;f)=\frac{m_1}{n}N(r,\beta_1;f)$ and $\overline{N}(r,\beta_1;\mathcal{L})=\frac{m_1}{n}N(r,\beta_1;\mathcal{L})$. Again using the second fundamental theorem in view of *Lemma 2.3* and (2.19) we have

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,\beta_1;f)+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+O(\log r).\\ &\leq \frac{2m_1}{n}T(r,f)+O(\log r), \quad (2.22)\end{aligned}$$

which is a contadiction as $n > 2m_1$. □

3. Proof Of the theorems

Proof Of the theorem 1.1. We prove the theorem step by step as follows.

<u>(**i**) $\Longrightarrow$ (**ii**)</u> : Suppose f is a non-constant meromorphic function and $\mathcal{L}$ is a non-constant L-function such that $E_f(S,t)=E_{\mathcal{L}}(S,t)$, where $t\in\mathbb{N}\cup\{0\}$. Consider $\mathcal{F}$ and $\mathcal{G}$ as defined by (2.4). Then for

(i) $t\geq 1$ and $n\geq 2s+3$, or
(ii) $t=0$ and $n\geq 2s+6$,

in view of the *Lemma 2.5* we get $\frac{1}{\mathcal{F}-1}=\frac{A}{\mathcal{G}-1}+B$, where $A(\neq 0), B\in\mathbb{C}$. Hence we have

$$T(r,\mathcal{F}) = T(r,\mathcal{G})+O(1). \quad (3.1)$$

Since

$$T(r,\mathcal{F}) = nT(r,f)+O(1) \ \textit{and} \ T(r,\mathcal{G}) = nT(r,\mathcal{L})+O(1). \quad (3.2)$$

So (3.1) implies that

$$T(r,f) = T(r,\mathcal{L})+O(1). \quad (3.3)$$

Now in view of *Lemma 2.3* using (3.2) and (3.3) we get

$$\begin{aligned} & \overline{N}(r,0;\mathcal{F})+\overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+\overline{N}(r,\infty;\mathcal{G}) \\ \leq\ & pT(r,f)+pT(r,\mathcal{L})+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;\mathcal{L}) \\ =\ & 2pT(r,f)+O(\log r)<\frac{2p+1}{n}T(r,\mathcal{F}) \\ \leq\ & T(r,\mathcal{F}) \quad [\because n\geq 2p+1]. \end{aligned}$$

So in view of *Lemma 2.7*, we have either $\mathcal{F}\mathcal{G}=1$ or $\mathcal{F}=\mathcal{G}$. Since $p\geq 2$, so in view of *Lemma 2.6* we have $\mathcal{F}\mathcal{G}\neq 1$. Hence $\mathcal{F}=\mathcal{G}$. That is, we get

$$P(f)=P(\mathcal{L}), \tag{3.4}$$

which by condition (i) implies $f=\mathcal{L}$.

<u>**(ii)** $\implies$ **(i)**</u> : Let $P(f)=P(\mathcal{L})$. That is,

$$\prod_{i=1}^{p}(f-\alpha_i)^{m_i}=\prod_{i=1}^{p}(\mathcal{L}-\alpha_i)^{m_i},$$

which implies f and $\mathcal{L}$ share (S,∞). Therefore, obviously f and $\mathcal{L}$ share (S,t) for $t\in\mathbb{N}\cup\{0\}$. Hence by condition (ii), we have $f=\mathcal{L}$.

Proof of the theorem 1.2. Let us consider $\mathbb{F}$ and $\mathbb{G}$ as defined by (2.10). Let f be a non-constant meromorphic function and $\mathcal{L}$ be a non-constant L-function such that $E_f(S,t)=E_{\mathcal{L}}(S,t)$, where $t\in\mathbb{N}\cup\{0\}$. Then $\mathbb{F}$, $\mathbb{G}$ share $(1,t)$. Now for

(i) $t\geq 1$ and $n\geq 2l+3$, or
(ii) $t=0$ and $n\geq 2l+6$,

in view of *Lemma 2.6* we have

$$\frac{1}{\mathbb{F}-1}=\frac{A}{\mathbb{G}-1}+B, \tag{3.5}$$

where $A(\neq 0), B\in\mathbb{C}$.

