UDC 512
Mathematics

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ON ISOTOPIC AND ISOMORPHIC SEMIRINGS

In this paper isotopic and isomorphic semirings is studied. It is proved that two M-isotopic semirings are isomorphic.
Keywords. *semiring , monoid, isotopy, isomorphic semirings, M-isotopic semirings, K-isotopic semirings.*

Լ.Աբրահամյան

ԿԻՍԱՕՂԱԿՆԵՐԻ ԻԶՈՏՈՊԻԱՅԻ ԵՎ ԻԶՈՄՈՐՖԻԶՄԻ ՄԱՍԻՆ

Սույն աշխատանքում դիտարկվում են իզոտոպ և իզոմորֆ կիսաօկները: Ապացուցվում է, որ, M-իզոտոպ կիսաօղակները իզոմորֆ են:
Բանալի բառեր` *Կիսաօղակ, մոնոիդ, իզոտոպիա, իզոմորֆ կիսաօղակներ, M-իզոտոպ կիսաօղակներ, K- իզոտոպ կիսաօղակներ:*

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ОБ ИЗОТОПИИ И ИЗОМОРФИЗМЕ ПОЛУКОЛЕЦ

В данной работе рассматриваются изотопные и изоморфные полукольца. Доказывается, что M-изотопные полукольца изоморфны.
Ключевые слова: *полукольцо, моноид, изотопия, изоморфизм, K-изотопные полукольца, M-изотопные полукольца.*

1. Introduction

A semiring is a set R equipped with two binary operations + and ·, called addition and multiplication, such as:

1.(R, +) is a commutative monoid with identity element 0:

$$(a + b) + c = a + (b + c)$$
$$0 + a = a + 0 = a$$
$$a + b = b + a$$

2.(R, ·) is a monoid with identity element 1:

$$(a\cdot b)\cdot c = a\cdot(b\cdot c)$$
$$1\cdot a = a\cdot 1 = a$$

3. Both multiplying left and right distribute over addition:

$$a\cdot(b + c) = (a\cdot b) + (a\cdot c)$$
$$(a + b)\cdot c = (a\cdot c) + (b\cdot c)$$

4. Multiplication by 0 annihilates R:

$$0\cdot a = a\cdot 0 = 0$$

A semiring $(R, +,\cdot)$ is called commutative, if $(R,\cdot)$ is commutative groupoid.
An idempotent semiring is a semiring $R = (R,+,\cdot)$, with identity $a + a = a$.

2. Examples

1. The motivating example of a semiring is a set of natural numbers **N** (including zero) under ordinary addition and multiplication. All these semirings are commutative.
2. The square *nxn* matrixes with non-negative entries form a (non-commutative) semiring under ordinary addition and multiplication of matrices. More generally, the same applies to the square matrices whith elements of any other given semiring *S*, and the semiring is generally non-commutative nevetheless *S* may be commutative.
3. If *A* is a commutative monoid, then the set *End (A)* of endomorphisms *f:A→A* form is a semiring, where addition is pointwise addition and multiplication is functional composition.

$$(f+g)x = f(x)+g(x)$$

$$(f\cdot g)x = g(f(x))$$

Zero morphism and identity are respective neutral elements

4. If $(Q,+,\cdot)$ is a semiring , then the set *End (Q)* of endomorphisms is a semiring under with of the following operations are:

$$(f+g)x = f(x)+g(x)$$

$$(f\cdot g)x = f(g(x))$$

5. The ideals of a ring is a semiring under addition and multiplication of ideals.
6. Any bounded, distributive lattice is a commutative, idempotent semiring under joining and meeting.

If $R=(R,+,\cdot)$ is a semiring, then we denote R^{+} =R(+).

3. Preliminary results

Two groupoids on G are called isotopic if there are permutations of G ρ, σ and τ, such as for any $a,b\in G$,

$$a\circ b=(a\rho\cdot b\sigma)\tau$$

where **.** and $\circ$ denotes the operation in these two groupoids. The isotopy relation is an equivalence relation for the binary operations. An isomorphism of two binary operations defined on the same set is a special case of an isotopy (with $\rho=\sigma=\tau^{-1}$).

In about quasigroups the following results are known[1,2]:

Theorem 1: (Albert,1943):Every groupoid that is isotopic to a quasigroup is a quasigroup itself.

Theorem 2: (Albert,1943): Every quasigroup is isotopic to some loop .

Theorem 3: (Albert,1943): If a loop (in particular, a group) is isotopic to some group, then they are isomorphic.

Theorem 5:(Bruck) If a groupoid with identity element is isotopic to a semigroup, then they are isomorphic, that is, they are both semigroups with identity.

The isotopy of rings with the same additive groups is defined by Albert in the following manner:

If $Q(+,\cdot)$ and $Q(+,\circ)$ rings are called k-isotopic if there exist bijective mappings $\alpha,\beta,\gamma:Q\to Q$ such as :

$$1)\alpha(x\cdot y)=\beta(x)\circ\gamma(y)$$

$$2)\alpha,\beta,\gamma\in Aut[Q(+)]\ .$$

Theorem6: (Albert [1,2], Kurosh [3]) If a ring with identity element is k-isotopic to an associative ring,then they are isomorphic.

