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INVESTIGATION OF CONVEX BODIES IN $\mathbf{R}^3$ BY SUPPORT PLANES

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Abstract. Let $\mathbf{R}^3$ be the 3-dimensional Euclidean space and $\mathbf{D}$ be a bounded convex body $D \subset \mathbf{R}^3$. Consider a family of support planes for which $\mathbf{D}$ is an envelope. How we can obtain information about D from the support planes? Conditions under which a given convex body is the envelope of a family of planes are obtained. Therefore the distances of these planes from the origin will be the support functions of this body D. In particular, we have cited expressions for the surface area and the volume of body $\mathbf{D}$ in terms of support planes.

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1. Introduction

Complicated geometrical patterns occur in many areas of science (see [1], [2] and [14]). Their analysis requires creation of mathematical models and development of special mathematical tools (see [2], [12] and [14]). The methods of form analysis are based on analysis of the objects as subsets of the n-dimensional Euclidean space $\mathbf{R}^n$. For these sets, geometrical characteristics are considered that are independent of the position and orientation of the sets (hence they coincide for congruent subsets). Classical examples are surface area and volume of a set (see [14] and [15]).

Let $\mathbf{D} \subset \mathbf{R}^n$ be a bounded convex body with inner points and the origin belongs to D. Reconstruction of a bounded convex body $\mathbf{D}$ is one of the main problem in stochastic geometry. Many authors have explored bounded convex bodes by covariogram, distribution function of the distance between two points and orientation-dependent chord length distribution function. It is known that we can reconstruct bodies in $\mathbf{R}^n$ by above mentioned methods when $n = 2$ (see [13]) and we can't do it when $n \geq 4$ (see [11], [12] and [15]), but when $n = 3$ it is an open problem. This problem has been investigated in [4] - [10]. Reconstruction of convex bodies by planes or lines in $\mathbf{R}^3$ is the interesting problem from stochastic geometry (see [1], [2] and [12]).

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We investigate bounded convex body **D** as the envelop of a family of support planes. Let's consider a family of planes in $\mathbf{R}^3$. The plane in $\mathbf{R}^3$ may be determined by its (p, θ, φ) spherical coordinates, that is p is the distance from the origin to the plane, θ is the angle between the normal to the plane and OZ-axis, and φ is the angle between the projection of the normal in XOY-plane and OX-axis. Then the equation of the plane in spherical coordinates have the following form:

$$x\,\sin\theta\,\cos\varphi + y\,\sin\theta\,\sin\varphi + z\,\cos\theta - p = 0, \qquad \theta \in [0,\pi], \quad \varphi \in S^1, \tag{1.1}$$

where S^1 is 1-dimensional sphere of radius 1 and centered at the origin (S^1 is the space of all directions in the plane).

If p is a function $p = p(\theta, \varphi)$, then (1.1) is represent a family of planes and we can find the envelope of this family of planes. If we assume that $p(\theta, \varphi)$ is differentiable, the envelope of the family is obtained from (1.1) and the partial derivatives

$$-x\sin\theta\,\sin\varphi + y\sin\theta\,\cos\varphi - \frac{\partial p}{\partial\varphi} = 0, \tag{1.2}$$

$$x\cos\theta\,\cos\varphi + y\cos\theta\,\sin\varphi - z\sin\theta - \frac{\partial p}{\partial\theta} = 0. \tag{1.3}$$

If we solve the system (1.1),(1.2) and (1.3) we get the parametric representation of the envelope of the families of planes (1.1), which are support planes for D:

$$x = p(\theta,\varphi)\,\sin\theta\,\cos\varphi - \frac{\sin\varphi}{\sin\theta}\,\frac{\partial p}{\partial\varphi} + \cos\theta\,\cos\varphi\frac{\partial p}{\partial\theta}, \tag{1.4}$$

$$y = p(\theta,\varphi)\,\sin\theta\sin\varphi + \frac{\cos\varphi}{\sin\theta}\,\frac{\partial p}{\partial\varphi} + \cos\theta\sin\varphi\frac{\partial p}{\partial\theta}, \tag{1.5}$$

$$z = p(\theta,\varphi)\,\cos\theta - \sin\theta\,\frac{\partial p}{\partial\theta}. \tag{1.6}$$

Formulae (1.1) - (1.3) give the coordinates $(x(\theta,\varphi); y(\theta,\varphi)); z(\theta,\varphi)))$ of point P at which the plane tangent the envelope D. Coordinates of the point H at which the perpendicular from the Origin O intersects the plane is $(p\sin\theta\, cos\varphi; p\sin\theta\,\cos\varphi;\, p\cos\theta)$. It follows from it, that the segment HP has the following length:

$$|HP| = \sqrt{\left(\frac{\partial p}{\partial\varphi}\right)^2\frac{1}{\sin^2\theta} + \left(\frac{\partial p}{\partial\theta}\right)^2}.$$

2. The main results

Let $S(D)$ be the surface area and $V(D)$ be the volume of D. It is known that the surface area and the volume of the envelope given by the coordinates (1.1) -

(1.3) can calculate by the formulas (see [3]):

$$S(D)=\int_0^{2\pi}\int_0^{\pi}\sqrt{EG-F^2}\,d\theta\,d\varphi \tag{2.1}$$

and

$$V(D)=\int_0^{2\pi}\int_0^{\pi}(x\,A+y\,B+z\,C)\,d\theta\,d\varphi, \tag{2.2}$$

where

$$E=\left(\frac{\partial x}{\partial\varphi}\right)^2+\left(\frac{\partial y}{\partial\varphi}\right)^2+\left(\frac{\partial z}{\partial\varphi}\right)^2, \tag{2.3}$$

$$G=\left(\frac{\partial x}{\partial\theta}\right)^2+\left(\frac{\partial y}{\partial\theta}\right)^2+\left(\frac{\partial z}{\partial\theta}\right)^2, \tag{2.4}$$

$$F=\frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\theta}+\frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\theta}+\frac{\partial z}{\partial\varphi}\frac{\partial z}{\partial\theta}, \tag{2.5}$$

$$A=\frac{\partial y}{\partial\varphi}\frac{\partial z}{\partial\theta}-\frac{\partial y}{\partial\theta}\frac{\partial z}{\partial\varphi}, \tag{2.6}$$

$$B=\frac{\partial z}{\partial\varphi}\frac{\partial x}{\partial\theta}-\frac{\partial z}{\partial\theta}\frac{\partial x}{\partial\varphi}, \tag{2.7}$$

$$C=\frac{\partial x}{\partial\varphi}\frac{\partial y}{\partial\theta}-\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\varphi}. \tag{2.8}$$

The main result of the present paper is the following statement.

We have a bounded convex body D which is the envelope of the family of planes and the function p is of class C^2 (recall that class C^n means n times continuously differentiable). If $p=p(\varphi\,,\,\theta)$, where $\theta\in[0,\pi]$ and $\varphi\in S^1$, then

$$S(D)=\int_0^{2\pi}\int_0^{\pi}|K(\theta,\varphi)|\,d\theta\,d\varphi, \tag{2.9}$$

$$V(D)=\int_0^{2\pi}\int_0^{\pi}p\,K(\theta,\varphi)\,d\theta\,d\varphi, \tag{2.10}$$

where

$$K(\theta,\varphi)=p^2\sin\theta-\frac{\cos^2\theta}{\sin^3\theta}\left(\frac{\partial p}{\partial\varphi}\right)^2+p\sin\theta\frac{\partial^2 p}{\partial\theta^2}+p\,\cos\theta\,\frac{\partial p}{\partial\theta}+p\,\frac{1}{\sin\theta}\,\frac{\partial^2 p}{\partial\varphi^2}+$$
$$+2\,\frac{\cos\theta}{\sin^2\theta}\,\frac{\partial p}{\partial\varphi}\frac{\partial^2 p}{\partial\theta\partial\varphi}-\frac{1}{\sin\theta}\left(\frac{\partial^2 p}{\partial\theta\partial\varphi}\right)^2+\frac{1}{\sin\theta}\frac{\partial^2 p}{\partial\theta^2}\frac{\partial^2 p}{\partial\varphi^2}+\cos\theta\,\frac{\partial p}{\partial\theta}\frac{\partial^2 p}{\partial\theta^2}.$$

Proof. Using (1.4)-(1.6) we can calculate $\frac{\partial x}{\partial\varphi}$, $\frac{\partial x}{\partial\theta}$, $\frac{\partial y}{\partial\varphi}$, $\frac{\partial y}{\partial\theta}$, $\frac{\partial z}{\partial\varphi}$, $\frac{\partial z}{\partial\theta}$ partial derivatives:

$$\frac{\partial x}{\partial\varphi}=-p\,\sin\theta\,\sin\varphi-\frac{\cos\varphi\cos^2\theta}{\sin\theta}\,\frac{\partial p}{\partial\varphi}- \tag{2.11}$$
$$-\cos\theta\,\sin\varphi\frac{\partial p}{\partial\theta}-\frac{\sin\varphi}{\sin\theta}\,\frac{\partial^2 p}{\partial\varphi^2}+\cos\theta\,\cos\varphi\frac{\partial^2 p}{\partial\theta\partial\varphi},$$

$$\frac{\partial x}{\partial \theta} = p\,\cos\theta\,\cos\varphi + \frac{\sin\varphi\cos\theta}{\sin^2\theta}\,\frac{\partial p}{\partial \varphi} + \cos\theta\,\cos\varphi\frac{\partial^2 p}{\partial \theta^2} - \frac{\sin\varphi}{\sin\theta}\,\frac{\partial^2 p}{\partial\varphi\partial\theta}, \tag{2.12}$$

$$\frac{\partial y}{\partial \varphi} = p\,\sin\theta\,\cos\varphi - \frac{\sin\varphi\cos^2\theta}{\sin\theta}\,\frac{\partial p}{\partial \varphi} + \tag{2.13}$$

$$\cos\theta\,\cos\varphi\frac{\partial p}{\partial \theta} + \frac{\cos\varphi}{\sin\theta}\,\frac{\partial^2 p}{\partial \varphi^2} + \cos\theta\,\sin\varphi\frac{\partial^2 p}{\partial\theta\partial\varphi},$$

$$\frac{\partial y}{\partial \theta} = p\,\cos\theta\,\sin\varphi - \frac{\cos\varphi\cos\theta}{\sin^2\theta}\,\frac{\partial p}{\partial \varphi} + \cos\theta\,\sin\varphi\frac{\partial^2 p}{\partial \theta^2} + \frac{\cos\varphi}{\sin\theta}\,\frac{\partial^2 p}{\partial\varphi\partial\theta}, \tag{2.14}$$

$$\frac{\partial z}{\partial \varphi} = \cos\theta\,\frac{\partial p}{\partial \varphi} - \sin\theta\,\frac{\partial^2 p}{\partial\theta\partial\varphi}, \tag{2.15}$$

$$\frac{\partial z}{\partial \theta} = -p\,\sin\theta - \sin\theta\,\frac{\partial^2 p}{\partial \theta^2}. \tag{2.16}$$

Using (2.11)-(2.16) and (2.3) - (2.8) we get the following representations for E, G, F, A, B and C.

$$E = \,p^2\sin^2\theta \,+\, 2\,p\,\frac{\partial^2 p}{\partial \varphi^2} \,+\, 2\,p\,\frac{\partial p}{\partial \theta}\,sin\theta\,\cos\theta \,+\, \frac{\cos^2\theta}{\sin^2\theta}\left(\frac{\partial p}{\partial \varphi}\right)^2 -$$

$$-\,2\,\frac{\cos\theta}{\sin\theta}\,\frac{\partial p}{\partial \varphi}\,\frac{\partial^2 p}{\partial\theta\partial\varphi} + \frac{1}{\sin^2\theta}\,\left(\frac{\partial^2 p}{\partial \varphi^2}\right)^2 + 2\,\frac{\cos\theta}{\sin\theta}\,\frac{\partial p}{\partial \theta}\,\frac{\partial^2 p}{\partial^2 \varphi} + \left(\frac{\partial^2 p}{\partial\theta\partial\varphi}\right)^2 + \cos^2\theta\,\left(\frac{\partial p}{\partial \theta}\right)^2,$$

$$G = p^2 + 2\,p\,\frac{\partial^2 p}{\partial \theta^2} + \frac{1}{\sin^2\theta}\,\left(\frac{\partial^2 p}{\partial\varphi\partial\theta}\right)^2 - 2\,\frac{\cos\theta}{\sin^3\theta}\,\frac{\partial p}{\partial \varphi}\,\frac{\partial^2 p}{\partial\varphi\partial\theta} + \frac{\cos^2\theta}{\sin^4\theta}\,\left(\frac{\partial p}{\partial \varphi}\right)^2 + \left(\frac{\partial^2 p}{\partial \theta^2}\right)^2,$$

$$F = p\,\frac{\partial^2 p}{\partial\varphi\partial\theta} \,+\, p\,\frac{\partial^2 p}{\partial\theta\partial\varphi} \,+\, \frac{\partial^2 p}{\partial\theta\partial\varphi}\,\frac{\partial^2 p}{\partial \theta^2} \,-\, 2\,\frac{\cos\theta}{\sin\theta}\,p\,\frac{\partial p}{\partial \varphi} \,-\, \frac{\cos\theta}{\sin\theta}\,\frac{\partial p}{\partial \varphi}\,\frac{\partial^2 p}{\partial \theta^2} +$$

$$+\,\frac{1}{\sin^2\theta}\,\frac{\partial^2 p}{\partial \varphi^2}\,\frac{\partial^2 p}{\partial\varphi\partial\theta} \,-\, \frac{\cos\theta}{\sin^3\theta}\,\frac{\partial p}{\partial \varphi}\,\frac{\partial^2 p}{\partial \varphi^2} \,+\, \frac{\cos\theta}{\sin\theta}\,\frac{\partial p}{\partial \theta}\,\frac{\partial^2 p}{\partial\varphi\partial\theta} \,-\, \frac{\cos^2\theta}{\sin^2\theta}\,\frac{\partial p}{\partial \theta}\,\frac{\partial p}{\partial \varphi},$$

