Известия НАН Армении, Математика, том 58, н. 2, 2023, стр. 82 – 96.

RIESZ MULTIRESOLUTION ANALYSIS ON LOCALLY COMPACT ABELIAN GROUPS: CONSTRUCTION AND EXCEPTIONS

SATYAPRIYA, RAJ KUMAR AND F. A. SHAH

University of Delhi, Delhi, India
Kirori Mal College, University of Delhi, Delhi, India
University of Kashmir, South Campus, Jammu and Kashmir, India[1]
E-mails: *kmc.satyapriya@gmail.com; rajkmc@gmail.com fashah@uok.edu.in*

Abstract. In this article, we construct a Riesz multiresolution analysis (MRA) on locally compact Abelian groups (LCA) starting from a suitably given scaling function. Subsequently, we investigate all the conditions under which a scaling function generates a Riesz MRA for $L^2(G)$. Besides, all the results are braced with illustrative examples. Towards the culmination, several exceptions are discussed regarding the non-existence of dilative automorphism α of G.

MSC2020 numbers: 42C40; 42C15; 43A70; 22B05.
Keywords: Riesz multiresolution analysis; scaling function; dilative automorphism; locally compact Abelian group; disconnected group.

1. Introduction

While investigating the linear quadrature mirror filters in signal processing, Mallat [1] pioneered an algorithm for the construction of orthonormal wavelet bases for $L^2(\mathbb{R})$, coined as the multiresolution analysis (MRA). Undoubtedly, the theory of MRA has attained a respectable status within the scientific and engineering communities in such a way that it is now considered as a nucleus of shared aspirations and ideas [2]. Some of the prominent wavelets obtained via an MRA include Shannon wavelet, Meyer wavelet, Franklin wavelet, spline wavelets, biorthogonal wavelets, nonuniform wavelets, harmonic wavelets, Daubechies wavelets and the Riesz wavelets [3, 4, 5, 6, 7].

The theory of MRA on locally compact abelian groups has grown at an exponential rate over the last two decades and is befitting for investigating deep problems in time-frequency analysis, owing to its ability to unify continuous and discrete theory, and to cover higher-dimensional problems without any notational complication. For instance, Dahlke [8] constructed orthonormal wavelet basis on LCA groups by means of the generalized B-splines and self-similar tiles, whereas Lang [9] adapted

[1]The first author is financially supported by HRDG-CSIR, Government of India, Grant No. 09/045(1653)/2019-EMR-I.

the well-known finite mask approach of Daubechies [2] to construct compactly supported wavelets on the Cantor dyadic groups. Later on, Kamyabi-Gol and Tousi [10, 11] investigated the conditions under which a function generates a multiresolution analysis on a locally compact abelian group using the theory of spectral functions and shift-invariant spaces. Subsequently, Yang and Taylor [12] introduced the notion of an MRA on certain non-abelian locally compact groups G with no regularity or decay constraints on the scaling functions and constructed the Haar-like wavelet bases for $L^2(G)$. Recently, Bownik and Jahan [13] constructed an MRA on compact Abelian groups with epimorphism as a dilation operator and characterize the scaling sequences of an MRA for $L^p(G), 1 \leq p < \infty$. Very recently, Kumar and Satyapriya [14] formulated the theory of frame multiresolution analysis on LCA groups and investigated certain properties of multiresolution subspaces $\{V_j : j \in \mathbb{Z}\}$ which provides the quantitative criteria for the construction of an FMRA for $L^2(G)$.
Motivated and inspired by the contemporary developments in the theory of MRA abreast the profound applicability of the unifying structure of locally compact Abelian groups, we construct a Riesz MRA for $L^2(G)$ starting from a given scaling function. Subsequently, we study all the conditions under which a scaling function ϕ generates a Riesz MRA for $L^2(G)$. Nevertheless, several illustrative examples are presented to facilitate a sound clarification of the constructed Riesz MRA. Towards the end, some exceptional cases have been discussed regarding the non-existence of dilative automorphism α of G.
The remainder of the article is structured as follows: Section 2 is entirely devoted for the exposition of the preliminaries including the definition of Fourier transform, dilative automorphism, uniform lattices and the Riesz basis on LCA groups. Section 3 exclusively deals with the construction of a Riesz MRA for $L^2(G)$. In section 4, some exceptional cases have been discussed briefly. Finally, a conclusion is extracted in Section 5.

2. Preliminaries and Fourier analysis on LCA groups

We shall start this section with a brief overview of the locally compact Abelian groups followed by some preliminary results concerning the Fourier transforms on LCA groups, which serves as a cornerstone for the subsequent developments of the Riesz bases for $L^2(G)$. Towards the culmination of the section, we present the definition and characterizations of Riesz bases in Hilbert spaces.

2.1. Basics of LCA groups. A group G equipped with a Hausdorff topology is called an LCA group, if it is metrizable, locally compact and can be written as a countable union of compact sets. The set of real number $\mathbb{R}$, integers $\mathbb{Z}$, unit disk $\mathbb{T}$ and $\mathbb{Z}_N$ (the integers modulo N) are some prominent examples of LCA groups. These groups along with their higher dimensional variants, are called elementary LCA groups. Moreover, the family of all continuous homomorphisms from the LCA group G to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is denoted by $\widehat{G}$ and also constitutes an LCA group under a suitable topology and the composition

$$(\omega + \omega')(x) = \omega(x)\, \omega'(x), \quad x \in G, \omega, \omega' \in \widehat{G}. \tag{2.1}$$

