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GENERALIZATIONS OF SOME DIFFERENTIAL INEQUALITIES FOR POLYNOMIALS

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Abstract. We consider polynomials of the form $P(z)=z^s\big(a_0+\sum\limits_{v=t}^{n-s}a_vz^v\big), t\geq 1, 0\leq s\leq n-1$ and prove some results for the estimate of the polar derivative $D_\alpha P(z):=nP(z)+(\alpha-z)P'(z)$ and generalize the results due to Aziz and Shah [*Indian J. Pure Appl. Math.,* **29**(1998), 163-173], Govil [*J. Approx. Theory,* **66**(1991), 29-35] and others.

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1. Introduction

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z):=\displaystyle\sum_{j=0}^{n}a_jz^j$ of degree atmost n over the field $\mathbb{C}$ of complex numbers.

If $P\in\mathcal{P}_n$ and P' be its derivative, then concerning the estimate $|P'(z)|$, in terms of $|P(z)|$ on $|z|=1$, we have the following famous sharp result due to Bernstein [7].

$$\max_{|z|=1}|P'(z)|\leq n\max_{|z|=1}|P(z)|. \tag{1.1}$$

Since equality holds in (1.1) if and only if P has all its zeros at the origin, it stands natural to ask what happens to inequality (1.1), if we impose restrictions on the location of zeros of P. In this connection the following inequalities are the earliest belonging to this domain of ideas which have a clear impact on the subsequent work carried forward since then.

If $P\in\mathcal{P}_n$ has all zeros in $|z|\geq 1$, then

$$\max_{|z|=1}|P'(z)|\leq \frac{n}{2}\max_{|z|=1}|P(z)|, \tag{1.2}$$

and if it has all zeros in $|z|\leq 1$, then

$$\max_{|z|=1}|P'(z)|\geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.3}$$

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Inequality (1.2) was conjectured by Erdös and latter verified by Lax [14], whereas inequality (1.3) is due to Turán [16]. Inequality (1.2) was generalized by Malik [15] to read as:

Theorem A. *If $P(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, where $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|.$$

Govil [13] also generalized inequality (1.2) in a different way. More precisely he proved the following.

Theorem B. *If $P(z)$ is a polynomial of degree n, such that $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ and $|Q'(z)|$ attain their maxima at the same point on the unit circle, where $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$.

It is worth mentioning that the Bernstein inequality has been generalized in different forms by replacing the underlying polynomial with more general class of functions. These inequalities have their own importance in the theory of approximation. The results we prove provide extensions, generalizations and refinements of various differential inequalities for polynomials. Before proceeding for the main results, we first define the polar derivative of a polynomial.

For a polynomial $P(z)$ of degree n, the polar derivative of $P(z)$ denoted by $D_\alpha P(z)$, is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

It is to be observed that

$$\lim_{|\alpha|\to\infty} \left| \frac{D_\alpha P(z)}{\alpha} \right| = P'(z).$$

Aziz [2] extended Theorem A to the polar derivative of a polynomial and proved the following.

Theorem C. *If $P(z)$ is a polynomial of degree n, such that $P(z)$ does not vanish in $|z| < k, k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n\left(\frac{k+|\alpha|}{1+k}\right) \max_{|z|=1} |P(z)|. \tag{1.4}$$

In this paper we prove.

Theorem 1.1. *If $P(z)$ is a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| > k, k \geq 1$ with s-fold zero at the origin, $0 \leq s < n$, then for every real or* *complex number* α *with* $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha|+k)}{1+k} + \frac{sk(|\alpha|-1)}{1+k} \right) \max_{|z|=1} |P(z)|. \tag{1.5}$$

The result is sharp for $s = 0$ and equality holds for the polynomial $P(z) = (z+k)^n$. For $s = 0$, inequality (1.5) reduces to a result due to Aziz [2, Theorem 3] whereas for $s = n-1$, we have the following.