4. The structure result

We introduce the following general concept of isotopy.

$Q(+_1,\cdot_1)$ and $Q'(+_2,\cdot_2)$ semrings are called K-isotopic, if there exist bijective mappings

$$\alpha,\beta,\gamma:Q\to Q'$$

such as

$$1)\alpha(x\cdot_1 y)=\beta(x)\cdot_2\gamma(y),$$

$$2)\ \alpha,\beta,\gamma:Q(+_1)\to Q'(+_2)$$

isomorphic mappings**:**

Theorem 1. If a ring with identity element is K-isotopic to an associative ring,then they are isomorphic.

Theorem 2. K-isotopic semirings are isomorphic.

The isotopy of algebras is defined as follows [4,5]:

Two algebras (Q,Ω) and (Q',Ω') with binary operations are called M-isotopic, if there exist bijective mappings $\alpha,\beta,\gamma:Q\to Q'$, $\psi:\Omega\to\Omega'$, such that $\psi:\Omega\to\Omega'$ preservs the arity of operations and

$$\alpha A(x,y)=(\psi A)(\beta x,\beta y)$$

for all $A\in\Omega$.

Theorem. Two M-isotopic semirings are isomorphic.

Proof. Let $A=(Q,\Sigma)$ $A'=(Q',\Sigma')$ is M-isotopic semirings, i.e.

$$\alpha A_i(x,y)=[\tilde{\psi}A_1](\beta x,\gamma y)\ ,$$

for all $A_i\in\Sigma$ operations and for all $x,y\in Q$ elements.

Since for all $x,y,z\in Q$ elements and for all $A_i\in\Sigma$ operations

$$A_i[A_i(x,y),z]=A_i[x,A_i(y,z)].$$

Then

$$A_i'\left[\beta\left(\alpha^{-1}A_i'(\beta x,\gamma y)\right),\gamma z\right]=A_i'\left[\beta x,\gamma\left(\alpha^{-1}A_i'(\beta y,\gamma z)\right)\right].$$

For $\beta x=\gamma r=e_i$ we obtain

$$\beta\left(\alpha^{-1}(\gamma y)\right)=\gamma\left(\alpha^{-1}(\beta y)\right),$$

for any $y\in Q$.

Futher, for $\beta x=e_i$ we obtain

$$A_i'\left[\beta\left(\alpha^{-1}(\gamma y)\right),\gamma z\right]=\gamma\left[\alpha^{-1}A_i'(\beta y,\gamma z)\right],$$

$$A_i'\left[\gamma\left(\alpha^{-1}(\beta y)\right), \gamma z\right] = \gamma\left[\alpha^{-1}A_i'(\beta y, \gamma z)\right],$$

Replacing βy to y and γz to z we obtain

$$A_i'\left[\gamma\left(\alpha^{-1}y\right), z\right] = \gamma\left[\alpha^{-1}A_i'(y, z)\right]:$$

Similarly, from condition $\gamma z = e_i$ we obtain

$$\beta\left[\alpha^{-1}A_i'(\beta x, \gamma y)\right] = A_i'\left[\beta x, \gamma\left(\alpha^{-1}(\beta y)\right)\right] = A_i'\left[\beta x, \beta\left(\alpha^{-1}(\gamma y)\right)\right],$$

Replacing βx to x and γy to y we obtain

$$\beta\left[\alpha^{-1}A_i'(x, y)\right] = A_i'\left[x, \beta\left(\alpha^{-1}y\right)\right]$$

for all $x, y \in Q$ elements.

Further,

$$\gamma\alpha^{-1}\beta A_i(x,y) = \gamma\alpha^{-1}\beta\left[\alpha^{-1}A_i'(\beta x, \gamma y)\right] = \alpha^{-1}\beta\left[\gamma\left[\alpha^{-1}A_i'(\beta x, \gamma y)\right]\right] = \alpha^{-1}\beta A_i'\left[\gamma\left(\alpha^{-1}(\beta x)\right), \gamma y\right] =$$

$$= \beta\left[\alpha^{-1}A_i'\left[\gamma\left(\alpha^{-1}(\beta x)\right), \gamma y\right]\right] = A_i'\left[\gamma\left(\alpha^{-1}(\beta x)\right), \beta\left(\alpha^{-1}(\gamma y)\right)\right] = A_i'\left[\beta\left(\alpha^{-1}(\gamma x)\right), \beta\left(\alpha^{-1}(\gamma y)\right)\right] =$$

$$= \left[\tilde{\psi}A_i\right]\left[\left(\gamma\alpha^{-1}\beta\right)x, \left(\gamma\alpha^{-1}\beta\right)y\right],$$

i.e. $(\gamma\alpha^{-1}\beta, \tilde{\psi})$ is an isomorphism from semiring A to semiring A'.

The theorem is proved.

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Հոդվածը տպագրության է երաշխավորել խմբագրական կոլեգիայի անդամ, ֆ.մ.գ.թ. Գ.Հ.Սահակյանը: