*Armenian Journal of Physics, 2023, vol. 16, issue 2, pp. 72-82*

# Generic Technique for Explicit Definition of Models with SPT Phases

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(Received: July 20, 2023; Revised: August 1, 2023; Accepted: August 10, 2023)

**Abstract:** This paper focuses on creation of a technique to derive systems with explicitly defined symmetry protected topological (SPT) phases on a triangular lattice, based on models featuring arbitrary symmetries. It results in emergence of effective edge models, which are massless in non-trivial SPT phases. When applied on non-interacting systems, the technique results in models with a translational invariance, which ensures independence from the local geometry.

**Keywords:** SPT phases, triangular lattice, induced edge model

DOI:10.54503/18291171-2023.16.2-72

## 1. Introduction

Symmetry protected topological (SPT) phases [1–3] have recently gained a huge research attention [4–10] both in terms of conceptual formulation and specific model studies. It is a relatively new concept for phase transitions strongly related to topological properties of the system, that is significantly different from classical Landau approach to the topic.

Systems with STP order are remotely similar to models described by Landau theory of phase transitions. They both rely on explicitly or spontaneous breaking of the system's symmetries as the inducer of different phases and are described by structure and properties of the symmetry groups. SPT ordered systems are also known [6, 11] to be short range entangled. However, SPT phases have topological origin, and the corresponding states are manifested on the edge of the system, similar to topologically ordered states. But the later are long-range entangled [12–17] unlike SPT.

An important feature of SPT order is its support for symmetry-protected gapless boundary phases, meaning it is able to be a topological insulator or a trivial insulator in different phases. This behavior is fundamental for topological quantum computation. Other remarkable properties of SPT ordered systems are the non-standard excitation algebra that emerges for edge states, sensitivity of the system to symmetry-breaking perturbations and others, that might later prove useful in some applications.

In the meantime, a significant amount of research has been done on this topic. That revealed the explicit connection between the symmetry classes and the SPT phases, which happens to be [6–10] a representation of the third group cohomology of the symmetry. This allows classification [1–3] and deeper understanding of SPT models and their variety. However, none of these models provide precise pathway of formulating an SPT system model or modifying a known model to SPT phase capable.

An outstanding example of explicitly written SPT model is presented in a paper [4], which is based on Ising paramagnet with $Z_2$ symmetry. The result is a model with two phases, one being a conventional insulator (with gapped spectrum), and the second being a topological insulator (with gapless spectrum created by edge states). Other such known models are [5, 18], which are basically extensions of [4] for $Z_3$ and $Z_3 \times Z_3$ symmetries. This paper is inspired by those works, and is trying to apply the ideas proposed there to formulate a similar model that is based on a more

complex system (namely the ferromagnetic phase of $Z_2$ quantum Ising model), that initially contains interaction, meanwhile developing a universal technique for doing so.

According to the conventional definition, SPT models have the following definitive properties. The system has to have a global symmetry $S$ which is not spontaneously broken in any of the phases. There is the so-called "trivial" phase which, generally speaking, has a gapped spectrum, and frequently has the simplest form of the Hamiltonian (it is usually possible to write the ground state as a direct product of different subsystem states). The other phases are separated from the trivial one and from each other. The phase separation is implemented as impossibility of continuously connecting (with any parametrization) states of different phases without breaking at least one symmetry. So, the phases are protected by symmetry. The non-trivial phases of SPT models tend to have gapless edge spectrum. In fact, this might be the only objective way of distinguishing between the trivial and non-trivial phases.

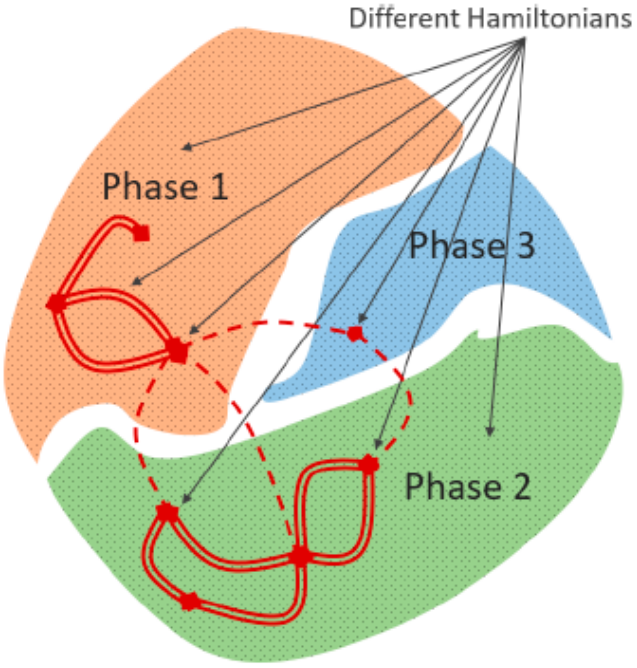



**Fig. 1.** Schematic representation of supposed SPT phase space. Double lines are continuous connections of Hamiltonians, and dashed lines are noncontinuous connections.

A phase space is generally defined as a set of Hamiltonians which describe the state of the system in different conditions. In Ginsburg-Landau theory those Hamiltonians differ by some parameters (for example a parameter describing an external magnetic field or a coupling constant), called critical parameters. There is also a critical point: a value of the parameter where the phase transition occurs.

In contrary, the phase space of SPT models can not be parametrized. For SPT models, the phase space is a set of Hamiltonians with some common symmetry $S$ in the same representation for all Hamiltonians. Notice, that $S$ is a symmetry in general, and might have multiple generators in given representation. Then a phase is defined as a "region" of those Hamiltonians, that can be continuously connected within the set (without breaking the symmetry), and two Hamiltonians, that don't have such a connection are said to be in different phases. By the very definition it is demanded that there is no such thing as "critical point" in this theory, as existence of one would mean continuity of Hamiltonian transformation inside the set of phase space's Hamiltonians, which are all symmetric. So, the "regions" of different phases don't "touch", as portrayed in Fig. 1.

The concept of a phase suggests that the Hamiltonians in the same phase should have identical spectra. In other words, they should be unitarily equivalent. And obviously, the unitary transformations connecting those Hamiltonians should also be symmetric under $S$. SPT principles suggest, that there should be no long-range interactions in the system, so the transformations should be local. Summing up, there is a set of continuously symmetric local unitary transformations that define a single phase.

## 2. Mathematical apparatus

The different phases occur as a result of the topology of the system: there would be no manifestation of different phases if the system didn't have an edge. In that case all the Hamiltonians would be equivalent, and all of them would be connected through an even larger set of continuously symmetric local unitary transformations. Yet some of them are no longer applicable for edged systems, as their symmetry might be broken. This leads to assumption, that transitions between different phases are done via transformations which are related to the unitary transformations whose symmetry is broken on edge of the system. Our approach is to explicitly restore the symmetry of those transformations, on the expense of their unitarity, and thus generate the Hamiltonians for non-trivial phases, as it is done in [4, 5, 18].

The SPT phases are known [1–3] to be classified by cohomologies of their symmetry group, and the concept of group cohomologies is heavily used during the construction of a model with SPT phases [6–10].

### 2.1. Group cohomologies

Group cohomologies are defined on functions that map multiple parameters of a specified group $G$ to some other group $F$, and they are symmetric under the generators of $G$ [19] . In other words

$$\nu_k: G^{k+1} \to F, \qquad \nu_k(g_0, g_1, \dots g_k) = \nu_k(0, -g_0 + g_1, \dots, -g_0 + g_k) \tag{1}$$

Here $g_i$ are any elements of group $G$ in additive representation. We will call $\nu_k$ a $k$-cochain or a $k$-form. Let's denote the space of all cochains by $C_k$.

We can define a so-called coboundary operator on cochains as

$$\delta\nu_{k-1}(g_0, g_1, \dots, g_k) = \prod_{i=0}^{k} \nu_{k-1}(g_0, g_1, \dots, \check{g}_i, \dots, g_k)^{(-1)^i} \tag{2}$$

where a "check" on the argument means that the argument is dropped. For example $\delta\nu_0(g_0, g_1) = \frac{\nu_0(g_1)}{\nu_0(g_0)} = 1$ because of the symmetric properties of cochains, or $\delta\nu_1(g_0, g_1, g_2) = \frac{\nu_1(g_1, g_2)\nu_1(g_0, g_1)}{\nu_1(g_0, g_2)}$. As you can see, the coboundary operation maps $C_k$ to $C_{n+1}$ as the number of arguments has increased, but the symmetry is not broken. The functions that can be generated with $\delta$ acting on $\nu_k$ are called coboundaries or exact forms (we will see why in a minute).

It can be easily shown, that alike the geometric boundary operator, $\delta^2$ is trivial. Indeed, each term of the resulting function would be missing two arguments, namely $i$ and $j$. Moreover, there will be two terms with same missing $(i, j)$, and depending on the order of their removal they will have opposite exponents. For term, where the greater index was removed first it will be $(-1)^{i+j}$, in other case it is $(-1)^{i+j-1}$, as when removing the argument with the grater index, it's index is reduced by 1.

And of course, there is the set of cochains, for which $\delta\nu_k = 1$. Those are called cocycles or closed forms. It is obvious, that all coboundaries are cocycles. The structure of action of $\delta$ on $C_k$ spaces is schematically shown on Fig. 2.

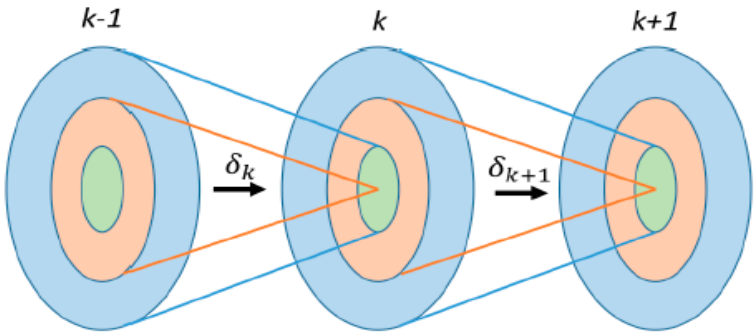



**Fig. 2.** Mapping by $\delta$ between $C_k$-s. Each set of invested disks represent a cochain space with the cochain index specified above. The green disks are coboundaries (exact forms), the reds are cocycles (closed forms), and the blues are cochains (all forms).

It is known, that $k$-cocycle set can be factorized by $k$-coboundary set. It will be easier to show in the physical implementation, so we skip a proof here. This Factor space is called $k$-th cohomology of $G \to F$ mapping and is denoted as

$$H^k(G,F) = \frac{k\text{-cocycles}}{k\text{-coboundaries}} = \frac{\mathrm{Ker}(\delta_{k+1})}{\mathrm{Im}(\delta_k)} \tag{3}$$

While studying this kind of functions it is useful to introduce functions

$$\begin{aligned} \omega_k(g_1,\dots,g_n) &= \nu_k(0, g_1, g_1+g_2,\dots, g_1+g_2+\dots+g_k) \\ \nu_k(g_0,g_1,\dots,g_n) &= \omega_k(-g_0+g_1, -g_1+g_2,\dots,-g_{k-1}+g_k) \end{aligned} \tag{4}$$

The $\omega_n$ have one less argument and no additional symmetry condition. One-to-one mapping between $\omega_n$ and $\nu_n$ (the whole symmetric equivalence class) is then guarantied. The downside of this is that the action of the coboundary operator becomes complicated. With straightforward calculations one can get

$$\delta\omega_k(g_0,g_1,\dots,g_k) = \omega_k(g_1,\dots,g_k)\cdot\prod_{i=1}^{k}\omega_k(g_0,g_1,\dots,g_{i-2},\ g_{i-1}+g_i,g_{i+1},\dots,g_k)^{(-1)^i}\cdot\omega_k(g_0,\dots,g_{k-1})^{(-1)^{k+1}} \tag{5}$$

The precise expressions for smaller values of $k$ are

$$\begin{aligned} \delta\omega_0(g_0) &= 1 \\ \delta\omega_1(g_0,g_1) &= \frac{\omega_1(g_1)\omega_1(g_0)}{\omega_1(g_0+g_1)} \\ \delta\omega_2(g_0,g_1,g_2) &= \frac{\omega_2(g_1,g_2)\omega_2(g_0,g_1+g_2)}{\omega_2(g_0+g_1,g_2)\omega_2(g_0,g_1)} \\ \delta\omega_3(g_0,g_1,g_2,g_3) &= \frac{\omega_3(g_1,g_2,g_3)\omega_3(g_0,g_1+g_2,g_3)\omega_3(g_0,g_1,g_2)}{\omega_3(g_0+g_1,g_2,g_3)\omega_3(g_0,g_1,g_2+g_3)} \end{aligned} \tag{6}$$

## 2.2. Unitary transformations

SPT phases protected by symmetry group $S$ are known [8-10] to be described by $H^{d+1}\big(N, U(1)\big)$. Here we will explicitly show how the cohomology is involved in the Hamiltonian construction, to provide intuitive understanding od the situation.

We will be working on a two-dimensional triangular lattice (this technique also works for $d$-dimensional lattices, such as a line in case $d = 1$, a tetrahedron lattice for $d = 3$, etc. Some insights about the differences will be provided alongside). Each node's state is given by an element of additive representation $N$ of group $S$.

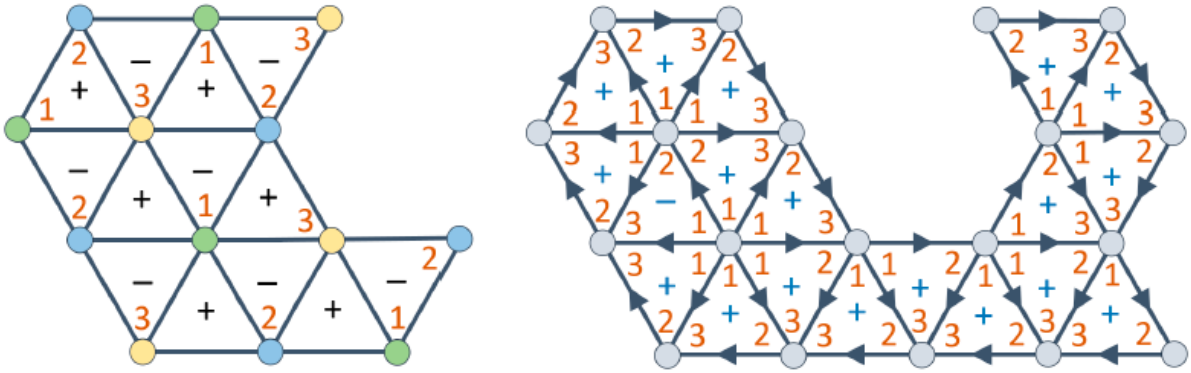
Suppose we have our trivial Hamiltonian, that is just a sum of some commutative elements over the lattice. This Hamiltonian will be symmetric under any $S_0$, that is a product of an arbitrary operator $s$ from $S$ over all the lattice. Now we need to look for the $S_0$-symmetric local unitary transformations to construct the whole phase space. Let's try to write $U$ as a product of $\big(S \to U(1)\big)$ 3-forms (or $d+1$-forms for $d$-dimensional case) over triangles as