Corollary 1.1. *If* $P(z)$ *is a polynomial of degree* n *having all* $n-1$ *zeros at the origin and one zero in* $|z| > k, k \geq 1$, *then for every* α *with* $|\alpha| \geq 1$, *we have*

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{1}{1+k} \left\{ \big(n(1+k) - k\big)|\alpha| + k \right\} \max_{|z|=1} |P(z)|.$$

On dividing both sides of above inequality by $|\alpha|$ and letting $|\alpha| \to \infty$, we get

$$\max_{|z|=1} |P'(z)| \leq \left(n - \frac{k}{1+k} \right) \max_{|z|=1} |P(z)|.$$

Remark 1.1. *Divide the two sides of inequality (1.5) by* $|\alpha|$ *and letting* $|\alpha| \to \infty$, *we get a result due to Aziz and Shah* [5].

Theorem 1.2. *Let* $P(z)$ *be a polynomial of degree* n, *such that all zeros of* $P(z)$ *lie in* $|z| > k, k \leq 1$ *with s-fold zeros at the origin, then for every real or complex number* α *with* $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha| + k^{n-s})}{1+k^{n-s}} + \frac{sk^{n-s}(|\alpha|-1)}{1+k^{n-s}} \right) \max_{|z|=1} |P(z)|, \tag{1.6}$$

provided $|P'(z)|$ *and* $|Q'(z)|$ *attain their maxima at the same point on the unit circle, where* $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

The result is sharp for $s = 0$ and equality holds for the polynomial $P(z) = z^n + k^n$.

On dividing both sides of inequality (1.6) by $|\alpha|$ and letting $|\alpha| \to \infty$, it reduces to a following result.

Corollary 1.2. *Let* $P(z)$ *be a polynomial of degree* n, *such that all zeros of* $P(z)$ *lie in* $|z| > k, k \leq 1$ *with s-fold zeros at the origin, then for every real or complex number* α *with* $|\alpha| \geq 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n + sk^{n-s}}{1+k^{n-s}} \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ *and* $|nP(z) - zP'(z)|$ *attain their maximum at the same points on* $|z| = 1$.

Remark 1.2. *For $s = 0$, Theorem 1.2 reduces to a result due to Chanam* [6, Theorem 1].

Remark 1.3. *By taking $s = 0$ and letting $|\alpha| \to \infty$ in (1.6), we get a result due to Govil* [13].

Theorem 1.3. *If $P(z)$ is a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| < k, k \leq 1$, with s-fold zeros at the origin. Then for every real or complex number α with $|\alpha| \leq 1$ and $|z| = 1$*

(1.7)
$$\max_{|z|=1}|D_\alpha P(z)| \leq \left(\frac{n(k+|\alpha|)}{1+k}+\frac{sk(|\alpha|-1)}{1+k}\right)\max_{|z|=1}|P(z)|-(n-s)\frac{1-|\alpha|}{k^{n-1}(1+k)}\min_{|z|=k}|P(z)|.$$

The result is sharp for $s = 0$ and equality holds for the polynomial $P(z) = (z+k)^n$.

Remark 1.4. *A result of Aziz and Shah* [4, Theorem 3] *follows from Theorem 1.3, if we take $s = 0$.*

Corollary 1.3. *For $\alpha = 0$, we get from (1.7),*

$$|nP(z) - zP'(z)| \leq \frac{nk-sk}{1+k}\max_{|z|=1}|P(z)| - (n-s)\frac{1}{k^{n-1}(1+k)}\min_{|z|=k}|P(z)|.$$

If $\max_{|z|=1}|P(z)| = |P(e^{i\phi})|$, then from above inequality, we get the following improvement of a result due to Aziz and Shah [5]

$$\max_{|z|=1}|P'(z)| \geq \frac{n+sk}{1+k}\max_{|z|=1}|P(z)| + \frac{(n-s)}{k^{n-1}(1+k)}\min_{|z|=k}|P(z)|. \tag{1.8}$$

We also prove the following results concerning the growth of polynomials.