This group is often referred as the dual group of G and its elements are called the characters of $\widehat{G}$. It is well-known that the double-dual group $\widehat{\widehat{G}} = G$ and as such $\omega(x)$ can be interpreted as either the action of $\omega \in \widehat{G}$ on $x \in G$ or the action of $x \in G$ on $\omega \in \widehat{G}$. For the sake of brevity, we shall use the following notation:

$$(\omega, x) = \omega(x), \quad x \in G, \omega \in \widehat{G}. \tag{2.2}$$

2.2. Fourier analysis on LCA groups

Let μ_G and $\mu_{\widehat{G}}$ be the Haar measures on LCA groups G and $\widehat{G}$, respectively. Based on the Haar measure, we define the spaces $L^p(G)$ and $L^p(\widehat{G}), 1 \leq p \leq \infty$ in the usual way. The Fourier transform of any arbitrary function $f \in L^1(G)$ is defined by

$$F : L^1(G) \to C_0(\widehat{G}), \quad F(f)(\omega) = \int_G f(x)\, \overline{(\omega, x)}\, d\mu_G(x), \tag{2.3}$$

where $C_0(\widehat{G})$ denotes the space of all continuous functions on $\widehat{G}$ vanishing at infinity. For the sake of our convenience, we will also use the notation $\hat{f}$ to denote the Fourier transform of the function f.

It is worth noticing that for a fixed Haar measure $d\mu_G(x)$, there exits a Haar measure $d\mu_{\widehat{G}}(x)$ on $\widehat{G}$ called the normalized Plancherel measure, such that the Fourier transform (2.3) is an isometric transform on $L^1(G) \cap L^2(G)$, and hence, it can be extended uniquely to a unitary isomorphism from $L^2(G)$ onto $L^2(\widehat{G})$ [15]. Therefore, each $f \in L^1(G)$ with $F(f)(\omega) \in L^1(\widehat{G})$ can be reconstructed via the following formula:

$$f(x) = \int_{\widehat{G}} \hat{f}(\omega)(\omega, x)\, d\mu_{\widehat{G}}(\omega), \quad x \in G. \tag{2.4}$$

Moreover, the Parseval's formula corresponding to (2.3) reads

$$\langle f, g\rangle = \int_G f(x)\, \overline{g(x)}\, d\mu_G(x) = \int_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, d\mu_{\widehat{G}}(\omega) = \langle \hat{f}, \hat{g}\rangle. \tag{2.5}$$

For typographical convenience, we shall denote the Haar measures $d\mu_G$ and $d\mu_{\widehat{G}}$ by dx and $d\omega$, respectively.

For $y \in G$, the generalized translation operator is defined by

$$T_y : L^2(G) \to L^2(G), \quad T_y f(x) = f(x - y), \quad x \in G. \tag{2.6}$$

Likewise, the generalized dilation operator D in $L^2(G)$ can be defined via the dilative automorphism introduced by Dahlke [8]. An automorphism $\alpha : G \to G$ is said to be dilative if there exists $N \in \mathbb{N}$ such that $K \subseteq \alpha^n(U)$, $\forall\, n \geq N$, where K is any compact set in G and U is an open neighbourhood at the origin. Therefore, for a dilative automorphism α, the dilation operator $D : L^2(G) \to L^2(G)$ is defined by

$$Df(x) = \Delta(\alpha)^{1/2} f\big(\alpha(x)\big), \quad x \in G, \tag{2.7}$$

where $\Delta(\alpha)$ is a positive constant such that

$$\int_G f(x)\, dx = \Delta(\alpha) \int_G f\big(\alpha(x)\big)\, dx. \tag{2.8}$$

2.3. Lattices and fundamental domains in LCA groups

A uniform lattice in an LCA group G is a discrete subgroup Λ for which the quotient group G/Λ is compact. In addition to this, we shall also assume that $\alpha(\Lambda) \subseteq \Lambda$. Corresponding to the lattice Λ, an annihilator $\Lambda^\perp$ is defined by

$$\Lambda^\perp = \Big\{\omega \in \widehat{G} : (x, \omega) = 1, x \in \Lambda\Big\}. \tag{2.9}$$

It is easy to verify that the annihilator $\Lambda^\perp$ is also a lattice in $\widehat{G}$ and $\hat{\alpha}\big(\Lambda^\perp\big) \subset \Lambda^\perp$, whenever $\alpha(\Lambda) \subset \Lambda$. For the classical case $G = \mathbb{R}$, we have $\Lambda = \Lambda^\perp = \mathbb{Z}$. Therefore, the inclusion $\alpha(\Lambda) \subset \Lambda$ always holds for the automorphism $x \mapsto 2x$ as $\alpha(\Lambda) = 2\mathbb{Z}$. Nevertheless, it is pertinent to mention that a lattice Λ in G can be used to obtain a splitting of the group G and $\widehat{G}$ into disjoint cosets [16].

Lemma 2.1. [16] *Let Λ be a lattice in an LCA group G. Then the following hold:*
(i). There exists a Borel measurable relatively compact set $\mathcal{Q} \subseteq G$ such that

$$G = \bigcup_{\lambda \in \Lambda} (\lambda + \mathcal{Q}), \;\; (\lambda + \mathcal{Q}) \cap (\lambda' + \mathcal{Q}) = \emptyset, \;\; \textit{for } \lambda \neq \lambda'; \;\; \lambda, \lambda' \in \Lambda. \tag{2.10}$$

(ii). There exists a Borel measurable relatively compact set $\mathcal{S} \subseteq \widehat{G}$ such that

$$\widehat{G} = \bigcup_{\omega \in \Lambda^\perp} (\omega + \mathcal{S}), \;\; (\omega + \mathcal{S}) \cap (\omega' + \mathcal{S}) = \emptyset, \;\; \textit{for } \omega \neq \omega'; \;\; \omega, \omega' \in \Lambda^\perp. \tag{2.11}$$