$$U = \Pi_\Delta U_\Lambda^{\epsilon_\Delta} = \Pi_\Delta \nu_3(0, n_1^\Delta, n_2^\Delta, n_3^\Delta)^{\epsilon_\Delta} \tag{7}$$

where $n_i^\Delta$ are states of particles on the triangle and $\epsilon_\Delta = \pm 1$ indicates the orientation of triangle based on the direction (clockwise or counterclockwise) of indexing.

We will discuss two of the ways of indexing a lattice. The first one is fairly simple. You just divide the lattice into three larger sub-lattices, usually denoted with different colors, and all the nodes of that sub-lattice have the same index, as shown on Fig. 3a.

In the other approach you first draw arrows on each link of the lattice (Fig. 3b) [7, 8]. The only restriction is that there should be no arrows making a cycle inside one triangle (or other structure block of the lattice, like tetrahedron in $d = 3$, etc.). Then the indexing is done for each triangle (or other structure block) independently, in a way that each arrow points from the smaller index to the larger. This is one-to-one mapping between arrows and indexing. Note, that the same node might have different indices in different triangles.



(a) Color-based indexing. (b) Arrows-based indexing.

**Fig. 3.** Indices of points and orientations of triangles for different ways of indexing the lattice.

Generally speaking, these transformations are not symmetric, but the situation changes if we demand for $\nu_3$ to be cocycles. This condition is written as

$$\delta\nu_3(n, n', n_1, n_2, n_3) = \frac{\nu_3(n', n_1, n_2, n_3)\nu_3(n, n', n_2, n_3)\nu_3(n, n', n_1, n_2)}{\nu_3(n, n_1, n_2, n_3)\nu_3(n, n', n_1, n_3)} = 1 \tag{8}$$

this can be rewritten as

$$\frac{\nu_3(n,n_1,n_2,n_3)}{\nu_3(n',n_1,n_2,n_3)}=\frac{\nu_3(n,n',n_1,n_2)\nu_3(n,n',n_2,n_3)}{\nu_3(n,n',n_1,n_3)} \tag{9}$$

Using this, the symmetry of $\nu$, and denoting $sns^\dagger = n_s + n$, we will get

$$S_0 U_\Delta S_0^\dagger = \nu_3(-n_s,n_1,n_2,n_3) = U_\Delta \frac{\nu_3(0,-n_s,n_1,n_2)\nu_3(0,-n_s,n_2,n_3)}{\nu_3(0,-n_s,n_1,n_3)} \tag{10}$$

So, $U_\Delta$ is not symmetric under $S$ by itself, but their product over all the triangles is. Indeed, in case we do our indexation the color-based way (Fig. 3a) a term $\nu_3(0,-n_s,n_i,n_j)$ will come from two neighboring triangles that have the $(i,j)$ link in common. As those triangles always have opposite orientations, the term appears both in the numerator and the denominator, and thus vanishes.

The case of arrow-based indexing (Fig. 3b) we also consider a link $IJ$ and the two triangles that contain it. Suppose the arrow points from $I$ to $J$. The possible index pairs $(i,j)$ of $I$ and $J$ are then $(1,2)$, $(2,3)$ and $(1,3)$ in both triangles (but not simultaneously). The triangle left to $I \to J$ will have orientations $\langle - \rangle$, $\langle - \rangle$ and $\langle + \rangle$ correspondingly, if we denote clockwise by $\langle + \rangle$. For the triangle on the right it will be $\langle + \rangle$, $\langle + \rangle$, $\langle - \rangle$. It is worth mentioning once again, that this doesn't mean always having opposite orientations in neighboring triangles as the index pair can be different in those triangles. One may notice, that for the right triangle the term $\nu_3(0,-n_s,n_i,n_j)$ will always end up in the numerator, and similarly, for the left triangle it's always in the denominator. Thus, those terms vanish.

In both cases we didn't take into account the links that are on the edge of the system thus don't have their counterparts from a neighboring triangle, as the symmetry is broken on the edge and this is exactly what we need. This kind of transformations are the ones that will be generating our Hamiltonians.

Notice, that in case of color-based indexing we are free to modify $\nu_3(0,n_1,n_2,n_3)$ by adding some terms to it that depend only on two of the $n$-s as they vanish immediately. Let's call those terms $\nu_{3(2)}$, indicating that formally they depend on three elements, but factually on two. So, we say the set of local symmetric $U$-s is given by {3-cocycles}×{non-cocycle $\nu_{3(2)}$-s}. Also notice, that in case of addition of such terms the action of symmetry on them wll no longer be given by (10) and should be calculated explicitly.

The feature that we get from the arrow-based indexing is that by choosing the arrow configuration we can make all the links on the edge have the same direction along the traversal, make all the edge links to have index pairs $(1,2)$ or $(2,3)$ and consequently all the edge triangles will have the same orientation as shown in Fig. 3b. In this case the source of local symmetric $U$ set is just {3-cocycles}. In case there are no thin regions (no edge nodes are connected via non edge link) we can even make all edge links to have the same index pair $(1,2)$ or $(2,3)$.

When we say, that a Hamiltonian $H_1$ is continuously connected to $H_2$ without breaking symmetry (let's call these Hamiltonians and the corresponding $U$-s equivalent), this means, that there is a set of symmetric $U^{(\alpha)}, \alpha \in [0,1]$ continuous on $\alpha$ with $U^{(0)} = 1$ and $U^{(1)} H_1 U^{(1)\dagger} = H_2$. The $U$-s that we have so far can not be parametrized that way.

The solution is to take $\nu_3$-s to be exact forms

$$\nu_3(0,n_1,n_2,n_3) = \delta\nu_2(0,n_1,n_2,n_3) = \nu_2(n_1,n_2,n_3)\frac{\nu_2(0,n_1,n_3)}{\nu_2(0,n_1,n_2)\nu_2(0,n_2,n_3)} \tag{11}$$

This has a structure similar to what we have seen in (9). Using a similar logic one can state, that only terms $\nu_2(n_1,n_2,n_3)$ will remain after in $U$. The major advantage over closed forms is that

$\nu_2$ has an intrinsic symmetry on $s$, which means that any $\nu_2(n_1, n_2, n_3)^\alpha$ is also symmetric under $s$. In their turn, $\nu_2^\alpha$ generate a continuous set of symmetric $U^\alpha$, where $U^0 = 1$ and $U^1 = U$.

So, the set of $U$-s equivalent to $1$ (trivial $U$-s) is given by {3-coboundaries}. In a similar way to what was suggested for just symmetric $U$-s, in case of color-based indexing we get an additional $\times$ {non-cocycle $\nu_{3(2)}$-s} here as well. Any two $U$-s that differ by a trivial $U$ are also obviously equivalent. So, the same-phase space is defined by trivial $U$-s, and the phases are

$$\{\text{different phases group}\} = \frac{\text{symmetric } U\text{-s}}{\text{trivial } U\text{-s}} = \frac{\text{3-cocycles}}{\text{3-coboundaries}} = H^3\big(N, U(1)\big) \qquad (12)$$

In case of $d$ dimensions it would have been $H^{d+1}\big(N, U(1)\big)$.

Derivation and usage of cohomology groups can be found in [8–10, 19, 20].

### 2.3. Finding a proper $\nu_3$

Now we need to find a closed but not exact 3-form which will be a source for a Hamiltonian from non-trivial phase. To this end we have to determine a basis for all possible functions. As it has been mentioned before, it's the best to look for specific functions in $\omega$-representation as there are no additional symmetry conditions on them. Also, it would be helpful if $\omega$ takes the familiar integers as arguments instead of $S$-elements (of course they should be a representation of S).

If group $S$ has $r$ generators $g_{\alpha_1}, g_{\alpha_2}, \dots, g_{\alpha_r}$ and $g_{\alpha_i}^{N_i} = 1$, then any element of $S$ can be given as $s = g_{\alpha_1}^{n_{\alpha_1}} g_{\alpha_2}^{n_{\alpha_2}} \dots g_{\alpha_r}^{n_{\alpha_r}}$, where $n_\alpha$-s are some integers. If we define addition for $n_{\alpha_i}$ to be by $\bmod N_i$, then $\{n_\alpha\}$ (or $\vec{n}$) becomes an additive representation of $S$ in integer numbers. Keep in mind, that $\vec{n}_1 + \vec{n}_2$ might be not a component-wise addition as there can be non-commutative generators in $S$, which should be taken into account. For example if we suppose the permutation group $S_3$ and take $g_{\alpha_1}$ as the generator of rotation and $g_{\alpha_2}$ as the generator of reflection, then for the vectors of additive representation we will have $(n_1, 0) + (n_2, n') = (n_1 + n_2, n')$ but $(n_1, 1) + (n_2, n') = (n_1 - n_2, 1 + n')$.

The $\omega$ functions basis (by saying basis we mean that any $\omega$ can be given as a product of these functions) can be given in a way similar to simple Taylor series expansion. The basic monomial functions are

$$\omega_K^{(c)}(\vec{n}_1, \vec{n}_2, \dots, \vec{n}_K) = \exp\frac{2\pi i}{N_c}\psi_K^{(c)}(\vec{n}\dots) \qquad \psi_K^{(c)}(\vec{n}\dots) = n_{1,\alpha_c}^{b_1} n_{2,\alpha_c}^{b_2} \dots n_{K,\alpha_c}^{b_K} \qquad (13)$$

where $c$ indicates the component of vectors and $b_k$ are some non-negative integers. As the $n$-s here are integers with no restrictions, the factor $2\pi i/N_c$ is there to ensure the equivalence of $n_{k,\alpha_c}$ and $n_{k,\alpha_c} + N_c$. There is only a finite number of these functions, as for any $N_c$, there is some value of $b \le N_c$, starting from which $n^b \equiv n^{b-p_c} (\bmod N_c)$, where $p_c > 0$. It is known as a simple consequence of Fermat-Euler theorem. For example, $n^4 \equiv n^2 (\bmod 4)$, $n^7 \equiv n (\bmod 7)$, $n^5 \equiv n^3 (\bmod 24)$, etc. The terms of higher order are no new terms.

Then the next step is to calculate the basis for all exact 3-forms. This is done by simply taking the whole basis of $\omega_2$ and applying $\delta$ as shown in (6) to it. The produced independent combinations as a whole are our basis.

Afterwards we need the basis for the closed 3-forms. Similarly, here $\delta$ should be applied to the whole basis of $\omega_3$ and the combinations that produce $1$ will be the basis. The part of this basis that is independent of exact 3-form basis is the basis for the factor space of 3-cocycles and 3-coboundaries. Though finding the cohomology is a mathematical problem and we just need one

non-exact $\omega_3$ to produce a non-trivial Hamiltonian, there will be no need in finding the whole basis, but rather a single element in it.

For simplicity of writing, we will be using $\psi$-s instead of $\omega$-s. Notice that they are defined right up to $\mathrm{mod}\, N_c$.

## 3. Construction of a Hamiltonian

Once we have a non-trivial $\nu_3$ and the corresponding $U$ it needs to be applied to our trivial Hamiltonian $H_0$, which is composed of local terms $t$ symmetric under $S$. The initial transformed Hamiltonian would then be written as

$$H_i = \sum_t UtU^\dagger \tag{14}$$

where $t$ runs through all the terms (there might be single-node terms, two-node terms defined on the links, etc.).

From the previous section we know, that this transformed Hamiltonian is not symmetric under $S$, particularly on the edge. In order to restore that symmetry, we can sum up all the possible symmetry transformations.

$$H = \frac{1}{\#S}\sum_{s\in S}\sum_t sUtU^\dagger s^\dagger \tag{15}$$

with $s$ taking all possible values from symmetry group $S$ and $\#S$ being the number of elements in it. Now this Hamiltonian is explicitly symmetric. What is left to do is to separate the edge part.

As already mentioned, $U$ is not symmetric on the edge of the system, and produces residual edge terms $\nu_3(0,-n_s,n_i,n_j)^{\pm 1}$ under action of symmetry $s$, as shown in (10) and the later interpretation. Here $n_s = sns^\dagger - n$ and $n_i$ and $n_j$ are correspond to nodes of links on the edge.

For any term $t$ given on node set $p_t$, all the $U_\Delta$-s that make up the $U$ will be commutative to it except the ones that have a common node with $p_t$. Let's call this "non-commutative with $t$" part of $U$ a $U_{p_t}$. The residual terms $R_{s,p_t}$ produced of action of $s$ on $U_{p_t}$ will be defined on the links on border of triangles that are contained in $U_{p_t}$. Only part of $R_{s,p_t}$, that is defined on links which contain a node form $p_t$ (we will call it $V_{s,p_t}$) is not commutative with $t$. Those will be system's edge links that contain a node from $p_t$. All of this is illustrated on Fig. 4. In mathematical formulation

$$\begin{aligned} sUtU^\dagger s^\dagger &= sU_{p_t}tU_{p_t}^\dagger s^\dagger = \\ = U_{p_t}R_{s,p_t}sts^\dagger R_{s,p_t}^\dagger U_{p_t}^\dagger &= U_{p_t}R_{s,p_t}tR_{s,p_t}^\dagger U_{p_t}^\dagger = \\ = U_{p_t}V_{s,p_t}tV_{s,p_t}^\dagger U_{p_t}^\dagger &= V_{s,p_t}\bar{t}V_{s,p_t}^\dagger \end{aligned} \tag{16}$$

where $\bar{t} = UtU^\dagger$. Notice that we have done noting but canceling out terms. So, the $V_{s,p_t}$ terms are exactly the ones shown in (10). Also, as it is clearly seen in Fig. 4, $V_{s,p_t}$ will only have terms on links $\langle qr\rangle$ on the edge $\partial$ of the system, that contain a node from $p_t$. So, the Hamiltonian can be written as

$$H = \frac{1}{\#S}\sum_t\sum_{s\in S} V_{s,p_t}\bar{t}V_{s,p_t}^\dagger \qquad V_{s,p_t} = \prod_{\substack{\langle qr\rangle\in\partial \\ i_q<i_r \\ q \text{ or } r\in p_t}} \nu_3\left(0,-n_s,n_q,n_r\right)^{-\epsilon_{qr}(-1)^{i_q+i_r}} \tag{17}$$

where $\epsilon_{qr}$ is the orientation of the triangle containing those points, $i_q$ and $i_r$ are the corresponding indices in that triangle, and the term $-(-1)^{i_q+i_r}$ in the exponent indicates the initial position (numerator vs denominator) of the $\nu_3$ in (10).

