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MENSHOV-TYPE THEOREM FOR DIVERGENCE SETS OF SEQUENCES OF LOCALIZED OPERATORS

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Abstract. For a multiparameter sequence of suitably localized operators $U_{\overline{n}} : L^1[0,1]^d \to M[0,1]^d$, $\overline{n} \in \mathbb{N}^p$, and a countable set $D \subset [0,1]^d$, we construct a function $f \in M[0,1]^d$ such that the sequence $\{U_{\overline{n}} f(x), \overline{n} \in \mathbb{N}^p\}$ diverges for $x \in D$ and converges for $x \in [0,1]^d \setminus D$, where the convergence is understood in rectangular sense. We also obtain a corresponding Menshov-type theorem. Examples of sequences of operators under consideration include tensor products of univariate orthonormal projections onto splines with arbitrary knots of order k, as well as tensor products of classical operators, as (C,α), $\alpha > 0$ means of Fourier partial sums or $(C,1)$ means of Walsh partial sums.

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1. Introduction

Let $\{f_n(\cdot)\}_{n=1}^{\infty}$ be a sequence of almost everywhere finite measurable functions defined on $[0,1]$. A set $D \subset [0,1]$ is called *a divergence set of the series* $\sum_{n=1}^{\infty} f_n(\cdot)$, if that series is divergent for all $x \in D$, and it is convergent for all $x \notin D$.

The case of particular interest is when the sequence $\{f_n(\cdot)\}_{n=1}^{\infty}$ is a sequence of terms of expantion of some $f \in L^2[0,1]$ with respect to some complete orthonormal sequence $\Phi = \{\phi_n, n \in \mathbb{N}\}$. That is, $f_n = (f,\phi_n)\phi_n$, where $(f,\phi_n) = \int_0^1 f(x)\phi_n(x)dx$. One of the classical cases is when Φ is the trigonometric system. Several classical results concerning pointwise convergence and divergence of Fourier series can be found e.g. in classical monographs A. Zygmund [1], or N.K. Bari [2], see also e.g. S. B. Stechkin [3], K. Zeller [4], L.V. Taikov [5], J.-P. Kahane, Y. Katznelson [6], V.V. Buzdalin [7], [8] for other results in this direction. Similar type of results for the Walsh series can be found e.g. in Sh.V. Kheladze [9], [10], U. Goginava [11].

Another classical system is the Haar system. It is well known that the behaviour of Fourier-Haar series of functions from $C[0,1]$ or $L^1[0,1]$ is different than respective

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Fourier or Fourier-Walsh series, in particular: (i) if $f \in C[0,1]$, then its Haar series converges uniformly to f on $[0,1]$, (ii) if $f \in L^1[0,1]$, then its Haar series converges to f almost everywhere on $[0,1]$. The question of characterization of divergence sets of Haar series has been completely solved by G. A. Karagulyan [12]: a set $D \subset [0,1]$ is a set of divergence of the Haar series of a function $f \in L^\infty[0,1]$ if and only if D is a $G_{\delta\sigma}$ set of measure 0. (Note that – because of (ii) and the fact that Haar functions are piecewise constant – a divergence set of the Haar series of any $f \in L^1[0,1]$ must be a $G_{\delta\sigma}$ set of measure 0, cf. the necessity part of Theorem 1 of [12].) Some earlier related results can be found in V. Prokhorenko [13], M.A. Lunina [14], V. M. Bugadze [15].

G.A. Karagulyan [16], [17] continued this line of investigation for sequences of operators with localization property. In particular, he proved the following:

Theorem A ([17], Theorem 1)**.** Let $M[0,1]$ be the space of bounded measurable functions on $[0,1]$, with the norm $\|f\|_M = \sup_{x\in[0,1]} |f(x)|$. Let $U_n : L^1[0,1] \to M[0,1]$ be a sequence of linear operators satisfying the following conditions:

- C1) $\rho_n = \|U_n : L^1[0,1] \to M[0,1]\| < \infty$ for each $n \in \mathbb{N}$.
- C2) $\rho = \sup_{n\in\mathbb{N}} \|U_n : L^\infty[0,1] \to M[0,1]\| < \infty$.
- C3) If $f \in M[0,1]$ is such that $f(x) = 1$ on $(a,b) \subset [0,1]$, then $U_n f(x) \to 1$ for all $x \in (a,b)$, where the convergence is uniform on compact subsets of (a,b).
- C4) $U_n f(x) \to f(x)$ a.e. for $f \in L^\infty[0,1]$.

Let D be a $G_{\delta\sigma}$ set of measure 0. Then there is a function $f \in L^\infty[0,1]$ such that $U_n(x,f) \to f(x)$ for $x \in [0,1] \setminus D$, and the sequence $\{U_n(x,f), n \geq 1\}$ is divergent for $x \in D$.