Theorem 1.4. *If $P(z) = a_n z^n + \sum_{v=\eta}^{n} a_{n-v}z^{n-v}, 1 \leq \eta < n$ is a polynomial of degree n, having all zeros on $|z| = k, k \leq 1$, then for every positive integer s*

$$\left\{M(P,\rho)\right\}^s \leq \frac{k^{n-2\eta+1}+k^{n-\eta+1}+\rho^{ns}-1}{k^{n-2\eta+1}+k^{n-\eta+1}}\left\{M(P,1)\right\}^s, \quad \rho \geq 1.$$

Remark 1.5. *For $\eta = 1$, we get a result due to Dewan et .al* [9, Theorem 1].

Also if we take $\eta = s = k = 1$, then Theorem 4 reduces to a result due to Ankeny and Rivlin [1].

Theorem 1.5. *If* $P(z) = a_n z^n + \sum\limits_{v=\eta}^{n} a_{n-v} z^{n-v}, 1 \leq \eta < n$ *is a polynomial of degree* n*, having all zeros on* $|z| = k, k \leq 1$*, then for every positive integer* s

$$\big\{M(P,\rho)\big\}^s \leq \frac{1}{k^{n-\eta+1}\big(\eta|a_{n-\eta}|(1+k^{\eta-1}) + n|a_n|k^{\eta-1}(1+k^{\eta+1})\big)}$$
$$\Bigg\{k^{n-\eta+1}\big(\eta|a_{n-\eta}|(1+k^{\eta-1}) + n|a_n|k^{\eta-1}(1+k^{\eta+1})\big)$$
$$+ (\rho^{ns}-1)\big(n|a_n|k^{2\eta} + \eta|a_{n-\eta}|k^{\eta-1}\Bigg)\Bigg\}\big\{M(P,1)\big\}^s,$$

where $\rho \geq 1$.

Remark 1.6. *For* $\eta = 1$*, we get a result due to Dewan et al* [9, Theorem 2].

2. Lemmas

Lemma 2.1. *If* $P(z) = a_n z^n + \sum\limits_{v=\eta}^{n} a_{n-v} z^{n-v}, 1 \leq \eta < n$ *is a polynomial of degree* n*, having all zeros on* $|z| = k, k \leq 1$*, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-2\eta+1} + k^{n-\eta+1}} \max_{|z|=1} |P(z)|.$$

The above Lemma is due to Dewan et al. [11].

Lemma 2.2. *If* $P(z) = a_n z^n + \sum\limits_{v=\eta}^{n} a_{n-v} z^{n-v}, 1 \leq \eta < n$ *is a polynomial of degree* n*, having all zeros on* $|z| = k, k \leq 1$*, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-\eta+1}} \left(\frac{n|a_n|k^{2\eta} + \eta|a_{n-\eta}|k^{\eta-1}}{\eta|a_{n-\eta}|(1+k^{\eta-1}) + n|a_n|k^{\eta-1}(1+k^{\eta+1})}\right) \max_{|z|=1} |P(z)|.$$

Lemma 2.2 is due to Dewan and Hans [10].

We also need the following lemma which is a simple consequence of maximum modulus principle.

Lemma 2.3. *If* $P(z)$ *is a polynomial of degree* n*, then for some* $\rho \geq 1$*, we have*

$$M(P,\rho) \leq \rho^n M(P,1),$$

where $M(P,\rho) = \max\limits_{|z|=\rho} |P(z)|$.

Lemma 2.4. *(see* [6]*). If* $P(z)$ *is a polynomial of degree* n*, such that* $P(z) \neq 0$ *in* $|z| < k, k \leq 1$*, then for every real or complex number* α *with* $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha P(z)| \leq n\left(\frac{k^n + |\alpha|}{1+k^n}\right) \max_{|z|=1} |P(z)|,$$

provided $|P'(z)|$ *and* $|Q'(z)|$ *attain their maxima at the same point on the unit circle, where* $Q(z) = z^n \overline{P\big(\frac{1}{\overline{z}}\big)}$.