It is obvious, that $V_{s,p_t}$ is $1$ for the $t$ that are fully emerged in the bulk, which reduces the Hamiltonian to

$$H = H_B + H_E \qquad H_B = \sum_{t\in\partial^*} \bar{t} \qquad H_E = \frac{1}{\#S}\sum_{t\in\partial}\sum_{s\in S} V_{s,p_t}\,\bar{t}V_{s,p_t}^{\dagger} \tag{18}$$

with $\partial^*$ denoting the bulk. One can notice that each bulk term is commutative with any bulk or edge term, however the edge terms are not commutative to each other in general. So, we got a non-trivial edge model that is separated from the bulk.

We might also want to have translational symmetry on the edge, i.e. ability to write all the components of $V_{s,p_t}$ in (17) without dependence in the point indices. Let's take a node $p$ and its next node along the traversal $p+1$. The $q$ in (17) is the one with the smaller index of the two. In case of arrow-based indexing there is nothing further to be done as we will always have $q = p$, $r = p+1$ and $-\epsilon_{qr}(-1)^{i_q+i_r} = 1$.

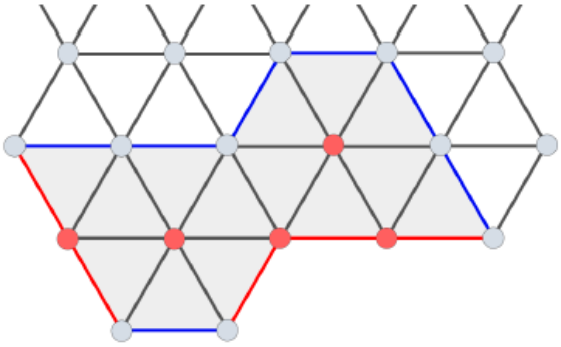


**Fig. 4.** The residual terms generated by action of *s* on $UtU^{\dagger}$. (In reality one is hardly going to have such a *t*, but it's easier to see the principle of *s*'s action here). Nodes in $p_t$ are marked red. Shaded triangles are ones with $[U_{\Delta}, t] \neq 0$ ($U_{p_t}$ triangles). The produced residual terms are on the links marked red or blue ($R_{s,t}$ links), and residual terms on blue links are commutative with *t*. So only the terms on red links remain ($V_{s,t}$ links).

In case of color-based indexing $\epsilon_{qr}$ is $1$ if $(i_p, i_{p+1}) \in \{(1,2),(2,3),(3,1)\}$ and is $-1$ otherwise. $(-1)^{i_q+i_r} = 1$ if $(i_p, i_{p+1}) \in \{(1,3),(3,1)\}$ and is $-1$ otherwise. So overall exponent is $1$ if $i_p < i_{p+1}$ and is $-1$ otherwise (if $i_p > i_{p+1}$). Our objective here is to make the expression under the product "antisymmetric" to permutation of the last two arguments as $f(0,-n_s,n_a,n_b) = f(0,-n_s,n_b,n_a)^{-1}$. This way the additional condition $i_q < i_r$ under the product can be removed, because the $-1$ from the exponent and the $-1$ from anti-symmetry will cancel each other. Here the freedom to add $\nu_{3(2)}$-s proves helpful. The addition of a $\nu_{3(2)}$ to the $U_{\Delta}$ adds a factor $s\nu_{3(2)}s^{\dagger}/\nu_{3(2)}$ to the right side of (10). In order not to break the existing symmetry for $i_p < i_{p+1}$ case, three terms $\nu_{3(2)}^{(1)}(0,n_1,n_2,n_3) = f_2(n_1,n_2)$, $\nu_{3(2)}^{(2)}(0,n_1,n_2,n_3) = f_2(n_2,n_3)$ and $\nu_{3(2)}^{(3)}(0,n_1,n_2,n_3) = f_2(n_1,n_3)^{-1}$ should be added at a time. This way new factors $f_2(n_s+n_i,n_s+n_j)f_2(n_i,n_j)^{-1}$ appear next to corresponding $\nu_3(0,-n_s,n_i,n_j)$. We can interpret it as modifying $\nu_3$ like

$$\nu_3(0,-n_s,n_a,n_b) \to \nu_3(0,-n_s,n_a,n_b)\frac{f_2(n_s+n_a,n_s+n_b)}{f_2(n_a,n_b)} \tag{19}$$

which has a chance to be anti-symmetrized. It should be possible, as set of $f_2$-s has a basis of size $(\#S)^2$, and there are only $\sim(\#S)^3/2$ conditions to be satisfied. Later we will mean the anti-symmetrized version when we refer to $\nu_3$, which will no longer have an intrinsic symmetry.

Afterwards for both indexing techniques $V_{s,p_t}$ can be written as

$$V_{s,p_t} = \prod_{(p,p+1)\cap p_t} \nu_3\left(0,-n_s,n_p,n_{p+1}\right) \tag{20}$$

The product runs through all edge terms that contain any point of $p_t$.

After choosing $\nu_3$, we are now able to explicitly write the expression for an edge Hamiltonian from a non-trivial SPT phase.

Summing up the information above, the steps for creating the model are:

1. Choose a symmetry.
2. Fix the multiplicative and corresponding additive representations of the symmetry.
3. Find a non-exact 3-cocycle (and symmetrize it if needed).
4. Write a symmetric trivial Hamiltonian (under the lattice product symmetry).
5. Calculate the transformed edge elements.

### 3.1. Specific example

The simplest application of this algorithm can be seen on the simplest symmetry group $Z_2$ with only generator $g_{\alpha_1}$, $g_{\alpha_1}^2 = 1$. We will be using a Hamiltonian that is symmetric under $Z_2$ in a representation where $g_{\alpha_1}$ is the $\sigma^x$ Pauli matrix. Then the corresponding additive representation is $n \in \{0,1\}$, $n = (\sigma^z+1)/2$. Mathematically we know that $H^3\big(Z_2,U(1)\big) = Z_2$, so there should be an SPT phase here.

We can first find the basis for exact forms by applying $\delta$ from (6) on each 2-form represented as in (13). It can be done in a straightforward way, using that $n^2 = n$, and the fact that $\psi$ are defined right up to $\mathbf{mod}\ 2$. The same can be done for $\delta$'s applications on $\psi_3$ to determine all the cocycles. It becomes clear, that the only non-exact 3-cocycle is given by $\psi_3(n_1,n_2,n_3) = n_1n_2n_3$ which corresponds to $\nu_3(0,n_1,n_2,n_3) = -1^{n_1n_2n_3-n_1n_3}$.

For the case of color-based indexing we want it to be antisymmetric on permutation of the last two arguments. It is to be done as mentioned in (19). $f_2$-s that depend on single argument change nothing in terms of symmetry, and the only other basic $f_2(n_1,n_2) = n_1n_2$ generates a term $-1^{n_1+n_1n_2+n_1n_3}$ next to $\nu_3$. So, adding half (of the exponent) of it will do the trick. Thereby the expressions for $\nu_3$ for arrow-based and color-based indexing will correspondingly be

$$\nu_3^{(a)}(0,-n_s,n_a,n_b) = -1^{n_s(n_a-1)n_b} \qquad \nu_3^{(c)}(0,-n_s,n_a,n_b) = i^{n_s(2n_an_b+n_a-n_b+1)} \tag{21}$$

The $\nu$ is trivial for $n_s = 0$. If the initial Hamiltonian just consists of elements $\sigma_p^x$ then each one of them on the edge will transform as

$$\sigma_p^x \to \tfrac{1}{2}\left[\bar{\sigma}_p^x + V_{1,p}\bar{\sigma}_p^x V_{1,p}^\dagger\right] \tag{22}$$

where $V$ is given by (20). The factor in front can be dropped, as it is some unimportant constant when we consider only the edge.

$V_{1,p}$ will only contain terms of links $\langle p-1,p\rangle$ and $\langle p,p+1\rangle$ and after straightforward calculations one can get the final expression of transformed edge $\sigma_p^x$

$$\sigma_p^x \to \bar{\sigma}_p^x - \bar{\sigma}_{p-1}^z \bar{\sigma}_p^x \bar{\sigma}_{p+1}^z \tag{23}$$

The bars explicitly indicate that the new operators satisfy Pauli matrix algebra just like the old ones, however they are given in a different representation. Here $\bar{\sigma}^z$ is the same as $\sigma^z$, due to $\sigma^z$-s being commutative with $U$ that makes the transformation to barred operators. Notice that we got the same Hamiltonian for both indexing techniques despite of having different $\nu_3$-s.

As it was supposed, we have obtained a translation-invariant edge Hamiltonian, which is written as just a sum of elements in (23) over the edge. The same Hamiltonian was derived by Levin and Gu using a very different approach in [4] through a rather complicated and arbitrary procedure.

## 4. Conclusions

We generalized the ideas used in [4, 5, 18] and presented a technique for creating SPT phase models as extensions on any two-dimensional systems with arbitrary non-trivial symmetry $S$. The technique relies on explicit application of specially defined symmetry-restored quasi-unitary transformations on the initial Hamiltonian. The transformations are based on $H^3\big(S,U(1)\big)$, and define the phase space of $S$-symmetric SPT models, which is also given by $H^3\big(S,U(1)\big)$ as suggested in [8]. The technique guaranties a creation of translation-invariant edge models once applied on non-interacting systems. An example of existing models was provided, to show the ease of use of the technique compared to their original derivation.

## Acknowledgements

I am grateful to V. Iugov, A. Sedrakyan, T. Hakobyan and T. Sedrakyan for the valuable discussions. The research was supported by Armenian SCS grants Nos. 20TTAT-QTa009 and 21AG-1C024.

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