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# SOME RESULTS ON NONLINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS

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Abstract. We describe transcendental entire solutions of certain nonlinear difference-differential equations of the forms:

$$f(z)^2 + f(z)[af'(z) + bf(z+c)] + q(z)e^{Q(z)}f(z+c) = u(z)e^{v(z)},$$

and

$$f(z)^n + f(z)^{n-1}[af'(z) + bf(z+c)] + q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z},$$

where $q(z), Q(z), u(z), v(z)$ are non-zero polynomials, $a, b, c, p_i, \lambda_i (i = 1, 2)$ are non-zero constants such that $\lambda_1 \neq \lambda_2$. Our results are improvements and complements of Li et al. ([8]). Some examples are given to illustrate our results are accurate.

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## 1. Introduction and main results

Considering a meromorphic function $f$ in the complex plane $\mathbb{C}$, we assume that the reader is familiar with the fundamental results and standard notation of Nevanlinna theory, such as the proximity function $m(r, f)$, the counting function $N(r, f)$, and the characteristic function $T(r, f)$, see, e.g., [3, 6, 18]. We denote by $S(r, f)$ any real function of growth $o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function $\alpha$ is said to be a small function of $f$, if $T(r, \alpha) = S(r, f)$.

In 1964, Hayman [3] considered the following non-linear differential equation

$$f(z)^n + Q_d(f(z)) = g(z), \tag{1.1}$$

where $Q_d(f)$ is a differential polynomial in $f$ with degree $d$ and obtained the following result.

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**Theorem 1.1.** *([3]) Suppose that $f(z)$ is a non-constant meromorphic function, $d \leq n-1$, and $f, g$ satisfy $N(r, f) + N(r, \frac{1}{g}) = S(r, f)$ in (1.1). Then we have $g(z) = (f(z) + \gamma(z))^n$, where $\gamma(z)$ is meromorphic and a small function of $f(z)$.*

Nowadays, there has been recent interest in connections between the Nevanlinna theory and the difference operator, as well as meromorphic solutions of difference and functional equations. Yang and Laine[16] then investigated finite order entire solutions $f$ of non-linear differential-difference equations of the form

$$f(z)^n + L(z, f) = h(z),$$

where $L(z, f)$ is a linear differential-difference polynomial in $f$ with meromorphic coefficients of growth $S(r, f)$, $h(z)$ is meromorphic, and $n \geq 2$ is an integer. Many authors have investigated this question by utilizing the Nevanlinna value distribution theory and its difference counterparts, see, e.g., [5, 9, 10, 11, 13, 15].

In 2016, Liu [12] investigated and classified the finite order entire solutions of the equation

$$f(z)^n + q(z)e^{Q(z)} f^{(k)}(z + c) = P(z), \tag{1.2}$$

where $q(z), Q(z), P(z)$ are polynomials, $n \geq 2, k \geq 1$ are integers and $c \in \mathbb{C}\backslash\{0\}$. Later, Chen [2] replaced $P(z)$ in (1.2) by $p_1 e^{\lambda_1} + p_2 e^{\lambda_2}$, where $p_1, p_2, \lambda_1, \lambda_2$ are non-zero constants, and studied its finite order entire solutions when $n \geq 3$.

By observing all the above equations, it is easy to see that the left side of these equations have only one dominant term $f^n$. It is nature to ask what can we get if the left side of these equations have two dominant terms. In 2021, Li [8] investigated non-linear differential-difference equations which may have two dominated terms on the left-hand side with the same degree:

$$f(z)^n + \omega f(z)^{n-1} f'(z) + q(z)e^{Q(z)} f(z + c) = P(z), \tag{1.3}$$

they replaced $P(z)$ in (1.3) by $u(z)e^{v(z)}$ or $p_1 e^{\lambda_1} + p_2 e^{\lambda_2}$ respectively, and obtained the following results.

**Theorem 1.2.** *([8]) Let $c, \widetilde{\omega} \neq 0$ be constants, $q, Q, u, v$ be polynomials such that $Q, v$ are not constants and $q, u \not\equiv 0$. Suppose that $f$ is a transcendental entire solution with finite order of*

$$f(z)^2 + \widetilde{\omega} f(z) f'(z) + q(z)e^{Q(z)} f(z + c) = u(z)e^{v(z)}, \tag{1.4}$$

*satisfying $\lambda(f) < \rho(f)$, then $\deg Q = \deg v$, and one of the following relations holds:*

*(1) $\rho(f) < \deg Q = \deg v$, and $f = Ce^{\frac{-z}{\widetilde{\omega}}}$*

*(2) $\rho(f) = \deg Q = \deg v$.*

**Theorem 1.3.** *([8]) Suppose that $n$ is a positive integer, $\omega$ is a constant and $c, \lambda_1, \lambda_2, p_1, p_2$ are non-zero constants, $q, Q$ are polynomials such that $Q$ is not a constant and $q \not\equiv 0$. If $f$ is a transcendental entire solution with finite order of*

$$f(z)^n + \omega f(z)^{n-1} f'(z) + q(z) e^{Q(z)} f(z+c) = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}, \tag{1.5}$$

*where $\lambda_2 \neq \pm\lambda_1$, then the following conclusions hold:*

*(1) If $n \geq 4$ for $\omega \neq 0$ and $n \geq 3$ for $\omega = 0$, then every solution $f$ satisfies $\rho(f) = \deg Q = 1$.*

*(2) If $n \geq 1$ and $f$ is a solution of (1.5) with $\lambda(f) < \rho(f)$, then*

$$f(z) = \left(\frac{p_2 n}{n + \omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 z}{n}}, \quad Q(z) = \left(\lambda_1 - \frac{\lambda_2}{n}\right) z + b_1,$$

*or*

$$f(z) = \left(\frac{p_1 n}{n + \omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 z}{n}}, \quad Q(z) = \left(\lambda_2 - \frac{\lambda_1}{n}\right) z + b_2,$$

*where $b_1, b_2 \in \mathbb{C}$ satisfy $p_1 = q\left(\frac{p_2 n}{n+\omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 c}{n} + b_1}$ and $p_2 = q\left(\frac{p_1 n}{n+\omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 c}{n} + b_2}$, respectively.*

In the following, inspired by the ideas of [8], we will investigate non-linear differential-difference equations which may have three dominated terms on the left-hand side of (1.4) and (1.5) and obtain the following results.

**Theorem 1.4.** *Let $a, b, c$ be non-zero constants, $q, Q, u, v$ be polynomials such that $q, u \not\equiv 0$. Suppose that $f$ is a transcendental entire solution with finite order of*

$$f(z)^2 + f(z)[af'(z) + bf(z+c)] + q(z) e^{Q(z)} f(z+c) = u(z) e^{v(z)}, \tag{1.6}$$

*satisfying $\lambda(f) < \rho(f)$, then one of the following relations holds:*

*(1) If $\rho(f) > \deg Q$, then $\rho(f) = \deg v = 1$, $Q$ reduces to a constant, and $f(z) = d(z)e^{a_1 z}$, where $d(z) = \frac{C_2 u(z-c)}{C_1 q(z-c)}$, here $C_1 = e^{Q + a_1 c}, C_2 = e^{v_0}$, $a_1, v_0$ are constants satisfying $1 + aa_1 + be^{a_1 c} = 0$.*

*(2) If $\rho(f) = \deg Q > \deg v$, then $\rho(f) = \deg Q = 1$, $v$ reduces to a constant, and $f(z) = d(z)e^{a_1 z}$, where $d(z) = \frac{C_4 u(z-c)}{C_3 q(z-c)}$, here $C_3 = e^{b_0 + a_1 c}, C_4 = e^{v}$, $a_1, b_0$ are constants satisfying $1 + aa_1 + be^{a_1 c} = 0$.*

*(3) If $\rho(f) < \deg Q$, then $\rho(f) = 1, \deg v = \deg Q$, $f(z) = d(z)e^{a_1 z}$, where $d(z)$ is an entire function with $\rho(d) < 1$, $a_1$ is a non-zero constant satisfying $1 + aa_1 + be^{a_1 c} = 0$.*

We exhibit some examples to show the existence of solutions in Theorem 1.4.

**Example 1.1.** *$f(z)=ze^z$ is a transcendental entire solution of the following differential-difference equation*

$$f(z)^2+f(z)\left[f'(z)-2e^{-\frac{1}{2}}f\left(z+\frac{1}{2}\right)\right]+f\left(z+\frac{1}{2}\right)=\left(z+\frac{1}{2}\right)e^{z+\frac{1}{2}}.$$

*Here $a=1, b=-2e^{-\frac{1}{2}}, v(z)=z+\frac{1}{2}$, and $0=\lambda(f)<\rho(f)=1$. Then we have $\deg v=\rho(f)=1>\deg Q=0$, and $a_1=1$ satisfy $1+aa_1+be^{a_1c}=0$.*

**Example 1.2.** *$f(z)=ze^z$ is a transcendental entire solution of the following differential-difference equation*

$$f(z)^2+f(z)\left[f'(z)-2e^{-\frac{1}{2}}f\left(z+\frac{1}{2}\right)\right]+e^{-z}f\left(z+\frac{1}{2}\right)=\left(z+\frac{1}{2}\right)e^{\frac{1}{2}}.$$

*Here $a=1, b=-2e^{-\frac{1}{2}}, Q(z)=-z$, and $0=\lambda(f)<\rho(f)=1$. Then we have $\deg Q=\rho(f)=1>\deg v=0$, and $a_1=1$ satisfy $1+aa_1+be^{a_1c}=0$.*

**Example 1.3.** *$f(z)=ze^z$ is a transcendental entire solution of the following differential-difference equation*

$$f(z)^2+f(z)\left[f'(z)-2e^{-\frac{1}{2}}f\left(z+\frac{1}{2}\right)\right]+e^{z^2}f\left(z+\frac{1}{2}\right)=\left(z+\frac{1}{2}\right)e^{z^2+z+\frac{1}{2}}.$$

*Here $a=1, b=-2e^{-\frac{1}{2}}, Q(z)=z^2, v(z)=z^2+z+\frac{1}{2}$, and $0=\lambda(f)<\rho(f)=1$. Then we have $2=\deg Q=\deg v>\rho(f)=1$, and $a_1=1$ satisfy $1+aa_1+be^{a_1c}=0$.*

**Theorem 1.5.** *Let $n$ is a positive integer, $a,b,c,\lambda_i,p_i (i=1,2)$ are non-zero constants, $q,Q$ are polynomials such that $Q$ is not a constant and $q\not\equiv 0$. If $f$ is a transcendental entire solution with finite order of*

$$f(z)^n+f(z)^{n-1}\left[af'(z)+bf(z+c)\right]+q(z)e^{Q(z)}f(z+c)=p_1e^{\lambda_1 z}+p_2e^{\lambda_2 z}, \tag{1.7}$$

*satisfying $\lambda(f)<\rho(f)$, then*

$$f(z)=\left(\frac{p_2 n}{n+a\lambda_2+nbe^{\frac{\lambda_2}{n}c}}\right)^{\frac{1}{n}}e^{\frac{\lambda_2}{n}z},\quad Q(z)=(\lambda_1-\frac{\lambda_2}{n})z+B_2,$$

*or*

$$f(z)=\left(\frac{p_1 n}{n+a\lambda_1+nbe^{\frac{\lambda_1}{n}c}}\right)^{\frac{1}{n}}e^{\frac{\lambda_1}{n}z},\quad Q(z)=(\lambda_2-\frac{\lambda_1}{n})z+B_2,$$

*where $B_2\in\mathbb{C}$ satisfy*

$$p_1=q\left(\frac{p_2 n}{n+a\lambda_2+nbe^{\frac{\lambda_2}{n}c}}\right)^{\frac{1}{n}}e^{\frac{\lambda_2}{n}c+B_2}\quad or\quad p_2=\left(\frac{p_1 n}{n+a\lambda_1+nbe^{\frac{\lambda_1}{n}c}}\right)^{\frac{1}{n}}e^{\frac{\lambda_1}{n}c+B_2},$$

*respectively.*

## 2. Some Lemmas

In order to prove results above, we need the following lemmas.

**Lemma 2.1** ([18], Theorem 1.51)**.** *Let $f_j(z)(j=1,\cdots,n)(n\geq 2)$ be meromorphic functions, and let $g_j(z)(j=1,\cdots,n)$ be entire functions satisfying*

*(i)$\sum_{j=1}^{n} f_j(z)e^{g_j(z)}\equiv 0$;*

*(ii) when $1\leq j<k\leq n$, then $g_j(z)-g_k(z)$ is not a constant;*

*(iii) when $1\leq j\leq n, 1\leq h<k\leq n$, then*

$$T(r,f_j)=o\left\{T\left(r,e^{g_h-g_k}\right)\right\}\quad (r\to\infty, r\notin E),$$

*where $E\subset(1,\infty)$ is of finite linear measure or logarithmic measure.*

*Then, $f_j(z)\equiv 0\,(j=1,\cdots,n)$.*

**Lemma 2.2** ([18], Theorem 1.62)**.** *Let $f_1,f_2,\cdots,f_n$ be non-constant meromorphic functions, and let $f_{n+1}\not\equiv 0$ be a meromorphic function such that $\sum_{j=1}^{n+1} f_j\equiv 1$. Suppose that there exists a subset $I\in\mathbb{R}^+$ with linear measure $mesI=\infty$, such that:*

$$\sum_{i=1}^{n+1} N\left(r,\frac{1}{f_i}\right)+n\sum_{i=1,i\neq j}^{n+1}\overline{N}(r,f_i)<(\sigma+o(1))T(r,f_j),\quad j=1,2,\cdots,n,$$

*as $r\in I$ and $r\to\infty$, where $\sigma$ is a real number satisfying $0\leq\sigma<1$. Then, $f_{n+1}=1$.*

**Lemma 2.3** ([7], Theorem 3.1)**.** *Let $f(z)$ be a meromorphic function with the hyper-order less than one, and $c\in\mathbb{C}\backslash\{0\}$. Then we have*

$$T(r,f(z+c))=T(r,f(z))+S(r,f).$$

**Lemma 2.4** ([18], Theorem 1.21)**.** *Suppose that $f(z)$ is meromorphic in the complex plane and $n$ is a positive integer. Then $f(z)$ and $f^{(n)}(z)$ have the same order.*

**Lemma 2.5** ([1], Lemma 3.3)**.** *Let $g$ be a transcendental meromorphic function of order less than 1, and let $h$ be a positive constant. Then there exists an $\varepsilon$-set $E$ such that as $\mathbb{C}\backslash E\ni z\to\infty$, one has*

$$\frac{g'(z+\eta)}{g(z+\eta)}\to 0,\quad \frac{g(z+\eta)}{g(z)}\to 1$$

*uniformly in $\eta$ for $\mid\eta\mid\leq h$. Further, the $\varepsilon$-set $E$ may be chosen so that for large $z$ not in $E$, the function $g$ has no zeros or poles in $\mid\zeta-z\mid\leq h$.*

**Lemma 2.6** ([17], Lemma 1)**.** *Let $f_1$ and $f_2$ be two meromorphic functions, and let $a,b_1$, $b_2$ be small functions of $f_1$ and $f_2$ satisfying $ab_1b_2\not\equiv 0$ and $b_1f_1+b_2f_2=a$.*

*Then one has*

$$T(r,f_1) \le \overline{N}(r,f_1) + \overline{N}\left(r,\frac{1}{f_1}\right) + \overline{N}\left(r,\frac{1}{f_2}\right) + S(r,f_1).$$

**Lemma 2.7** ([18], Theorem 1.57)**.** *Let $f_j(z), j = 1,2,3$ be meromorphic functions and $f_1(z)$ is not a constant. If $\sum_{j=1}^{3} f_j(z) \equiv 1$, and*

$$\sum_{j=1}^{3} N\left(r,\frac{1}{f_j}\right) + 2\sum_{j=1}^{3}\overline{N}(r,f_j) < (\lambda + o(1))T(r), \quad r \in I,$$

*where $\lambda < 1$, $T(r) = \max_{1\le j\le 3}\{T(r,f_j)\}$ and $I$ represents a set of $r \in (0,\infty)$ with infinite linear measure. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.*

**Lemma 2.8** ([18], Theorem 1.42, Theorem 1.44)**.** *Let $f(z)$ be a non-constant meromorphic function in the complex plane. If $0,\infty$ are Picard exceptional values of $f(z)$, then $f(z) = e^{h(z)}$, where $h(z)$ is a non-constant entire function. Moreover, $f(z)$ is of normal growth, and*

*(i) if $h$ is a polynomial of degree $p$, then $\rho(f) = p$;*

*(ii) if $h$ is a transcendental entire function, then $\rho(f) = \infty$.*

**Lemma 2.9** ([18], Theorem 1.22)**.** *Suppose $f(z)$ is a non-constant meromorphic function in the complex plane and $k$ is a positive integer, and let $\Psi(z) = \sum_{i=0}^{k} a_i(z) f^{(i)}(z)$, where $a_1(z), a_2(z), \cdots, a_k(z)$ are small functions of $f(z)$. Then*

$$T(r,\Psi) \le T(r,f) + k\overline{N}(r,f) + S(r,f).$$

## 3. Proof of Theorem 1.4

Let $f$ be a transcendental entire solution with finite order of equation (1.6) satisfying $\lambda(f) < \rho(f)$. Then, by the Hadamard factorization theorem, we can factorize $f(z)$ as

$$f(z) = d(z)e^{h(z)}, \tag{3.1}$$

where $h$ is a polynomial with $\deg h = \rho(f)$, $d$ is the canonical product formed by zeros of $f$ with $\rho(d) = \lambda(f) < \rho(f)$. Obviously, $h$ is a non-constant polynomial. In fact, if $h$ is a constant, then from (3.1), we will have $\rho(f) = \rho(d) = \lambda(f)$, a contradiction. Thus we have that $\deg h \ge 1$. Let $\deg h = m (\ge 1)$, and $h(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots$, where $a_m \ne 0$.

We rewrite (1.6) as

$$f^2 + f(af' + b\overline{f}) + qe^{Q}\overline{f} = ue^{v}, \tag{3.2}$$

where $\overline{f} = f(z+c)$, for simplicity.