Lemma 2.5. *If $P(z)$ is a polynomial of degree n, such that all zeros of $P(z)$ lie in $|z| < k, k \leq 1$, then for every real or complex number α with $|\alpha| \leq 1$*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left\{ \frac{k+|\alpha|}{1+k} \max_{|z|=1} |P(z)| - \frac{1-|\alpha|}{k^{n-1}(1+k)} \min_{|z|=k} |P(z)| \right\}.$$

The above lemma is due to Aziz and Shah [4].

3. Proofs of theorems

Proof of Theorem 1.1. Since $P(z) = z^s \phi(z)$, where $\phi(z)$ is a polynomial of degree $n - s$, which does not vanish in $|z| < k$. Applying inequality (1.4) to the polynomial $\phi(z)$, we get

$$\max_{|z|=1} |D_\alpha \phi(z)| \leq (n-s) \left(\frac{k+|\alpha|}{1+k} \right) \max_{|z|=1} |\phi(z)|. \tag{3.1}$$

Since

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) = nz^s \phi(z) + (\alpha - z)(sz^{s-1}\phi(z) + z^s \phi'(z))$$
$$= z^s D_\alpha \phi(z) + \alpha s z^{s-1} \phi(z),$$

where $D_\alpha \phi(z) = (n-s)\phi(z) + (\alpha - z)\phi'(z)$.
Therefore,

$$zD_\alpha P(z) = z^{s+1} D_\alpha \phi(z) + \alpha s z^s \phi(z).$$

Hence for $|z| = 1$, we have

$$\max_{|z|=1} |D_\alpha P(z)| \leq \max_{|z|=1} |D_\alpha \phi(z)| + |\alpha| s \max_{|z|=1} |\phi(z)|.$$

Using $\max_{|z|=1} |\phi(z)| = \max_{|z|=1} |P(z)|$, we get

$$\max_{|z|=1} |D_\alpha P(z)| \leq \max_{|z|=1} |D_\alpha \phi(z)| + |\alpha| s \max_{|z|=1} |P(z)|. \tag{3.2}$$

Considering (3.1) in (3.2), we obtain

$$\max_{|z|=1} |D_\alpha P(z)| \leq (n-s) \left(\frac{k+|\alpha|}{1+k} \right) \max_{|z|=1} |P(z)| + |\alpha| s \max_{|z|=1} |P(z)|.$$

Equivalently

$$\max_{|z|=1} |D_\alpha P(z)| \leq \left(\frac{n(|\alpha| + k)}{1+k} + \frac{sk(|\alpha| - 1)}{1+k} \right) \max_{|z|=1} |P(z)|.$$

This completely proves Theorem 1.1. □

Proof of Theorem 1.2. Since $P(z) = z^s \phi(z)$. On applying Lemma 2.4 to $\phi(z)$, the proof follows similarly as that of Theorem 1.1. □

Proof of Theorem 1.3. Since $P(z)$ is a polynomial with a zero of multiplicity s at origin, therefore, we write it as $P(z) = z^s\phi(z)$, where $\phi(z)$ is a polynomial of degree $n-s$. Applying Lemma 2.5 to $\phi(z)$, we get

$$\max_{|z|=1}|D_\alpha\phi(z)| \le (n-s)\left\{\frac{k+|\alpha|}{1+k}\max_{|z|=1}|\phi(z)| - \frac{1-|\alpha|}{k^{n-s-1}(1+k)}\min_{|z|=k}|\phi(z)|\right\}. \tag{3.3}$$

Using (3.3) in (3.2), we get

(3.4)

$$\max_{|z|=1}|D_\alpha P(z)| \le (n-s)\left\{\frac{k+|\alpha|}{1+k}\max_{|z|=1}|\phi(z)| - \frac{1-|\alpha|}{k^{n-s-1}(1+k)}\min_{|z|=k}|\phi(z)|\right\} + |\alpha| s\max_{|z|=1}|P(z)|.$$

Since $\max_{|z|=1}|P(z)| = \max_{|z|=1}|\phi(z)|$ and $\min_{|z|=k}|\phi(z)| = \frac{1}{k^s}\min_{|z|=k}|P(z)|$. we have from (3.4)

$$\max_{|z|=1}|D_\alpha P(z)| \le \left(\frac{n(k+|\alpha|)}{1+k} + \frac{sk(|\alpha|-1)}{1+k}\right)\max_{|z|=1}|P(z)| - (n-s)\frac{1-|\alpha|}{k^{n-1}(1+k)}\min_{|z|=k}|P(z)|.$$

This completely proves Theorem 1.3. □

Proof of Theorem 1.4. If we write $\max_{|z|=1}|P(z)| = M(P,1)$, where $P(z) = a_n z^n + \sum_{v=\eta}^{n} a_{n-v}z^{n-v}, 1 \le \eta < n$ is a polynomial of degree n having all zeros on $|z| = k, k \le 1$, then by Lemma 2.1

$$|P'(z)| \le \frac{n}{k^{n-2\eta+1} + k^{n-\eta+1}}M(P,1), \quad \text{for} \quad |z| = 1. \tag{3.5}$$

Since $P'(z)$ is a polynomial of degree $n-1$, it follows from (3.5) by an application of maximum modulus principle that for $r \ge 1$ and $0 \le \phi < 2\pi$,

$$|P'(re^{i\phi})| \le \frac{nr^{n-1}}{k^{n-2\eta+1} + k^{n-\eta+1}}M(P,1). \tag{3.6}$$

Hence for some $\rho \ge 1$ and for each $\phi, 0 \le \phi < 2\pi$

$$\begin{aligned}\big\{P(\rho e^{i\phi})\big\}^s - \big\{P(e^{i\phi})\big\}^s &= \int_1^\rho \frac{d}{du}\big\{P(ue^{i\phi})\big\}^s du \\ &= \int_1^\rho s\big\{P(ue^{i\phi})\big\}^{s-1}P'(ue^{i\phi})e^{i\phi}du.\end{aligned}$$

This implies

$$\big|\big\{P(\rho e^{i\phi})\big\}^s - \big\{P(e^{i\phi})\big\}^s\big| \le s\int_1^\rho \big|P(ue^{i\phi})\big|^{s-1}\big|P'(ue^{i\phi})\big|du.$$

Using (3.5) and Lemma 2.3, we get

$$\big|\big\{P(\rho e^{i\phi})\big\}^s - \big\{P(e^{i\phi})\big\}^s\big| \le \frac{ns}{k^{n-2\eta+1} + k^{n-\eta+1}}\big\{M(P,1)\big\}^s\int_1^\rho u^{ns-1}du.$$

Equivalently,

$$\left|\left\{P(\rho e^{i\phi})\right\}^s\right| \leq \left|\left\{P(e^{i\phi})\right\}^s\right| + \frac{\rho^{ns}-1}{k^{n-2\eta+1}+k^{n-\eta+1}}\left\{M(P,1)\right\}^s$$
$$\leq \left\{M(P,1)\right\}^s + \frac{\rho^{ns}-1}{k^{n-2\eta+1}+k^{n-\eta+1}}\left\{M(P,1)\right\}^s.$$

This in particular gives,

$$\left\{M(P,\rho)\right\}^s \leq \frac{k^{n-2\eta+1}+k^{n-\eta+1}+\rho^{ns}-1}{k^{n-2\eta+1}+k^{n-\eta+1}}\left\{M(P,1)\right\}^s,$$

where $M(P,\rho)=\max\limits_{|z|=\rho}|P(z)|$. This completely proves Theorem 1.4. $\square$

Proof of Theorem 1.5. The proof of Theorem 1.5 follows on the same lines as that of Theorem 1.4 by using Lemma 2.2 instead of Lemma 2.1. $\square$

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