The restriction that D is a $G_{\delta\sigma}$ set is natural, as the range of operators U_n may be a subspace of $C[0,1]$, see also Theorem 2 of [17] for more general situation.

We are interested in extending Theorem A to the setting of operators $U_{\overline{n}} = U^{(1)}_{n_1} \otimes \ldots \otimes U^{(d)}_{n_d}$, where $\overline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, and for each $1 \leq i \leq d$ the sequence of operators $\{U^{(i)}_{n_i}, n_i \in \mathbb{N}\}$ satisfies some variant of localization conditions analogous to those of Theorem A, and the convergence is understood in rectangular sense, i.e. as $\min(\overline{n}) \to \infty$.

For comparison, let us remind some facts concerning pointwise convergence of expansions of functions on $[0,1]^d$ with respect to tensor products of univariate complete ortonormal systems. In case of the tensor product Haar system, both rectangular and spherical partial sums of $f \in L(\log^+ L)^{d-1}$ converge to f a.e., and this result is sharp. For the rectangular partial sums, the convergence results follow by B. Jessen, J. Marcinkiewicz, A. Zygmund [18], while the divergence results are contained in S. Saks [19]. For the spherical partial sums, the convergence results can be found in G.G. Kemkhadze [20], while the divergence results can be found in G.E. Tkebuchava [21] in case of $d = 2$ and G. Oniani [22] for general $d \geq 2$, the latter paper containing several other divergence results. Note that in the one-dimensional case the Fourier-Haar series of every $f \in L^1[0, 1]$ converges almost everywhere on $[0, 1]$. The Fourier series of any function $f(x) \in L^2[\mathbb{T}]$ converges almost everywhere on $\mathbb{T}$. On the other hand, for the trigonometric system on $\mathbb{T}^2$, C. Fefferman [23] showed the existence of $f \in C(\mathbb{T}^2)$ with rectangular partial sums diverging everywhere on $\mathbb{T}^2$. More results in this direction can be found in G. Gát, G. Karagulyan [24].

In this note, we present a partial result in the direction of extending Theorem A to the multivariate and multiparameter setting, in the very special case that the divergence set $D \subset [0, 1]^d$ is countable. This is the content of the main results of this note, Theorems 1.1 and 1.2. Recall that a construction of a function $f \in C(\mathbb{T})$ with its Fourier series divergent exactly on a given countable set $D \subset \mathbb{T}$ is one of classical results, cf. e.g. [2], chapter IV, section 21, or [1], vol 1, chapter VIII, a remark following the proof Theorem 1.16. In case of the Haar system, construction of $f \in L^\infty[0, 1]$ with its Haar series divergent on exactly a given countable set $D \subset [0, 1]$ is one of the results in V.I. Prokhorenko [13].

The paper is organized as follows. In section 1.1, we formulate the setting of the problem and the main results, i.e. Theorems 1.1 and 1.2. Then, in section 2, we give examples of sequences of kernels and corresponding operators satisfying assumptions of Theorems 1.1 and 1.2. We begin with one-parameter examples. In the univariate and one-parameter case, we are mostly interested in sequences of orthogonal projections on spline spaces with arbitrary knots. We describe this example in detail in section 2.1, in particular in Proposition 2.1. Some other classical one-parameter examples are mentioned in section 2.2. Then, by the usual tensor product procedure, starting from low-dimensional and low-parameter kernels satisfying assumptions of Theorems 1.1 and 1.2, it is possible to get higher-dimensional and higher-parameter examples. This is described in section 2.3. Finally, in section 3, we proceed with the proofs of Theorems 1.1 and 1.2.

1.1. **Setting of the problem and formulation of the result.** To formulate the result, we need to introduce some notation. By $\mu = \mu_d$ we denote the Lebesgue measure on $[0, 1]^d$ (or on $\mathbb{R}^d$, $\mathbb{T}^d$, when we switch to the setting of $\mathbb{R}^d$ or $\mathbb{T}^d$), and

$\|\cdot\|$ denotes the euclidean norm on $\mathbb{R}^d$. For $E \subset [0,1]^d$, we put $E^c = [0,1]^d \setminus E$. By $M[0,1]^d$ we denote the space of bounded functions on $[0,1]^d$, with the norm $\|f\|_M = \sup_{x\in[0,1]^d} |f(x)|$. By $\mathrm{supp}(f)$ we denote the set-theoretic support of function f, i.e.

$$\mathrm{supp}(f) := \{x\colon f(x) \neq 0\}. \tag{1.1}$$

For $\overline{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$, denote $\min(\overline{n}) = \min(n_1, \ldots, n_d)$.

Consider a family of d-variate kernels, parametrized by $\overline{n} \in \mathbb{N}^p$,

$$K_{\overline{n}} : [0,1]^d \times [0,1]^d \to \mathbb{R}, \quad \overline{n} \in \mathbb{N}^p \tag{1.2}$$

with the following localization properties:

(k.1) Each K_n is measurable with respect to the σ-field on $[0,1]^d \times [0,1]^d$, which is a product of d-variate Lebesgue σ-fields on $[0,1]^d$.