Obviously, we have $\rho(\overline{f}) = \rho(f) = \rho(f')$ by Lemma 2.3 and Lemma 2.4. So from (3.2), by the order property, we get

$$\deg v = \rho(ue^v) \leq \max\{\rho(f') = \rho(f) = \rho(\overline{f}), \rho(e^Q), \rho(q)\} = \max\{\deg h, \deg Q\}. \tag{3.3}$$

By substituting (3.1) into (3.2), we get

$$\left[d^2 + ad(d' + dh')\right] e^{2h} + bd\overline{d}e^{\overline{h}+h} + q\overline{d}e^{Q+\overline{h}} = ue^v. \tag{3.4}$$

**Case 1.** $\rho(f) > \deg Q$, then we have $\deg h > \deg Q$, and $\deg v \leq \deg h$ from (3.3).

**Subcase 1.1.** $\deg h > \deg v$. From (3.4) we have

$$\left\{\left[d^2 + ad(d' + dh')\right] e^{h_1} + bd\overline{d}e^{h_2}\right\} e^{2a_m z^m} + q\overline{d}e^{h_3}e^{a_m z^m} = ue^v, \tag{3.5}$$

where $h_1 = 2a_{m-1}z^{m-1} + \cdots$, $h_2 = (2a_{m-1} + a_m mc)z^{m-1} + \cdots$ and $h_3 = Q + (a_m mc + a_{m-1})z^{m-1} + \cdots$ are all polynomials with degree at most $m-1$. So, combining with $\rho(d') = \rho(d) = \rho(\overline{d}) < m$, by using Lemma 2.1 to (3.5), we have $q\overline{d} \equiv 0$, which yields a contradiction. Thus $\deg h > \deg v$ can not hold.

**Subcase 1.2.** $\deg h = \deg v$. Let $v(z) = v_m z^m + v_{m-1}z^{m-1} + \cdots$, where $v_m \neq 0$. From (3.4) we have

$$\left\{\left[d^2 + ad(d' + dh')\right] e^{h_1} + bd\overline{d}e^{h_2}\right\} e^{2a_m z^m} + q\overline{d}e^{h_3}e^{a_m z^m} = ue^{h_4}e^{v_m z^m}, \tag{3.6}$$

where $h_4 = v_{m-1}z^{m-1} + \cdots$ is a polynomial with degree at most $m-1$, $h_1$, $h_2$ and $h_3$ are defined as in Subcase 1.1.

If $v_m \neq 2a_m$ and $v_m \neq a_m$, combining with $\rho(d') = \rho(d) = \rho(\overline{d}) < m$, by using Lemma 2.1 to (3.6), we get $u \equiv 0$, a contradiction.

If $v_m = 2a_m$, then (3.6) can be reduced to

$$\left\{\left[d^2 + ad(d' + dh')\right] e^{h_1} + bd\overline{d}e^{h_2} - ue^{h_4}\right\} e^{2a_m z^m} + q\overline{d}e^{h_3}e^{a_m z^m} = 0,$$

by using Lemma 2.1, we have $q\overline{d} \equiv 0$, a contradiction. Thus, we have

$$v_m = a_m. \tag{3.7}$$

Rewriting (3.6) as

$$\left\{\left[d^2 + ad(d' + dh')\right] e^{h_1} + bd\overline{d}e^{h_2}\right\} e^{2a_m z^m} + (q\overline{d}e^{h_3} - ue^{h_4})e^{a_m z^m} = 0.$$

Similarly as above, by Lemma 2.1, we get

$$\begin{cases} [d^2 + ad(d' + dh')]e^{h_1} + bd\overline{d}e^{h_2} \equiv 0, \\ q\overline{d}e^{h_3} - ue^{h_4} \equiv 0. \end{cases}$$

By observing the expressions of $h_1$ and $h_2$, we have $h_2 = h_1 + h_5$, where $h_5 = a_m mcz^{m-1} + \cdots$. Noting that $d \not\equiv 0$, then above equations can be rewrote as

$$\begin{cases} d + a(d' + dh') + b\overline{d}e^{h_5} \equiv 0, \\ q\overline{d}e^{h_3} - ue^{h_4} \equiv 0. \end{cases} \tag{3.8}$$

By the second equation of (3.8), we get

$$\overline{d} = \frac{u}{q} e^{h_4 - h_3}. \tag{3.9}$$

**Subcase 1.2.1.** $\deg h \geq 2$. Firstly, we claim that $h_4 - h_3$ is a non-constant polynomial. In fact, if $h_4 - h_3$ is a constant, then $\overline{d}$ reduces to a rational function, by the first equation of (3.8) and Lemma 2.1, we have $\overline{d} \equiv 0$, a contradiction.

Secondly, we claim that $\overline{h_4} - \overline{h_3} + \overline{h_5}$ is a non-constant polynomial. Substituting (3.9) into the first equation of (3.8), we get

$$\left\{ \left(1 + a\overline{h'}\right) \frac{u}{q} + a \left[ \left( \frac{u}{q} \right)' + \frac{u}{q}(h_4' - h_3') \right] \right\} e^{h_4 - h_3} + b \frac{\overline{u}}{\overline{q}} e^{\overline{h_4} - \overline{h_3} + \overline{h_5}} = 0. \tag{3.10}$$

If $\overline{h_4} - \overline{h_3} + \overline{h_5}$ is a constant, say $c_1$. Then (3.10) becomes

$$\left\{ (1 + a\overline{h'}) \frac{u}{q} + a \left[ \left( \frac{u}{q} \right)' + \frac{u}{q}(h_4' - h_3') \right] \right\} e^{h_4 - h_3} + b \frac{\overline{u}}{\overline{q}} e^{c_1} = 0,$$

that gives $\overline{u} \equiv 0$, a contradiction.

Thus, we get $h_4 - h_3$ and $\overline{h_4} - \overline{h_3} + \overline{h_5}$ are non-constant polynomials. If $\overline{h_4} - \overline{h_3} + \overline{h_5} - (h_4 - h_3)$ is not a constant, by (3.10) and Lemma 2.1, we have $\overline{u} \equiv 0$, a contradiction. If $\overline{h_4} - \overline{h_3} + \overline{h_5} - (h_4 - h_3)$ is a constant, say $c_2$, then (3.10) reduces to

$$(1 + a\overline{h'}) \frac{u}{q} + a \left[ \left( \frac{u}{q} \right)' + \frac{u}{q}(h_4' - h_3') \right] + b \frac{\overline{u}}{\overline{q}} e^{c_2} = 0,$$

that is,

$$(1 + a\overline{h'}) uq\overline{q} + a\overline{q}[u'q - q'u + uq(h_4' - h_3')] + b\overline{u}q^2 e^{c_2} = 0, \tag{3.11}$$

then it can be verified that the term $a\overline{h'}uq\overline{q}$ would have a higher degree of $z$ than all the other terms in (3.11), we obtain $a = 0$, which is impossible.

**Subcase 1.2.2.** $\deg h = 1$.

Noting that $\deg v = \deg h$, so we have $\deg v = 1$. Suppose that $h(z) = a_1 z + a_0$, $v(z) = v_1 z + v_0$, from (3.4) and (3.7), we have

$$\left\{ [d^2 + ad(d' + da_1)] e^{2a_0} + bd\overline{d} e^{2a_0 + a_1 c} \right\} e^{2a_1 z} + \left\{ q\overline{d} e^{Q + a_1 c + a_0} - ue^{v_0} \right\} e^{a_1 z} = 0. \tag{3.12}$$

Since $1 = \deg h > \deg Q$, we have $Q$ is a non-zero constant, by Lemma 2.1, we get

$$\begin{cases} d + a(d' + da_1) + b\overline{d} e^{a_1 c} \equiv 0, \\ q\overline{d} c_3 - uc_4 \equiv 0, \end{cases} \tag{3.13}$$

where $c_3 = e^{Q + a_1 c + a_0}$, $c_4 = e^{v_0}$, by the second equation of (3.13), we get $\overline{d} = \frac{c_4}{c_3} \frac{u}{q}$ is a rational function, then by the first equation of (3.13), we have $1 + aa_1 + be^{a_1 c} = 0$ as $z \to \infty$.

Thus, we get $f(z) = d(z)e^{a_1 z + a_0}$, where $d(z) = \frac{c_4}{c_3}\frac{u(z-c)}{q(z-c)}$, $a_0, a_1 (\neq 0)$ are constants satisfying $1 + aa_1 + be^{a_1 c} = 0$.

**Case 2.** $\rho(f) = \deg Q > \deg v$. Suppose that $Q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots$, where $b_m \neq 0$. From (3.4) we have

$$\left\{[d^2 + ad(d' + dh')]e^{h_1} + bd\overline{d}e^{h_2}\right\} e^{2a_m z^m} + q\overline{d}e^{h_6} e^{(a_m + b_m) z^m} = ue^v, \tag{3.14}$$

where $h_6 = (a_m mc + a_{m-1} + b_{m-1}) z^{m-1} + \cdots$ is a polynomial with degree at most $m-1$, and $h_1, h_2$ are defined as in Subcase 1.1.

If $b_m \neq \pm a_m$, combining with $\rho(d') = \rho(d) = \rho(\overline{d}) < m$, by using Lemma 2.1 to (3.14), we get $u \equiv 0$, which yields a contradiction.

If $b_m = a_m$, then (3.14) can be reduced to

$$\left\{[d^2 + ad(d' + dh')]e^{h_1} + bd\overline{d}e^{h_2} + q\overline{d}e^{h_6}\right\} e^{2a_m z^m} = ue^v,$$

by using Lemma 2.1, we have $u \equiv 0$, a contradiction.

Thus, we have

$$b_m = -a_m. \tag{3.15}$$

Rewriting (3.14) as

$$\left\{[d^2 + ad(d' + dh')]e^{h_1} + bd\overline{d}e^{h_2}\right\} e^{2a_m z^m} = ue^v - q\overline{d}e^{h_6}. \tag{3.16}$$

Similarly as above, by Lemma 2.1, we get

$$\begin{cases} d^2 + ad(d' + dh') + bd\overline{d}e^{h_5} \equiv 0, \\ q\overline{d}e^{h_6} - ue^v \equiv 0. \end{cases} \tag{3.17}$$

By the second equation of (3.17), we get

$$\overline{d} = \frac{u}{q} e^{h_6 - v}. \tag{3.18}$$

**Subcase 2.1.** $\deg h \geq 2$.

Firstly, we claim that $h_6 - v$ is a non-constant polynomial. Otherwise, if $h_6 - v$ is a constant, then $\overline{d}$ reduces to a rational function, by the first equation of (3.17) and Lemma 2.1, we have $\overline{d} \equiv 0$, a contradiction.

Secondly, we claim that $\overline{h_6} - \overline{v} + \overline{h_5}$ is a non-constant polynomial. Substituting (3.18) into the first equation of (3.17), we get

$$\left\{(1 + a\overline{h'})\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(h_6' - v')\right]\right\} e^{h_6 - v} + b\frac{\overline{u}}{\overline{q}} e^{\overline{h_6} - \overline{v} + \overline{h_5}} = 0. \tag{3.19}$$

If $\overline{h_6} - \overline{v} + \overline{h_5}$ is a constant, say $c_5$. Then (3.19) becomes

$$\left\{(1 + a\overline{h'})\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(h_6' - v')\right]\right\} e^{h_6 - v} + b\frac{\overline{u}}{\overline{q}} e^{c_5} = 0,$$

that gives $\overline{u} \equiv 0$, a contradiction.

Thus, we get $h_6 - v$ and $\overline{h_6} - \overline{v} + \overline{h_5}$ are non-constant polynomials. If $\overline{h_6} - \overline{v} + \overline{h_5} - (h_6 - v)$ is not a constant, by (3.19) and Lemma 2.1, we have $\overline{u} \equiv 0$, a contradiction. If $\overline{h_6} - \overline{v} + \overline{h_5} - (h_6 - v)$ is a constant, say $c_6$, by (3.19), we get

$$(1 + a\overline{h'})\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}\left(h_6' - v'\right)\right] + b\frac{\overline{u}}{\overline{q}}e^{c_6} = 0,$$

that is,

$$(1 + a\overline{h'})uq\overline{q} + a\overline{q}\left[u'q - q'u + uq\left(h_6' - v'\right)\right] + b\overline{u}q^2e^{c_6} = 0, \tag{3.20}$$

then it can be verified that the term $a\overline{h'}uq\overline{q}$ would have a higher degree of $z$ than all the other terms in (3.20), we obtain $a = 0$, which is impossible.