The sets $\mathcal{Q}$ and $\mathcal{S}$ appearing in (2.10) and (2.11) are called a fundamental domains or the tiles associated with the lattices Λ and $\Lambda^\perp$, respectively.
We now discuss the periodic functions on G. For a given set $H \subset G$, a function $f : G \to \mathbb{C}$ is said to be H-periodic if

$$f(x + h) = f(x), \quad \forall\, x \in G, h \in H. \tag{2.12}$$

In particular, if we take $H = \Lambda$, then by virtue of Λ-periodicity of the functions defined on G, we can determine the space $L^2(G/\Lambda)$. Similarly, we can define $\Lambda^{\perp}$-periodic functions on $\widehat{G}$ and hence, the space $L^2(\widehat{G}/\Lambda^{\perp})$ can be determined accordingly [17, 14]. If we assume that $G = \mathbb{R}$ and $\Lambda = \mathbb{Z}$, then both the quotient spaces $L^2(G/\Lambda)$ and $L^2(\widehat{G}/\Lambda^{\perp})$ can be identified with the space $L^2(\mathbb{T})$. Note that a function $F \in L^2(\widehat{G}/\Lambda^{\perp})$ if and only if there exists a sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that [14]

$$F(\omega) = \sum_{\lambda\in\Lambda} c_\lambda(\omega, \lambda), \quad \forall\, \omega \in \widehat{G}. \tag{2.13}$$

2.4. The Riesz basis in $L^2(G)$

Considering the lattice Λ as a countable index set, we define a Riesz basis for the space $L^2(G)$. For a detailed study on Riesz bases and related topics, we refer to [16].

Definition 2.1. A family $\{f_\lambda : \lambda \in \Lambda\}$ is called a Riesz basis for $L^2(G)$ if there exist a bounded bijective operator $U : L^2(G) \to L^2(G)$ and an orthonormal basis $\{e_\lambda : \lambda \in \Lambda\}$ of $L^2(G)$ such that $f_\lambda = Ue_\lambda$, for each $\lambda \in \Lambda$.

It is worth noticing that for a Riesz basis $\{f_\lambda : \lambda \in \Lambda\}$ of $L^2(G)$, there exists positive constant $0 < A \leq B < \infty$ such that [16]

$$A\left\|f\right\|^2 \leq \sum_{\lambda\in\Lambda} \left|\langle f, f_\lambda\rangle\right|^2 \leq B\left\|f\right\|^2, \quad \forall f \in L^2(G). \tag{2.14}$$

The numbers A and B are called lower and upper Riesz bounds, respectively. In the optimal case, we have

$$A = \left\|U^{-1}\right\|^{-2}, \quad \text{and} \quad B = \left\|U\right\|^2. \tag{2.15}$$

In most cases, it is extremely strenuous to determine the existence of A and B or an operator U to verify whether the family $\{f_\lambda : \lambda \in \Lambda\}$ is a Riesz basis or not. In such cases, an alternate characterization of the Riesz bases is given below.

Lemma 2.2. [16] *Let Λ be a lattice in LCA group G. A sequence $\{f_\lambda : \lambda \in \Lambda\}$ in $L^2(G)$ is a Riesz basis for $L^2(G)$ if and only if the map $T : l^2(\Lambda) \to L^2(G)$ given by*

$$T(\{c_\lambda\}) = \sum_{\lambda\in\Lambda} c_\lambda f_\lambda, \tag{2.16}$$

is well defined and bijective.

3. Construction of Riesz MRA on LCA groups

In this section, we construct the Riesz MRA on locally compact Abelian groups by first choosing an appropriate scaling function $\phi \in L^2(G)$ and then obtaining the subspace V_0 by taking the linear span of translates of ϕ. Consequently, the other subspaces $V_j, j \in \mathbb{Z}$ can be generated as the scaled adaptations of V_0. Besides, we shall present several conditions under which a scaling function generates a Riesz MRA for $L^2(G)$. Prior to that, we shall formally introduce the notion of a Riesz MRA in $L^2(G)$ by slight modifications of the classical MRA.

Definition 3.1. A Riesz multiresolution analysis of $L^2(G)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(G)$ satisfying the following properties:
(i). Nested Property: $V_j \subseteq V_{j+1}$, for all $j \in \mathbb{Z}$;
(ii). Density Property: $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(G)$;
(iii). Separation Property: $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv). Dilation Property: $f(\cdot) \in V_j$ if and only if $f(\alpha(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(v). Translation Invariant: $f \in V_j \implies T_\lambda f \in V_j$ for all $\lambda \in \Lambda, j \in \mathbb{Z}$;
(vi). Riesz Basis: the sequence $\{T_\lambda\phi(\cdot) = \phi(\cdot - \lambda) : \lambda \in \Lambda\}$ is a Riesz basis for V_0.

Definition 3.1 allows us to make the following observations.
(i). The function ϕ appearing in (vi) is called as the scaling function or the father wavelet of a Riesz MRA, where as the subspaces V_j's are known as approximation spaces or multiresolution subspaces.
(ii). Condition (vi) also implies that

$$V_0 = \overline{\operatorname{span}}\{T_\lambda\phi : \lambda \in \Lambda\}, \tag{3.1}$$

whereas condition (iv) yields

$$V_j = \overline{\operatorname{span}}\{D^j\, T_\lambda\phi : \lambda \in \Lambda\}, \quad j \in \mathbb{Z}. \tag{3.2}$$

(iii). Using Definition 3.1 together with the fact that D^j is an unitary operator, it follows that $\{D^jT_\lambda\phi : \lambda \in \Lambda\}$ is a Riesz basis for $V_j, j \in \mathbb{Z}$ with same bounds as that of the Reisz sequence $\{T_\lambda\phi : \lambda \in \Lambda\}$. Therefore, every function $f \in V_j$ can be expressed as

$$f(x) = \sum_{\lambda\in\Lambda} c_\lambda\, D^jT_\lambda\phi(x), \quad \forall x \in G. \tag{3.3}$$

(iv). Implementation of the Fourier transform of (3.3) yields

$$\hat{f}\big(\hat{\alpha}^j(\gamma)\big) = F(\gamma)\,\hat{\phi}(\gamma), \quad \forall\, \gamma \in \widehat{G}, \tag{3.4}$$

where $F(\gamma) \in L^2(\widehat{G}/\Lambda^\perp)$.