From (3.5) we easily obtain

$$T(r,f)=T(r,\mathcal{L})+S(r,f). \tag{3.6}$$

Now in view of *Lemma 2.3*, (3.6) and from the construction of $\mathbb{F}$ and $\mathbb{G}$ we get

$$\begin{aligned} & \overline{N}(r,0;\mathbb{F})+\overline{N}(r,\infty;\mathbb{F})+\overline{N}(r,0;\mathbb{G})+\overline{N}(r,\infty;\mathbb{G}) \\ \leq\ & \overline{N}(r,0;f)+\sum_{j=1}^{k}\overline{N}(r,\beta_j;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;\mathcal{L})+\sum_{j=1}^{k}\overline{N}(r,\beta_j;\mathcal{L})+\overline{N}(r,\infty;\mathcal{L}) \\ \leq\ & (1+k)T(r,f)+(1+k)T(r,\mathcal{L})+O(\log r)=2(1+k)T(r,f)+O(\log r) \\ <\ & \frac{2k+3}{n}T(r,\mathbb{F})\leq T(r,\mathbb{F}) \ \ [\because\ \ n\geq 2k+3]. \end{aligned}$$

So in view of *Lemma 2.7*, we have either $\mathbb{F}\mathbb{G} \equiv 1$ or $\mathbb{F} \equiv \mathbb{G}$. Again in view of *Lemma 2.9* we have $\mathbb{F}\mathbb{G} \not\equiv 1$. Thus $\mathbb{F} \equiv \mathbb{G}$; i.e., $R(f) = R(\mathcal{L})$.

Therefore we find that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$, whenever $R(f) = R(\mathcal{L}) \implies f = \mathcal{L}$. That is $(i) \implies (ii)$.

To show $(ii) \implies (i)$, suppose that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$. Let $R(f) = R(\mathcal{L})$, then we have $R(f) - 1 = R(\mathcal{L}) - 1$; i.e., $\dfrac{P(f)}{\phi(f)} = \dfrac{P(\mathcal{L})}{\phi(\mathcal{L})}$. Therefore f and $\mathcal{L}$ share (S,∞) and which implies $E_f(S,t) = E_{\mathcal{L}}(S,t)$, hence $f = \mathcal{L}$.

4. Applications

In this section, we prove that all the existing results can be improved as an application of our results. Moreover, there exist other polynomials providing better results than the existing ones including those polynomials which are still uncertain to be critically injective or non-critically injective. Furthermore, in this section we have also exhibited a similar result for non-critically injective polynomials which is yet not considered in this literature. In a word, by executing the following examples we prove the far reaching applications of *Theorem 1.1* and *Theorem 1.2*.

First of all we exhibit examples of critically injective polynomials as the applications of *Theorem 1.1*.

Example 4.1. *Let us consider the following polynomial.*

$$P(z) = z^n + az^{n-m} + bz^{n-2m} + c, \tag{4.1}$$

where $a, b, c \in \mathbb{C}^*$ be such that $P(z)$ has no multiple root, $gcd(m,n) = 1$ and $\frac{a^2}{4b} = \frac{n(n-2m)}{(n-m)^2}$, $c \neq \frac{\beta_i\beta_j}{\beta_i+\beta_j}$. Here $\beta_i = -(c_i^n + ac_i^{n-m} + bc_i^{n-2m})$, where c_i are the roots of the equation $nz^{2m} + (n-m)az^m + b(n-2m) = 0$, for $i = 1, 2, \ldots, 2m$. Suppose S denotes the set of zeros of (4.1).

Obviously, $P(z)$ has only simple zeroes and it is critically injective [6, see Lemma 2.7]. From (4.1) we have

$$P'(z) = z^{n-2m-1}[nz^{2m} + a(n-m)z^m + b(n-2m)] \tag{4.2}$$

$$= nz^{n-2m-1}\left(z^m + \frac{n(n-m)}{2n}\right)^2. \tag{4.3}$$

From (4.1) and (4.2) we find that

$$p = 2m + 1 \quad \text{and} \quad s = m + 1.$$

In [6, see proof of Theorem 1.1] it is also proved that $P(f) = P(g)$ implies $f = g$ for $n \geq 2m+4$, where $f$and g are non-constant meromorphic functions. Hence for a non-constant meromorphic function f having finitely many poles and an L-function $\mathcal{L}$ we have $P(f) = P(\mathcal{L}) \implies f = \mathcal{L}$ when $n \geq 2m+4$. Thus $P(z)$ satisfies the condition (i) of Theorem 1.1 of the present paper and hence $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ for

(1) $n \geq \max\{4m+3, 2m+5\} \geq 7$ when $t \geq 1$, and
(2) $n \geq \max\{4m+3, 2m+8\} \geq 10$ when $t = 0$.

Remark 4.1. *Note that the polynomial*

(4.4)
$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c, \quad \text{where } n \geq 6, \quad c \neq 0, 1.$$

introduced by Frank-Reinders [7] comes as the special case of (4.1) for $m = 1$, $a = -\frac{2n}{n-1}$, $b = \frac{n}{n-2}$ *and* $c \in \mathbb{C} - \{0, \frac{-1}{(n-1)(n-2)}\}$. *Hence* $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ *as* $n \geq 7$ *when* $t \geq 1$ *and* $n \geq 10$ *when* $t = 0$, *where* S *denotes the set of zeros of* (4.4) *and* f, $\mathcal{L}$ *are non-constant meromorphic function having finitely many poles and a non-constant L-function respectively.*

Example 4.2. *Consider the polynomial*

$$P(z) = z^n + az^{n-m} + b = z^{n-m}(z^m + a) + b, \tag{4.5}$$

where n, m *are relatively prime inegers and* a, b *are non-zero constants such that the polynomial has no multiple zero. Suppose* $S = \{z : P(z) = 0\}$. *Here*

$$p = m + 1 \geq 2$$

and

$$P'(z) = z^{n-m-1}(nz^m + a(n-m)); \tag{4.6}$$

i.e., $s = m + 1$.

Suppose that $P(f) = P(\mathcal{L})$, for any non-constant meromorphic function f having finitely many poles and a non-constant L-function $\mathcal{L}$, then we have

$$f^n - \mathcal{L}^n = -a(f^{n-m} - \mathcal{L}^{n-m}). \tag{4.7}$$

If $f^n \not\equiv \mathcal{L}^n$, then we can rewrite (4.7) as

$$\mathcal{L}^m = -a\frac{(h-v)(h-v^2)...(h-v^{n-m-1})}{(h-u)(h-u^2)...(h-u^{n-1})}, \tag{4.8}$$

where $h = \frac{f}{\mathcal{L}}$, $u = exp(2\pi i/n)$ and $v = exp(2\pi i/(n-m))$. Noting that n and $(n-m)$ are relatively prime positive integers, then the numerator and denominator of (4.8) have no common factors. Since $\mathcal{L}$ has atmost one pole at $z = 1$ in the complex plane, and whenever $n \geq 5$ we can see that there exists at least three distinct roots of $h^n = 1$ such that they are Picard exceptional values of h, and so it follows by (4.8) that h and thus $\mathcal{L}$ are constants, which is impossible.

Therefore, we must have $f^n = \mathcal{L}^n$. Then by (4.7) we also have $f^{n-m} = \mathcal{L}^{n-m}$. Since n and $(n-m)$ are relatively prime positive integers, we deduce that $f = \mathcal{L}$. Thus we see that $P(f) = P(\mathcal{L}) \implies f = \mathcal{L}$, when $n \geq 5$.

Now we apply Theorem 1.1 to find the minimum value of n for which we can say that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$.

Therefore, $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ for

(1) $n \geq max\{2m+3, 2m+5\} = 2m+5$ when $t \geq 1$, and
(2) $n \geq max\{2m+3, 2m+8\} = 2m+8$ when $t = 0$.

In the next example we explore a non-critically injective polynomial in the direction of Theorem 1.1.