**Subcase 2.2.** $\deg h = 1$.

Noting that $\deg Q = \rho(f) = \deg h$, so we have $\deg Q = \deg h = 1$. Suppose that $h(z) = a_1z + a_0$, $Q(z) = b_1z + b_0$, from (3.4) and (3.15), we have

$$\left\{[d^2 + ad(d' + da_1)]e^{2a_0} + bd\overline{d}e^{2a_0+a_1c}\right\}e^{2a_1z} + q\overline{d}e^{a_1c+a_0+b_0} = ue^v. \tag{3.21}$$

Since $1 = \deg h > \deg v$, we have $v$ is a non-zero constant, by Lemma 2.1, we get

$$\begin{cases} d + a(d' + da_1) + b\overline{d}e^{a_1c} \equiv 0, \\ q\overline{d}c_7 - uc_8 \equiv 0, \end{cases} \tag{3.22}$$

where $c_7 = e^{a_1c+a_0+b_0}$, $c_8 = e^v$, by the second equation of (3.22), we get $\overline{d} = \frac{c_8}{c_7}\frac{u}{q}$ is a rational function, then by the first equation of (3.22), we have $1 + aa_1 + be^{a_1c} = 0$ as $z \to \infty$.

Thus $f(z) = d(z)e^{a_1z+a_0}$, where $d(z) = \frac{c_8}{c_7}\frac{u(z-c)}{q(z-c)}$, $a_0, a_1 (\neq 0)$ are constants satisfying $1 + aa_1 + be^{a_1c} = 0$.

**Case 3.** $\rho(f) < \deg Q$, then we have $T(r, f) = S(r, e^Q)$. Thus we get $T(r, f') = S(r, e^Q)$ from Lemma 2.9 and $T(r, \overline{f}) = S(r, e^Q)$ from Lemma 2.3. Therefore, by (3.2), we have

$$\begin{aligned} T(r, e^Q) + S(r, e^Q) &= T(r, f^2 + f(af' + b\overline{f}) + q\overline{f}e^Q) \\ &= T(r, ue^v) = T(r, e^v) + S(r, e^v). \end{aligned}$$

Therefore, $\deg Q = \deg v$. Differentiating (3.2) yields

$$2ff' + f'(af' + b\overline{f}) + f(af'' + b\overline{f'}) + Ae^Q = (u' + uv')e^v, \tag{3.23}$$

with $A = q'\overline{f} + q\overline{f'} + q\overline{f}Q'$.

Eliminating $e^v$ from (3.2) and (3.23) to get

$$B_1e^Q + B_2 = 0, \tag{3.24}$$

where

$$B_1 = uA - q\overline{f}(u' + uv'),$$

$$B_2 = 2uff' + uf'(af' + b\overline{f}) + uf(af'' + b\overline{f'}) - (u' + uv')[f^2 + f(af' + b\overline{f})].$$

Noticing that $\rho(\overline{f}) = \rho(f) < \deg Q$, and $\rho(f'') = \rho(f') = \rho(f) < \deg Q$ from Lemma 2.4, thus by Lemma 2.1, we get $B_1 \equiv B_2 \equiv 0$. It follows from $B_1 \equiv 0$ that

$$\frac{q'}{q} + \frac{\overline{f'}}{\overline{f}} + Q' = \frac{u'}{u} + v',$$

by integrating, we have $q\overline{f}e^{Q} = c_9 ue^{v}$, where $c_9$ is a non-zero constant.

**Subcase 3.1.** $c_9 = 1$. By substituting $q\overline{f}e^{Q} = ue^{v}$ into (3.2), we see that

$$f + af' + b\overline{f} = 0. \tag{3.25}$$

**Subcase 3.1.1.** $\deg h \geq 2$. Then $\deg v = \deg Q > \rho(f) = \deg h \geq 2$. By substituting $\overline{f} = \frac{u}{q}e^{v-Q}$ into (3.25), we can get

$$\left\{\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(v' - Q')\right]\right\}e^{v-Q} + b\frac{\overline{u}}{\overline{q}}e^{\overline{v}-\overline{Q}} = 0. \tag{3.26}$$

If $\overline{v} - \overline{Q} - (v - Q)$ is a constant, say $c_{10}$. Then (3.26) becomes

$$\frac{u}{q} + a\left[\left(\frac{u}{q}\right)' + \frac{u}{q}(v' - Q')\right] + b\frac{\overline{u}}{\overline{q}}e^{c_{10}} = 0,$$

that is,

$$uq\overline{q} + a\overline{q}[u'q - q'u + uq(v' - Q')] + b\overline{u}q^2e^{c_{10}} = 0, \tag{3.27}$$

we claim that $v' - Q'$ is not a constant, otherwise $v - Q$ is linear, then $\deg h = \rho(f) = \rho(\overline{f}) = \rho(e^{v-Q}) = 1$, which contradicts with $\deg h \geq 2$. It can be verified that the term $auq\overline{q}(v' - Q')$ would have a higher degree of $z$ than all the other terms in (3.27), we obtain $a = 0$, which is impossible.

If $\overline{v} - \overline{Q} - (v - Q)$ is not a constant, by (3.26) and Lemma 2.1, we have $\overline{u} \equiv 0$, a contradiction.

**Subcase 3.1.2.** $\deg h = 1$.

By substituting $f = de^{h}$ into (3.25), we can get

$$[d + a(d' + dh')]e^{h} + b\overline{d}e^{\overline{h}} = 0, \tag{3.28}$$

substituting $h(z) = a_1z + a_0$ into (3.28), we have

$$1 + a\left(\frac{d'}{d} + a_1\right) + b\frac{\overline{d}}{d}e^{a_1c} = 0.$$

Noting that $\rho(d) < \rho(f) = \deg h = 1$, by Lemma 2.5, we have $1 + aa_1 + be^{a_1c} = 0$ as $z \to \infty$.

Thus $f(z) = d(z)e^{a_1z+a_0}$, where $d(z)$ is an entire function with $\rho(d) < 1$, $a_0, a_1(\neq 0)$ are constants satisfying $1 + aa_1 + be^{a_1c} = 0$.

**Subcase 3.2.** $c_9 \neq 1$.

In this case, we have $\overline{f}=c_9\frac{u}{q}e^{v-Q}$ . By substituting it into (3.2), we get

$$\begin{aligned}\left\{c_9^2\frac{u^2}{q^2}+ac_9^2\frac{u}{q}\left[\left(\frac{u}{q}\right)'+\frac{u}{q}(v'-Q')\right]\right\}e^{2(v-Q)}\\ +bc_9^2\frac{u}{q}\frac{\overline{u}}{\overline{q}}e^{\overline{v}-\overline{Q}+v-Q}=(1-c_9)\overline{u}e^{\overline{v}},\end{aligned}\tag{3.29}$$

we can easily get $v-Q$ is not a constant because $f$ is transcendental, and so $\overline{v}-\overline{Q}+v-Q$ is not a constant.

If $\overline{v}-\overline{Q}-(v-Q)$ is a constant, say $c_{11}$. Then (3.29) becomes

$$c_9^2\frac{u^2}{q^2}+ac_9^2\frac{u}{q}\left[\left(\frac{u}{q}\right)'+\frac{u}{q}(v'-Q')\right]+bc_9^2e^{c_{11}}\frac{u}{q}\frac{\overline{u}}{\overline{q}}=(1-c_9)\overline{u}e^{\overline{v}-2(v-Q)}.$$

Since $\deg Q=\deg v>\rho(f)=\deg(v-Q)$, we can easily deduce a contradiction by the fact that $c_9\neq 1$ and $\overline{u}\not\equiv 0$.

If $\overline{v}-\overline{Q}-(v-Q)$ is not a constant, note that $\deg Q=\deg v>\rho(f)=\deg(v-Q)$, so $\overline{v}-2(v-Q)$ and $v-Q-\overline{Q}$ are not constants, by Lemma 2.1, we can also deduce a contradiction by the fact that $c_9\neq 1$ and $\overline{u}\not\equiv 0$.

The proof of Theorem 1.4 is now completed.

## 4. Proof of Theorem 1.5

Suppose that $f$ is a transcendental entire solution with finite order of equation (1.7) with $\lambda(f)<\rho(f)$. Then, by the Hadamard factorization theorem, we can factorize $f(z)$ as

$$f(z)=d(z)e^{h(z)},\tag{4.1}$$

where $h$ is a polynomial with $\deg h=\rho(f)$, $d$ is the canonical products formed by zeros of $f$ with $\rho(d)=\lambda(f)<\rho(f)$. Similarly as in the proof of Theorem 1.4, we have $\rho(f)=\deg h\geq 1$.

We rewrite (1.7) as

$$f^n+f^{n-1}(af'+b\overline{f})+qe^Q\overline{f}=p_1e^{\lambda_1 z}+p_2e^{\lambda_2 z},\tag{4.2}$$

where $\overline{f}=f(z+c)$, for simplicity.

By substituting (4.1) into (4.2), we get

$$d^{n-1}[d+a(d'+dh')]e^{nh}+b\overline{d}d^{n-1}e^{(n-1)h+\overline{h}}+q\overline{d}e^{Q+\overline{h}}=p_1e^{\lambda_1 z}+p_2e^{\lambda_2 z}.\tag{4.3}$$

**Case 1.** $\deg h\geq 2$.

**Subcase 1.1.** $\deg(Q+\overline{h})\leq 1$. Rewriting (4.3) as:

$$d^{n-1}[d+a(d'+dh')]e^{nh}+b\overline{d}d^{n-1}e^{(n-1)h+\overline{h}}=p_1e^{\lambda_1 z}+p_2e^{\lambda_2 z}-q\overline{d}e^{Q+\overline{h}}.\tag{4.4}$$

Denote that $\alpha = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z} - q\overline{d}e^{Q+\overline{h}}$, since $\rho(f) = \deg h \geq 2$, we have $T(r,\alpha) = S(r,e^h)$. Next, we claim that $\alpha \not\equiv 0$. Otherwise, (4.4) reduces to

$$d + a(d' + dh') + b\overline{d}e^{\overline{h}-h} \equiv 0,$$

it implies $\overline{d} \equiv 0$ because $\deg h \geq 2$, a contradiction.

From (4.4), we have

$$d^{n-1}[d + a(d' + dh')]e^{nh} + b\overline{d}d^{n-1}e^{(n-1)h+\overline{h}} = \alpha. \tag{4.5}$$

Obviously, $d^{n-1}[d+a(d'+dh')] \not\equiv 0$. Otherwise, if $d^{n-1}[d+a(d'+dh')] \equiv 0$, we will have $d = C_1e^{-\frac{z}{a}-h}$, so $\rho(d) = \deg h = \rho(f)$, which contradicts with the fact that $\rho(d) < \rho(f)$. Then from (4.5) and Lemma 2.6, we get

$$T(r,e^{nh}) \leq \overline{N}(r,e^{nh}) + \overline{N}\left(r,\frac{1}{e^{nh}}\right) + \overline{N}\left(r,\frac{1}{e^{(n-1)h+\overline{h}}}\right) + S(r,e^{nh}) = S(r,e^{nh}),$$

a contradiction.

**Subcase 1.2.** $\deg(Q+\overline{h}) \geq 2$. Dividing both sides of (4.3) by $p_2e^{\lambda_2 z}$, we obtain

$$\sum_{j=1}^{4} f_j = 1, \tag{4.6}$$

where

$$f_1 = \frac{d^{n-1}[d + a(d' + dh')]}{p_2}e^{nh-\lambda_2 z}, \quad f_2 = \frac{b\overline{d}d^{n-1}}{p_2}e^{(n-1)h+\overline{h}-\lambda_2 z},$$
$$f_3 = \frac{q\overline{d}}{p_2}e^{Q+\overline{h}-\lambda_2 z}, \quad f_4 = -\frac{p_1}{p_2}e^{\lambda_1 z-\lambda_2 z}.$$

Since $\deg(nh - \lambda_2 z) \geq 2$, $\deg((n-1)h + \overline{h} - \lambda_2 z) \geq 2$, $\deg(Q + \overline{h} - \lambda_2 z) \geq 2$, $\lambda_1 \neq \lambda_2$, it is obvious to see $f_j (j = 1,2,3,4)$ are not constants. Thus, we deduce:

$$\begin{aligned}\sum_{j=1}^{4} N\left(r,\frac{1}{f_j}\right) &\leq O\left(N\left(r,\frac{1}{\overline{d}}\right)\right) + O\left(N\left(r,\frac{1}{d}\right)\right) + O(\log r)\\ &\leq O(T(r,d)) + O(\log r) = o(T(r,f_j)), (j = 1,2,3),\end{aligned}$$

and

$$\sum_{j=1}^{4} \overline{N}(r,f_j) \leq O(\log r) = o(T(r,f_j)), (j = 1,2,3),$$

as $r \in I$ and $r \to \infty$.

Thus by (4.6) and Lemma 2.2, we deduce $f_4 = \frac{p_1}{p_2}e^{\lambda_1 z-\lambda_2 z} \equiv 1$, which is impossible.

**Case 2.** $\deg h = 1$. In this case, we claim that $\deg Q = 1$. Otherwise, suppose that $\deg Q \geq 2$, by (1.7), we obtain $q\overline{f}e^Q = H$, where

$$H = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z} - f^n - f^{n-1}(af' + b\overline{f}).$$

Note that $\rho(\overline{f}) = \rho(f') = \rho(f) = \deg h = 1 < \deg Q$, then by combining with Lemma 2.1, we get $q\overline{f} \equiv H \equiv 0$, a contradiction. So we have $\deg Q = \deg h = 1$.

Set $h(z) = A_1 z + B_1$, $Q(z) = A_2 z + B_2$, where $A_1(\neq 0), A_2(\neq 0)$ and $B_1, B_2$ are constants. By substituting these into (4.3) and dividing both sides by $p_2 e^{\lambda_2 z}$, we have

$$f_1 + f_2 + f_3 = 1, \tag{4.7}$$

where

$$\begin{aligned} f_1 &= -\frac{p_1}{p_2} e^{(\lambda_1 - \lambda_2)z}, \\ f_2 &= \frac{e^{nB_1} d^{n-1}[d + a(d' + dh') + b\overline{d}e^{A_1 c}]}{p_2} e^{(nA_1 - \lambda_2)z}, \\ f_3 &= \frac{q\overline{d}e^{A_1 c + B_1 + B_2}}{p_2} e^{(A_1 + A_2 - \lambda_2)z}. \end{aligned}$$

Obviously, $f_1$ is not a constant since $\lambda_1 \neq \lambda_2$. We set

$$T(r) = \max\{T(r, f_1), T(r, f_2), T(r, f_3)\} = T(r, \mathrm{e}^z).$$

Since $\rho(d) < 1$, then we have

$$N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_2}\right) + N\left(r, \frac{1}{f_3}\right) \leq O(T(r, d)) + O(\log r) = o(T(r)),$$

and

$$\overline{N}(r, f_1) + \overline{N}(r, f_2) + \overline{N}(r, f_3) \leq O(\log r) = o(T(r)),$$

as $r \to \infty$. Therefore, by using Lemma 2.7, we can deduce that $f_2 \equiv 1$ or $f_3 \equiv 1$. If $f_2 \equiv 1$, that is

$$e^{nB_1} d^{n-1} \left[d + a(d' + dh') + b\overline{d}e^{A_1 c}\right] e^{(nA_1 - \lambda_2)z} = p_2. \tag{4.8}$$

We assert that $A_1 = \frac{\lambda_2}{n}$. Otherwise, suppose that $A_1 \neq \frac{\lambda_2}{n}$, then from $\rho(d') = \rho(d) < 1 = \deg[(nA_1 - \lambda_2)z]$, by using Lemma 2.1 to (4.8), we get $p_2 = 0$, a contradiction. Thus $h = \frac{\lambda_2}{n} z + B_1$. By substituting it into (4.8), we have

$$d^{n-1} \left[d + a\left(d' + d\frac{\lambda_2}{n}\right) + b\overline{d}e^{\frac{\lambda_2}{n}c}\right] = p_2 e^{-nB_1}. \tag{4.9}$$

Next, we assert that $d$ is a constant. Otherwise, if $d$ is a non-constant entire function, then from (4.9) we get that 0 is a Picard exceptional value of $d$. Thus by Lemma 2.8, we have $d = e^{\alpha}$, where $\alpha$ is a non-constant polynomial, which contradicts with the assumption that $\rho(d) < 1$. So we have that $d$ is a non-zero constant, and (4.9) reduces to

$$d^n e^{nB_1} \left(1 + a\frac{\lambda_2}{n} + be^{\frac{\lambda_2}{n}c}\right) = p_2.$$

Therefore,

$$f(z) = de^{h(z)} = de^{B_1} e^{\frac{\lambda_2}{n} z} = \left( \frac{np_2}{n + a\lambda_2 + nbe^{\frac{\lambda_2}{n} c}} \right)^{\frac{1}{n}} e^{\frac{\lambda_2}{n} z}.$$

Moreover, from $f_2 \equiv 1$ and (4.7), we also have $f_1 + f_3 \equiv 0$. That is

$$q\overline{d}e^{A_1 c + B_1 + B_2} e^{(A_1 + A_2) z} = p_1 e^{\lambda_1 z},$$

which implies that

$$A_2 = \lambda_1 - A_1 = \lambda_1 - \frac{\lambda_2}{n}, \text{ i.e. } Q(z) = \left( \lambda_1 - \frac{\lambda_2}{n} \right) z + B_2,$$

where $B_2$ satisfies $p_1 = q \left( \frac{np_2}{n + a\lambda_2 + nbe^{\frac{\lambda_2}{n} c}} \right)^{\frac{1}{n}} e^{\frac{\lambda_2}{n} c + B_2}$.

If $f_3 \equiv 1$, by using the similar methods as in the case $f_2 \equiv 1$, we get

$$f(z) = \left( \frac{np_1}{n + a\lambda_1 + nbe^{\frac{\lambda_1}{n} c}} \right)^{\frac{1}{n}} e^{\frac{\lambda_1}{n} z},$$

then from (4.7) we have $f_1 + f_2 = 0$. This gives that

$$Q(z) = \left( \lambda_2 - \frac{\lambda_1}{n} \right) z + B_2,$$

where $B_2$ satisfies $p_2 = q \left( \frac{np_1}{n + a\lambda_1 + nbe^{\frac{\lambda_1}{n} c}} \right)^{\frac{1}{n}} e^{\frac{\lambda_1}{n} c + B_2}$.

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