$$A = -\cos\varphi\,\sin\theta\,K(\theta,\varphi),$$

$$B = -\sin\varphi\,\sin\theta\,K(\theta,\varphi),$$

$$C = -\cos\theta\,K(\theta,\varphi).$$

Therefore, we use (2.1) and (2.2) to calculate $S(D)$ and $V(D)$. We need to calculate $\sqrt{EG-F^2}$ and $x\,A + y\,B + z\,C$. Using before mentioned representations of E, G, F, A, B, C and (1.4) - (1.6) we get

$$EG - F^2 \,=\, K^2(\theta,\varphi) \quad \text{and} \quad x\,A + y\,B + z\,C \,=\, p\,K(\theta,\varphi).$$

Inserting these results in (2.1) and (2.2) we get (2.9) and (2.10) for surface area $S(D)$ and volume $V(D)$. □

In particular case, where $p(\varphi,\theta) = p(\theta)$

$$S(D) = 2\pi\int_0^\pi \left(p^2\,\sin\theta - \sin\theta\left(\frac{\partial p}{\partial \theta}\right)^2 + cos\theta\,\frac{\partial p}{\partial \theta}\frac{\partial^2 p}{\partial \theta^2}\right)\,d\theta.$$

$$V(D) = 2\pi \int_0^\pi \left(p^3 \sin\theta + p^2 \frac{\partial^2 p}{\partial\theta^2} \sin\theta + p^2 \frac{\partial p}{\partial\theta} \cos\theta + p \frac{\partial p}{\partial\theta} \frac{\partial^2 p}{\partial\theta^2} \cos\theta \right) d\theta.$$

In the case, where p is constant the envelope of the planes is the ball K_p of radius p and $S(K_p)$ is the sphere, we get $S(K_p) = 4\pi p^2$ and $V(K_p) = \frac{4}{3}\pi p^3$. Note that for the planar case, the corresponding results can be found in [1].

Список литературы

[1] L. A. Santalo, Integral Geometry and Geometric Probability, Addision-Wesley, Reading (2004).

[2] R. Schneider and W. Weil, Stochastic and Integral Geometry, Springer, Berlin (2008).

[3] V. A. Zalgaller, The Theory of Envelopes [In Russian], Moscow (1975).

[4] N. G. Aharonyan and V. H. Khalatyan, “Distribution of the distance between two random points in a body from R^n”, Journal of Contemporary Mathematical Analysis, **55**, no. 6, 329 -–334 (2020).

[5] V. K. Ohanyan and D. M. Martirosyan, “Orientation-dependent chord length distribution function for right prisms with rectangular or right trapezoidal bases”, Journal of Contemporary Mathematical Analysis, **55**, no. 6, 344 -– 355 (2020).

[6] N. G. Aharonyan and H. O. Harutyunyan, “Geometric probability calculation for a triangle”, Proceedings of the YSU. Physical and Mathematical Sciences, **51**, 211 -– 216 (2017).

[7] H. S. Harutyunyan and V. K. Ohanyan, “The chord length distribution function for regular polygons”, Advances in Applied Probability, **41**, 358 -– 366 (2009).

[8] A. G. Gasparyan and V. K. Ohanyan, “Covariogram of a parallelogram”, Journal of Contemporary Mathematical Analysis, **49**, 194 -– 206 (2014).

[9] H. S. Harutyunyan and V. K. Ohanyan, “Covariogram of a cylinder”, Journal of Contemporary Mathematical Analysis, **49**, 366 -– 375 (2014).

[10] A. G. Gasparyan and V. K. Ohanyan, “Recognition of triangles by covariogram”, Journal of Contemporary Mathematical Analysis, **48**, 110 -– 122 (2013).

[11] G. Averkov and G. Bianchi, “Retrieving convex bodies from restricted covariogram functions”, Advances in Applied Probability, **39**, 613 – 629 (2007).

[12] R. J. Gardner, Geometric Tomography, Cambridge University Press, Cambridge, UK, 2ed edition (2006).

[13] G. Bianchi and G. Averkov, “Confirmation of Matheron’s conjecture on the covariogram of a planar convex body”, Journal of the European Mathematical society **11**, 1187 – 1202 (2009).

[14] D. Stoyan and H. Stoyan, Fractals, Random Shapes and Point Fields, Methods of Geometrical Statistics, Chichester, New York, John Wiley Sons XIV, 389 p. (1994).

[15] R. J. Gardner, P. Gronche and C. Zong, “Sums, projections, and sections of lattice sets, and the discrete covariogram”, Discrete Comput. Geom, **34**, 391 – 409 (2005).

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