Example 4.3. *Let*

$$P(z) = z^n + 2z^{n-1} + z^{n-2} + c, \tag{4.9}$$

where $n(\geq 5)$ is odd, $c \in \mathbb{C}$ such that $P(z)$ does not have any multiple zero. Also we have

$$P'(z) = z^{n-3}\left(nz^2 + 2(n-1)z + (n-2)\right). \tag{4.10}$$

Here $P(z)$ is a non-critically injective polynomial and we see that

$$p = 2, \quad s = 3.$$

Suppose $S = \{z : P(z) = 0\}$. Let f and $\mathcal{L}$ be two non-constant meromorphic and L-function respectively such that

$$P(f) = P(\mathcal{L}).$$

Since $\mathcal{L}$ has at most one pole in $\mathbb{C}$, hence proceeding in the same line of proof of as done in Example 4.4 *of* [15] *for uniqueness polynomial of entire function we also get here $f = \mathcal{L}$.*

Therefore $P(z)$ satisfies condition (i) of Theorem 1.1. Hence we conclude that $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ when

(1) $n \geq max\{2.2+1, 2.3+3\} = 9$ *for* $t \geq 1$, *and*
(2) $n \geq max\{2.2+1, 2.3+6\} = 13$ *for* $t = 0$.

Now we apply *Theorem 1.2* for rest of the examples where in the first example we have considered a polynomial which is still uncertain to be critically injective or non-critically injective [3, see section 5] and the polynomial used in the second example is critically injective.

Example 4.4. *Consider the polynomial*

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2,$$

where $n(\geq 6)$ is an integer and a, b are two non-zero complex numbers satisfying $ab^{n-2} \neq 1, 2$. Suppose $S = \{z : P(z) = 0\}$. It is obvious that $n(n-1)z^2 - 2n(n-2)bz + (n-1)(n-2)b^2 = 0$; has two distinct roots, say α_1 and α_2. Here

$$R(z) = \frac{az^n}{n(n-1)(z-\alpha_1)(z-\alpha_2)}. \tag{4.11}$$

Hence $S = \{z : R(z) - 1 = 0\}$. From (4.11) we have

$$R'(z) = \frac{(n-2)az^{n-1}(z-b)^2}{n(n-1)(z-\alpha_1)^2(z-\alpha_2)^2}. \tag{4.12}$$

Let f a non-constnat meromorphic function having finitely many poles and $\mathcal{L}$ be a non-constant L-function. Since every L-function is meromorphic in $\mathbb{C}$, so $R(f) = R(\mathcal{L}) \implies f = \mathcal{L}$ for $n \geq 6$ directly follows from [1, see page 67].

We also find that in this case $l = 2,\ \ k = 2$. Since $P(z)$ satisfies condition (i) of Theorem 1.2. Hence we obtain $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$ for

(1) $n \geq max\{2k+3, 2l+3\} = 7$ when $t \geq 1$, and
(2) for $n \geq max\{2k+3, 2l+6\} = 10$ when $t = 0$.

Example 4.5. *Consider the polynomial*

$$P(z) = z^n + az^m + b, \tag{4.13}$$

where m and n are positive integers such that $n \geq m+4$, a and b are finite non-zero complex numbers with $\dfrac{b^{n-m}}{a^m} \neq \dfrac{(-1)^n m^m (n-m)^{n-m}}{n^n}$. Then $P(z)$ has only simple zeros. Let S denotes the set of zeros of $P(z)$. Suppose

$$R(z) = -\frac{z^n}{az^m + b}. \tag{4.14}$$

Then we find that $S = \{z : R(z) - 1 = 0\}$. From (4.14) we have

$$R'(z) = -\frac{z^{n-1}[a(n-m)z^m + bn]}{(az^m+b)^2}. \tag{4.15}$$

Now for a non-constant meromorphic function f and a non-constant L-function $\mathcal{L}$ consider $R(f) = R(\mathcal{L})$. Then we have

$$\frac{f^n}{af^m+b} = \frac{\mathcal{L}^n}{a\mathcal{L}^m+b} \implies a(f^n\mathcal{L}^m - f^m\mathcal{L}^n) - b(\mathcal{L}^n - f^n) = 0. \tag{4.16}$$

Let $h = \dfrac{f}{\mathcal{L}}$. Suppose that h is a non-constant meromorphic function. Then from (4.16) we have

$$(4.17)\quad ah^m\mathcal{L}^{n+m}(h^{n-m}-1)+b\mathcal{L}^n(h^n-1)=0$$
$$\implies \mathcal{L}^m = -\frac{b(h^n-1)}{ah^m(h^{n-m}-1)} = -\frac{b}{a}\frac{(h-u)(h-u^2)...(h-u^{n-1})}{h^m(h-v)(h-v^2)...(h-v^{n-m-1})},$$

where $u = exp(2\pi i/n)$, and $v = exp(2\pi i/(n-m))$. Since n and m are co-prime, so is n and $(n-m)$. Hence the numerator and denominator of (4.17) have no common factors. Further, the function $\mathcal{L}$ has atmost one pole in the complex plane, it follows that h has atleast $(n-m-1)$ picard exceptional values among $\{0, v, v^2, ..., v^{n-m-1}\}$.

Clearly this is a contradiction as $n \geq m+4$. Hence h is constant. Thus from (4.17) we must have $h^n = 1 = h^{n-m}$, which in turn implies $h = 1$; i.e., $f = \mathcal{L}$. Therefore we obtain that $R(z)$ satisfies condition (i) of Theorem 1.2.

Now we count the cardinality of the set S for which $E_f(S,t) = E_{\mathcal{L}}(S,t) \implies f = \mathcal{L}$. In this case, for $R(z)$ we have

$$l = m+1, \quad k = m.$$

Therefore the condition (ii) of the Theorem 1.2 is satisfied if

(1) $n \geq max\{2m+3, 2m+5\} = 2m+5$ for $t \geq 1$ and
(2) $n \geq max\{2m+3, 2m+8\} = 2m+8$ for $t = 0$.

Remark 4.2. *Observe that* Example 4.5 *and* Example 4.2 *answer* Question 1.4 *and* Question 1.5 *with threefold improvement to* Theorem G-H *as discussed in* Remark 1.2 *which inturn improve* Theorem D-H *by relaxing the nature of sharing of the sets or reducing the least cardinalities of the sets or both.*

Remark 4.3. *Further note that* Example 4.1*,* Example 4.3 *and* Example 4.4 *answer* Question 1.2 *and* Question 1.3 *affirmatively. Moreover,* Example 4.1 *and* Example 4.4 *improves* Theorem D-H *either by relaxing the nature of sharing of the sets or reducing the least cardinalities of the sets or both.*

5. Conclusion and an open question

Observe that if we consider *Example 4.2* in the direction of *Theorem 1.1*, then we would obtain the same conclusion for $n \geq \max\{2(n-m)+3, 2(n-m+1)+3\} = \max\{2n-2m+3, 2n-2m+5\}$; i.e., $m \geq \frac{n+5}{2}$, which is absurd. So, *Theorem 1.2* is not applicable for *Example 4.2*, whereas *Theorem 1.1* is applicable for the same. Similarly we would have problems in counting the cardinality of the set if we apply *Theorem 1.2* in case of *Example 4.1* and *Example 4.3*.

Conversely the conclusion of *Example 4.4* and *Example 4.5* can not be obtained as the application of *Theorem 1.1* but *Theorem 1.2*. That is why, we have have proved two theorems in this paper in the most general setting to justify all the existing results as well as to include all the variants of polynomials for the uniqueness of f and $\mathcal{L}$.

Last but not the least, observing *Theorem 1.1-1.2* and *Example 4.1-4.5* carefully, it is obvious that for any polynomial if one can find $P(f) = P(\mathcal{L})$ or $R(f) = R(\mathcal{L})$ implies $f = \mathcal{L}$, then at instant we would be able to find out the set with least possible cardinality and sharing condition. Hence under this circumstances, the following question become indispensable for the uniqueness of f and $\mathcal{L}$.

Question 5.1. *Can one find general criterion(s) for any general polynomial given by (1.6) so that $P(f) = P(\mathcal{L})$ or $R(f) = R(\mathcal{L})$ implies $f = \mathcal{L}$?*

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