We now state a lemma which provide a complete characterization of the scaling functions $\phi \in L^2(G)$ so that the condition (vi) holds good.

Lemma 3.1. [17] *For any arbitrary function $\phi \in L^2(G)$, the family of translates $\big\{T_\lambda \phi(x) : \lambda \in \Lambda\big\}$ forms a Riesz sequence for V_0 with bounds A and B if and only if*

$$A \le \Phi(\gamma) \le B, \quad \forall\, \gamma \in \widehat{G}, \tag{3.5}$$

where $\Phi(\gamma) = \sum_{\omega \in \Lambda^\perp} \big|\hat{\phi}(\gamma+\omega)\big|^2$.

To construct a Riesz MRA on locally compact Abelian groups, the foremost requirement is to choose an appropriate function $\phi \in L^2(G)$ such that $\big\{T_\lambda \phi : \lambda \in \Lambda\big\}$ constitutes a Riesz basis for V_0. Once we choose ϕ and define the subspaces V_j's by (3.2), the separation property (iii) of the subspaces becomes redundant. Moreover, it follows that V_j satisfy the dilation and translating properties of Definition 3.1. Nevertheless, the separation property associated with the MRA based wavelet frames for $L^2(G)$ is proved in [14]. However, for the sake of courtesy, we shall state this result for a Riesz MRA on LCA groups below.

Lemma 3.2. *Let $\phi \in L^2(G)$ be such that the family of translates $\big\{T_\lambda \phi : \lambda \in \Lambda\big\}$ forms a Riesz sequence for V_0 and let $\big\{V_j : j \in \mathbb{Z}\big\}$ be the sequence of closed subspaces of $L^2(G)$ as defined in* (3.2). *Then, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.*

In order to verify that the ladder spaces V_j's generated by ϕ constitute a Riesz MRA for $L^2(G)$, it is sufficient to show that the following properties also hold:

- The subspaces V_j are nested;
- $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(G)$.

Since the subspaces V_j defined via (3.2) exhibit the scaling property and D^j is a unitary operator, thus, the only inclusion $V_0 \subseteq V_1$ is required to show that the subspaces V_j's are nested.

Theorem 3.1. *Let $\phi \in L^2(G)$ be such that the family of translates $\big\{T_\lambda \phi : \lambda \in \Lambda\big\}$ forms a Riesz sequence for V_0 and let $\big\{V_j : j \in \mathbb{Z}\big\}$ be the sequence of closed subspaces of $L^2(G)$ as defined in* (3.2). *Then, the following statements are equivalent:*

(i). $V_0 \subseteq V_1$

(ii). *There exist a $\Lambda^\perp$-periodic function $H_0 \in L^\infty(\widehat{G}/\Lambda^\perp)$ such that*

$$\hat{\phi}\big(\hat{\alpha}(\gamma)\big) = H_0(\gamma)\, \hat{\phi}(\gamma), \quad \forall\, \gamma \in \widehat{G}. \tag{3.6}$$

Proof. We first assume that $V_0 \subseteq V_1$. Since $V_1 = DV_0$ and $\phi \in V_0$, hence, it follows that $D^{-1}\phi \in V_0$. By virtue of (3.4), there exists a function $F \in L^2(\widehat{G})/\Lambda^\perp$ such that

$$\widehat{D^{-1}\phi}(\gamma) = F(\gamma)\,\hat{\phi}(\gamma), \quad \forall\, \gamma \in \widehat{G},$$

which in turn implies that

$$\hat{\phi}\big(\hat{\alpha}(\gamma)\big) = H_0(\gamma)\,\hat{\phi}(\gamma), \quad \gamma \in \widehat{G}, \tag{3.7}$$

where $H_0(\gamma) = \Delta_\alpha^{-1/2} F(\gamma)$, which asserts (3.6). It remains to show that $H_0(\gamma)$ is essentially bounded on $\widehat{G}$. Using representation (3.3) and the definition of Φ, we have

$$\Phi(\gamma) = \sum_{\omega\in\Lambda^\perp} \Big|\hat{\phi}(\gamma+\omega)\Big|^2 = \sum_{\omega\in\Lambda^\perp} \Big|H_0\big(\hat{\alpha}^{-1}(\gamma+\omega)\big)\hat{\phi}\big(\hat{\alpha}^{-1}(\gamma+\omega)\big)\Big|^2. \tag{3.8}$$

Since $\hat{\alpha}(\Lambda^\perp) \subseteq \Lambda^\perp$, relation (3.8) becomes

$$\Phi(\gamma) \geq \sum_{\omega\in\hat{\alpha}(\Lambda^\perp)} \Big|H_0\big(\hat{\alpha}^{-1}(\gamma+\omega)\big)\hat{\phi}\big(\hat{\alpha}^{-1}(\gamma+\omega)\big)\Big|^2.$$

Applying the $\Lambda^\perp$-periodicity of H_0, we obtain

$$\Phi(\gamma) \geq \Big|H_0\big(\hat{\alpha}^{-1}(\gamma)\big)\Big|^2 \Phi\big(\hat{\alpha}^{-1}(\gamma)\big).$$

Since the collection $\{T_\lambda\phi : \lambda \in \Lambda\}$ forms a Riesz sequence for V_0 with bounds A and B (say), therefore, it is easy to obtain the following inequality:

$$\big|H_0(\gamma)\big| \leq \sqrt{\frac{B}{A}}, \quad \forall\, \gamma \in \widehat{G}.$$

Thus, we conclude that the $\Lambda^\perp$-periodic function H_0 is also essentially bounded. Conversely, suppose that there exists a $\Lambda^\perp$-periodic function H_0 such that (3.6) holds. Then, for an arbitrary function $f \in V_0$, relation (3.4) implies that there exist some $F \in L^2(\widehat{G}/\Lambda^\perp)$ such that $\hat{f}(\gamma) = F(\gamma)\,\hat{\phi}(\gamma)$ for all $\gamma \in \hat{G}$. Therefore, by virtue of (3.6), above relation yields

$$\hat{f}\big(\hat{\alpha}(\gamma)\big) = F\big(\hat{\alpha}(\gamma)\big)\,H_0(\gamma)\,\hat{\phi}(\gamma), \quad \forall\, \gamma \in \widehat{G},$$

which further implies that $\hat{f}\big(\hat{\alpha}(\gamma)\big) = H_1(\gamma)\,\hat{\phi}(\gamma)$ for all $\gamma \in \widehat{G}$, where $H_1(\gamma) = F\big(\hat{\alpha}(\gamma)\big)\,H_0(\gamma)$ and belongs to $L^2(\widehat{G}/\Lambda^\perp)$ as $F \in L^2(\widehat{G}/\Lambda^\perp)$ and $H_0 \in L^\infty(\widehat{G}/\Lambda)$. Again by virtue of (3.4), it follows that $f \in V_1$. This completes the proof of Theorem 3.1. □

Corollary 3.1. *Let $\phi \in L^2(G)$ be such that the family of translates $\{T_\lambda\phi : \lambda \in \Lambda\}$ forms a Riesz sequence for V_0 and let $\{V_j : j \in \mathbb{Z}\}$ be the sequence of closed subspaces of $L^2(G)$ as defined in* (3.2)*. Then, the refinement mask H_0 associated with the scaling function ϕ is unique.*

It's just a matter of demonstrating that the union of the subspaces $\{V_j : j \in \mathbb{Z}\}$ is dense in $L^2(G)$. Several authors have used different approaches to demonstrate this property. For instance, Daubechies [2] determine this property by knowing the behaviour of $\hat{\phi}$ around the neighbourhood of $0 \in \widehat{G}$. Kamyabi and Tousi [10] employed the machinery of spectral radius and shift invariant spaces to verify this property for LCA groups. On the other hand, the property of α-substantiality of the scaling function ϕ has be utilized to prove this property [14, 12]. We shall also use the property of α-substantial to prove that the union of V_j is dense in $L^2(G)$. We recall that a function $f \in L^2(G)$ is said to be α-substantial if there exists a non-zero function $g \in L^2(G)$ such that

$$D^j f * g = 0, \quad \forall\, j \in \mathbb{Z} \implies g = 0.$$

Lemma 3.3. [14] *If $\phi \in L^2(G)$ is such that $|\hat{\phi}| > 0$ on a neighbourhood of $0 \in \widehat{G}$, then ϕ is α-substantial.*

We now state the union theorem in terms of α-substantial of the scaling function ϕ.

Theorem 3.2. [14] *Let ϕ be a refinable function in $L^2(G)$ and $\{V_j : j \in \mathbb{Z}\}$ be defined by* (3.2). *Then, the following conditions are equivalent:*
(i). $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$
(ii). *ϕ is α-substantial.*

We now assemble all the conditions under which a function $\phi \in L^2(G)$ generates a Riesz MRA for $L^2(G)$.

Theorem 3.3. *A function $\phi \in L^2(G)$ generate a Riesz MRA for $L^2(G)$ if the following conditions are satisfied:*
(i). *The family of translates $\{T_\lambda \phi : \lambda \in \Lambda\}$ forms a Riesz sequence.*
(ii). *The subspaces V_j are defined by* (3.2).
(iii). *The function $\hat{\phi}$ is nonzero on a neighbourhood of $0 \in \widehat{G}$.*
(iv). *There exists a $\Lambda^\perp$-periodic function $H_0 \in L^\infty(\widehat{G}/\Lambda^\perp)$ such that* (3.6) *holds.*

We now present some examples for the lucid illustration of the proposed construction.

Example 3.1. Let $G = \mathbb{R}^+$ be an LCA group of of all positive real numbers with Haar measure

$$\mu_G(\mathcal{B}) = \int_{\mathcal{B}} \frac{(\log 2)^{-1}}{t}\, dt,$$

where $\mathcal{B} \subseteq G$ is any Borel set in G. Then, it is easy see that $\Lambda = \{2^n : n \in \mathbb{Z}\}$ is a uniform lattice in G and the corresponding quotient group $\Lambda/\alpha(\Lambda)$ has the following representation

$$\frac{\Lambda}{\alpha(\Lambda)} = \Big\{\alpha(\Lambda), 2\,\alpha(\Lambda)\Big\},$$

where $\alpha : x \mapsto x^2$ is a dilative automorphism of G. For $x, \omega \in G$, the map $x \to x^{i \log \omega}$ is a continuous homomorphism from G to the unit interval $\mathbb{T}$. Subsequently, the characters of G may be defined by $(\omega, x) = \phi_\omega(x)$ so that dual group of $\mathbb{R}^+$ is $\mathbb{R}^+$; i.e, $\widehat{G} = G$. The measure $\mu_{\hat{G}}$ is normalized appropriately so that the inversion formula and the Parseval formula hold. Moreover, the annihilator $\Lambda^\perp$ of Λ and the automorphism $\hat{\alpha}$ of $\widehat{G}$ can be derived accordingly. We choose the sets

$$\mathcal{Q} = \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right) \quad \text{and} \quad \mathcal{S} = \left[e^{-\frac{\pi}{\log 2}}, e^{\frac{\pi}{\log 2}}\right)$$

as the fundamental domains in G and $\widehat{G}$, respectively. Then, we observe that $\mu_G(\mathcal{Q}) = 1 = \mu_{\hat{G}}(\mathcal{S})$. We now define a function $\phi \in L^2(G)$ via its Fourier transform as

$$\hat{\phi}(\gamma) = \mathcal{X}_{A_1}(\gamma) + 2\mathcal{X}_{A_2}(\gamma), \qquad \gamma \in \widehat{G}, \tag{3.9}$$

where

$$A_1 = \left[e^{-\frac{\pi}{\log 4}}, e^{\frac{\pi}{\log 4}}\right), \quad \text{and} \quad A_2 = \left[e^{-\frac{\pi}{\log 2}}, e^{-\frac{\pi}{\log 4}}\right) \bigcup \left[e^{\frac{\pi}{\log 4}}, e^{\frac{\pi}{\log 2}}\right). \tag{3.10}$$

Then, it is quite evident that for any $\gamma \in \mathcal{S}$, we have

$$\sum_{\omega \in \Lambda^\perp} \left|\hat{\phi}(\gamma\omega)\right|^2 = \mathcal{X}_{A_1}(\gamma) + 2\mathcal{X}_{A_2}(\gamma),$$

which further implies that

$$1 \leq \Phi(\gamma) \leq 2, \quad \forall\, \gamma \in \widehat{G}. \tag{3.11}$$

Using Lemma 3.2, it follows that the family of translates $\{T_\lambda \phi : \lambda \in \Lambda\}$ forms a Riesz sequence. In particular, if we choose the subspaces V_j as defined in (3.2), then the collection$\{T_\lambda \phi : \lambda \in \Lambda\}$ constitutes a Riesz basis for V_0. From the relation (3.9), we observe that $\hat{\phi}(\gamma)$ is continuous in any neighbourhood of $0 \in \widehat{G}$ with $\hat{\phi}(0) = 1$ and is contained in A_1 , this verifies the density property (ii) of a Riesz MRA on a LCA group G. Let H_0 be the $\Lambda^\perp$-periodic extension of the function

$$\mathcal{X}_{A_3}(\gamma) + 2\mathcal{X}_{A_4}(\gamma), \quad \gamma \in \mathcal{S},$$

where

$$A_3 = \left[e^{-\frac{\pi}{\log 16}}, e^{\frac{\pi}{\log 16}}\right), \quad A_4 = \left[e^{-\frac{\pi}{\log 4}}, e^{-\frac{\pi}{\log 16}}\right) \bigcup \left[e^{\frac{\pi}{\log 16}}, e^{\frac{\pi}{\log 4}}\right).$$

Therefore, we have

$$\hat{\phi}\big(\hat{\alpha}(\gamma)\big) = \hat{\phi}(\gamma^2) = H_0(\gamma)\,\hat{\phi}(\gamma), \quad \forall\, \gamma \in \widehat{G}.$$

Thus, we can say that ϕ is refineable and hence, the subspaces V_j are also nested. Using Theorem 3.3, it follows that ϕ generates a Riesz MRA for $L^2(G)$.

Example 3.2. Let $G = \mathbb{R}$ be the Euclidean group of real numbers. We choose the Lebesgue measure dx as the Haar measure on $\mathbb{G}$, then it is easy verify that the dual of $\mathbb{R}$ itself $\mathbb{R}$. Consider $\Lambda = \mathbb{Z}$ as a uniform lattice in G and the map $x \mapsto 3x$ as a dilative automorphism of $\mathbb{R}$. Then, it turns out that $\Lambda^{\perp} = \Lambda$ and $\hat{\alpha} = \alpha$.
We now define a function $\phi \in L^2(G)$ via its Fourier transform by

$$\hat{\phi}(x) = \begin{cases} \left(\dfrac{x^4}{24} + \dfrac{5x^3}{12} + \dfrac{25x^2}{16} + \dfrac{125x}{48} + \dfrac{625}{384}\right)^{1/2}, & x \in \left[-\dfrac{5}{2}, -\dfrac{3}{2}\right) \\ \left(-\dfrac{x^4}{6} - \dfrac{5x^3}{6} - \dfrac{5x^2}{4} - \dfrac{5x}{24} + \dfrac{55}{96}\right)^{1/2}, & x \in \left[-\dfrac{3}{2}, -\dfrac{1}{2}\right) \\ \left(\dfrac{x^4}{4} - \dfrac{5x^2}{8} + \dfrac{115}{192}\right)^{1/2}, & x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right) \\ \left(-\dfrac{x^4}{6} + \dfrac{5x^3}{6} - \dfrac{5x^2}{4} + \dfrac{5x}{24} + \dfrac{55}{96}\right)^{1/2}, & x \in \left[\dfrac{1}{2}, \dfrac{3}{2}\right) \\ \left(\dfrac{x^4}{24} - \dfrac{5x^3}{12} + \dfrac{25x^2}{16} - \dfrac{125x}{48} + \dfrac{625}{384}\right)^{1/2}, & x \in \left[\dfrac{3}{2}, \dfrac{5}{2}\right). \end{cases} \tag{3.12}$$

Define the subspaces V_j via the relation (3.2). Then, it trivially satisfies the separation condition (ii) of a Riesz MRA. Since, $\hat{\phi}(\gamma) \neq 0$ in the neighbourhood of $0 \in \widehat{G}$, therefore, by virtue of Lemma 3.3, it follows that ϕ is α-substantial and subsequently, Theorem 3.2 implies that V_j is dense in $L^2(G)$. We also observe that $\Phi(\gamma) = 1$, $\forall\, \gamma \in \widehat{G}$, which means that the family of translates $\big\{T_\lambda \phi : \lambda \in \Lambda\big\}$ constitutes a Riesz basis for V_0 with Riesz bounds equal to 1. Finally, taking $H_0(\gamma) = \hat{\phi}(3\gamma)/\hat{\phi}(\gamma)$ as the refinement mask on $[-1/2, 1/2)$, which can be extended periodically to the whole real line $\mathbb{R}$, then all the conditions of Theorem 3.3 are satisfied and hence, ϕ generates a Riesz MRA for $L^2(G)$.

4. Some exceptional cases

Since the construction of Riesz MRA discussed in the previous section entirely relies upon the existence of a dilative automorphism α of the locally compact Abelian group G. There arises a serious question: Does there always exist a dilative automorphism α for every LCA group G? In this section, we show that there exist LCA groups for which there exist no dilative automorphism.

Example 4.1. Let $G = \mathbb{T}$ be the circle group equipped with the topology of $\mathbb{R}^2$. Taking $\Lambda = \{1\}$ as a uniform lattice of G, then the quotient group G/Λ can be identified as G itself, which is compact. Let $\alpha : x \mapsto x^n$ be an automorphism of G. Assume that $U = \{e^{ix} : -\pi/4 < x < \pi/4\}$ is an open neighbourhood of the identity element $1 \in G$. Then, for the case $G = K$, we have $\alpha^{-n}(K) = K$ for all $n \in \mathbb{N}$, and hence, the condition for α to be dilative can never be achieved. Therefore, we conclude that there exist no dilative automorphism α on G and hence we can't construct a Riesz MRA on the space $L^2(G)$ using the methods given in this paper. But, if we use an epimorphism instead of automorphism, then using the methods of [13], the construction of a Riesz MRA is possible on the space $L^2(G)$.

Theorem 4.1. *For a non-trivial compact abelian group G, there does not exist a dilative automorphism α of G.*

Proof. Let α be an automorphism of a compact Abelian group G. Let U be any proper open neighbourhood of the origin in G. Take $K = G$ as a compact set in G and assume that α is dilative. Then, there exist some $n_0 \in \mathbb{N}$ such that $K \subseteq \alpha^n(U)$, $n \geq n_0$. This implies that

$$\alpha^{-n}(K) \subset U, \quad \forall\, n \geq n_0. \tag{4.1}$$

Hence, $\alpha^{-n}(K) = G$ as $K = G$. Using the fact that $U \subset G$ and (4.1), it follows that $G = U$, which can not be true as U is a proper subset of G. Thus, we conclude that there exist no dilative automorphism α for a non-trivial LCA group G. □

Remark 4.1. The paper [13] deals with the construction of an MRA on the space $L^2(G)$, G being a compact Abelian group. The main difference in our paper and [13] lies in the associated structures of lattices and dilation operator. In [13], the authors have used an epimorphism A to define the dilation operator for $L^2(G)$. Further, in [13] a nonstationary like case has been dealt with as there are different scaling functions ϕ_j for different multiresolution spaces V_j (although it is valid for the stationary case as well.). They have also used different sets for translation on different multiresolution levels. To sum up, the analogous of condition (vi) of Definition 3.1 reads as: there exist functions $\phi_j \in V_j$ such that the family $\{T_\beta \phi_j : \beta \in ker(V_j)\}$ forms an orthonormal basis for V_j.

In the next example, we show that there also exist no dilative automorphism α on the discrete group $\mathbb{Z}$.

Example 4.2. Let $G = \mathbb{Z}$ be the group of integers under addition and equipped with the discrete topology. For a fixed $m \in \mathbb{Z}$, the quotient group $\mathbb{Z}/m\mathbb{Z}$ is finite and compact, so we can take $\Lambda = m\mathbb{Z}$ as a uniform lattice for G. Let $\alpha : x \mapsto px$, $p \in \mathbb{Z}$, be an automorphism of G, then $pm\mathbb{Z} \subseteq m\mathbb{Z}$. We claim that α is not dilative. By taking $U = \{0\}$ and $K = \{-1, 0, 1\}$, we observe that U is an open neighbourhood of the origin and K is compact in $\mathbb{Z}$. Therefore, $\alpha^n(U) = \{p^n \cdot 0\} = \{0\}$, $n \in \mathbb{N}$ and thus, K can not be contained within $\widehat{\alpha}^n(U)$, for any $n \in \mathbb{Z}$. Hence, there exist no dilative automorphism α of $\mathbb{Z}$.

In the following theorem, we generalize above assertion for any arbitrary discrete group G.

Theorem 4.2. *Let G be a discrete LCA group with atleast two elements. Then, there does not exist any dilative automorphism α of G.*

Proof. Let α be an automorphism of a discrete LCA group G with atleast two elements. Let $U = \{0\}$ and $K = \{a_1, a_2, \ldots, a_N\}$ such that atleast one a_k's is non-zero. Clearly, U is an open neighbourhood of $0 \in G$ and K is compact n G. Therefore, $\alpha^n(0) = 0$, for any $n \in \mathbb{N}$. This means that

$$\alpha^n(U) = U, \quad \forall\, n \in \mathbb{N}. \tag{4.2}$$

Since K is not contained in U, so K can not be a subset of $\alpha^n(U)$, for any $n \in \mathbb{N}$, which implies that α can not be dilative. □

Remark 4.2. It is noteworthy that if the discrete group G is finite (for instance the group of integers modulo n, i.e. $_n$), then G is also compact in its topology and hence we can apply the theory of [13] to construct an MRA or a Riesz MRA for such groups. We also find it necessary to mention here that for the groups except the ones mentiond in this section (for instance the Euclidean group $\mathbb{R}^n$), the theory of [13] fails and our theory needs to be applied.

Since a discrete group can be considered as a special case of a disconnected group, therefore, it is worthwhile to generalize the above result for disconnected groups. We recall that the component $C(x)$ of $x \in G$ is the union of all connected subsets of G which contain x [18]. Thus, we can say that a component $C(x)$ is the maximal connected subset of G or in other words, it is not properly contained in any connected subset of G.

Theorem 4.3. *Let G be the disconnected LCA group with atleast two components. Then, there exist no dilative automorphism α of G.*

Proof. Let G_1 and G_2 be two components of the disconnected LCA group G. Without loss of generality, we assume that $0 \in G_1$. Let U be an open neighbourhood of $0 \in G$ contained in G_1 and K be a compact subset of G entirely contained within G_2. Then, for any automorphism α of G, $\alpha^n(U), n \in \mathbb{N}$ is always contained in G_1. This means that

$$K \cap \alpha^n(U) = \emptyset, \quad \forall\, n \in \mathbb{N}. \tag{4.3}$$

Hence, there exist no dilative automorphism α of G. $\square$

For the lucid illustration of the above result, we present an example of a disconnected LCA group G having four components.

Example 4.3. Consider the set

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R},\ abc > 0 \right\}.$$

Then, it is easy to verify that G is LCA group under the matrix multiplications and induced topology of the Euclidean space $\mathbb{R}^3$. Moreover, under this topology, the group G is disconnected having the following components:

$$G_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R},\ a, b, c > 0 \right\},$$

$$G_2 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R},\ a, b < 0, c > 0 \right\},$$

$$G_3 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R},\ a, c < 0, b > 0 \right\},$$

$$G_4 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R},\ a > 0, b, c < 0 \right\}.$$

We also observe that the given group G is algebraically isomorphic and topologically homeomorphic to a subset of $\mathbb{R}^3$ with the following representation:

$$G_1 = \big\{(x, y, z) \in \mathbb{R}^3 : x, y, z > 0\big\}, \qquad G_2 = \big\{(x, y, z) \in \mathbb{R}^3 : x, y < 0, z > 0\big\},$$
$$G_3 = \big\{(x, y, z) \in \mathbb{R}^3 : x, z < 0, y > 0\big\}, \quad G_4 = \big\{(x, y, z) \in \mathbb{R}^3 : x > 0, y, z < 0\big\}.$$

It is quite evident that the identity element $e \sim (1, 1, 1)$ of G belongs to G_1. Let U be an open neighbourhood of e in G entirely contained in G_1 with the following identification in $\mathbb{R}^3$:

$$U \sim \left\{ (x, y, z) \in \mathbb{R}^3 : \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2} < \frac{1}{2} \right\}.$$

If α is any automorphism of G, then α maps any connected set onto connected set in G, that is;

$$\alpha^n(U) \subset G_1, \quad \forall\, n \in \mathbb{Z}. \tag{4.4}$$

Let K be any set in G with the following identification:

$$K \sim \left\{(x, y, z) \in \mathbb{R}^3 : \max\big\{|x+1|, |y+1|, |z-1|\big\} \le \frac{1}{2}\right\}.$$

Clearly, K is compact in G and entirely contained within G_2. Thus, $K \cap \alpha^n(U) = \emptyset$, $n \in \mathbb{Z}$, which means that α can not be dilative automorphism on G.

Список литературы

[1] S. Mallat, “Multiresolution approximations and wavelet orthonormal basis of L2(R)”, Trans. Amer. Math. Soc. **315**, 69 – 87 (1989).

[2] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia (1992).

[3] L. Debnath and F. A. Shah, Wavelet Transforms and Their Applications, Birkhäuser, New York (2015).

[4] M. Bownik, “Riesz wavelets and generalized multiresolution analyses”, Appl. Comput. Harmon. Anal., **14**(3), 181 – 194 (2003).

[5] R. A. Zalik, “On MRA Riesz wavelets”, Proc. Amer. Math. Soc., **135**(3), 787 – 793 (2007).

[6] F. A. Shah and Abdullah, “Nonuniform multiresolution analysis on local fields of positive characteristic”, Complex Anal. Opert. Theory, **9**, 1589 – 1608 (2015).

[7] H. M. Srivastava, F. A. Shah and W. Z. Lone, “Fractional nonuniform multiresolution analysis in $L^2(\mathbb{R})$”, Math. Methods Appl. Sci., **44**(11), 9351 – 9372 (2021).

[8] S. Dahlke, “Multiresolution analysis and wavelets on locally compact Abelian groups”, Wavelets, Images, and Surface Fitting, eds. P.G., Laurent et al.; A. K. Peters, Wellesley, 141 – 156, (1993).

[9] W. C. Lang, “Orthogonal wavelets on the Cantor dyadic group”, SIAM J. Math. Anal., **27**, 305 – 312 (1996).

[10] R. A. Kamyabi-Gol and R. R. Tousi, “Some equivalent multiresolution conditions on locally compact Abelian groups”, Proc. Indian Acad. Sci. Math. Sci., **120**(3), 317 – 331 (2010).

[11] R. A. Kamyabi-Gol and R. R. Tousi, “The structure of shift-invariant spaces on a locally compact Abelian group”, J. Math. Anal. Appl., **340**, 219 – 225 (2008).

[12] Q. Yang and K. F. Taylor, “Multiresolution analysis and Haar-like wavelet bases on locally compact groups”, J. Appl. Funct. Anal., **7**(4), 413 – 439 (2012).

[13] M. Bownik and Q. Jahan, “Wavelets on compact Abelian groups”, Appl. Comput. Harmon. Anal., **49**(2), 471 – 494 (2020).

[14] R. Kumar and Satyapriya, “Construction of a frame multiresolution analysis on locally compact Abelian groups”, Aust. J. Math. Anal. Appl. **18**, Article 5. 19 pages (2021).

[15] G. B. Folland, A Course in Abstract Harmonic Analysis, CRC Press, Boca Raton, Florida (1995).

[16] O. Christensen, An introduction to Frames and Riesz Bases, second edition, Birkhäuser, Boston (2016).

[17] C. Cabrelli and V. Paternostro, “Shift-invariant spaces on LCA groups”, J. Funct. Anal., **258**, no. 6, 2034 – 2059 (2010).

[18] T. B. Singh, Introduction to Topology, Springer, Singapore (2013).

Поступила 16 ноября 2021

После доработки 11 февраля 2022

Принята к публикации 13 февраля 2022