(k.2) Boundedness: for each $\overline{n} \in \mathbb{N}^p$, there is $\gamma_{\overline{n}}$ such that $|K_{\overline{n}}(x,y)| \leq \gamma_{\overline{n}}$ for all $x, y \in [0,1]^d$.

(k.3) For each $x \in [0,1]^d$, there is a number $w(x) \in (0,\infty)$ such that $\int_{[0,1]^d} |K_{\overline{n}}(x,y)|dy \leq w(x)$ for all $\overline{n} \in \mathbb{N}^p$.

(k.4) Partition of unity: $\int_{[0,1]^d} K_{\overline{n}}(x,y)dy = 1$ for all $\overline{n} \in \mathbb{N}^p$ and $x \in [0,1]^d$.

(k.5) Localization: for each $x \in [0,1]^d$ and $\delta > 0$

$$\lim_{\min(\overline{n})\to\infty} \int_{\{y\in[0,1]^d : \|x-y\|>\delta\}} |K_{\overline{n}}(x,y)|dy = 0,$$

where for $\overline{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p$, we denote $\min(\overline{n}) = \min(n_1, \ldots, n_d)$.

Consider sequence of operators $U_{\overline{n}}$, corresponding to kernels $K_{\overline{n}}$, given by formula

$$U_{\overline{n}}f(x) = \int_{[0,1]^d} K_{\overline{n}}(x,y)f(y)dy. \tag{1.3}$$

The above properties of kernels $K_{\overline{n}}$ imply the following:

(v.1) Each $U_{\overline{n}} : L^1[0,1]^d \to M[0,1]^d$ is a bounded linear operator with

$$\|U_{\overline{n}} : L^1[0,1]^d \to M[0,1]^d\| \leq \gamma_{\overline{n}}.$$

(v.2) If $x \in [0,1]^d$ is a continuity point of $f \in M[0,1]^d$, then $\lim_{\min(\overline{n})\to\infty} U_{\overline{n}}f(x) = f(x)$.

Indeed, (v.1) follows by (k.1) and (k.2), while (v.2) is a consequence of (k.3), (k.4) and (k.5).

In this setting, we prove the following:

Theorem 1.1. *Let $\{K_{\overline{n}}, \overline{n} \in \mathbb{N}^p\}$ be a sequence of kernels as in* (1.2)*, satisfying conditions (k.1)-(k.5), and let $\{U_{\overline{n}}, \overline{n} \in \mathbb{N}^p\}$ be a sequence of corresponding operators given by formula* (1.3)*. Let $D \subset [0,1]^d$ be a countable set. Let $\varepsilon > 0$, and fix a* *sequence* $\{\overline{n}_r, r \geq 1\}$ *of elements of* $\mathbb{N}^p$ *with* $\lim_{r\to\infty} \min(\overline{n}_r) = \infty$*. Then there is a real-valued function* $f \in M[0,1]^d$ *such that*

- $\mu\{x \in [0,1]^d \colon f(x) \neq 0\} < \varepsilon$,
- f *is continuous at each point* $x \notin D$*, and consequently,* $U_{\overline{n}} f(x) \to f(x)$ *as* $\min(\overline{n}) \to \infty$,
- *the sequence* $\{U_{\overline{n}_r} f(x), r \geq 1\}$ *is divergent for all* $x \in D$.

Concerning the countinuity properties of f as in Theorem 1.1, observe that because of (v.2), any function $f \in M[0,1]^d$ such that the sequence $\{U_{\overline{n}} f(x), \overline{n} \in \mathbb{N}^p\}$ is divergent, cannot be continuous at x.

Because of (v.2), Theorem 1.1 implies the following Menshov-type theorem:

Theorem 1.2. *Let* $\{K_{\overline{n}}, \overline{n} \in \mathbb{N}^p\}$ *be a sequence of kernels as in* (1.2)*, satisfying conditions (k.1)-(k.5), and let* $\{U_{\overline{n}}, \overline{n} \in \mathbb{N}^p\}$ *be a sequence of corresponding operators given by formula* (1.3)*. Let* $D \subset [0,1]^d$ *be a countable set. Let* g *be a measurable, almost everywhere finite function, defined on* $[0,1]^d$ *and* $\varepsilon > 0$*. Fix a sequence* $\{\overline{n}_r, r \geq 1\}$ *of elements of* $\mathbb{N}^p$ *with* $\lim_{r\to\infty} \min(\overline{n}_r) = \infty$*. Then there is* $f \in M[0,1]^d$ *such that*

- $\mu\{x \in [0,1]^d \colon f(x) \neq g(x)\} < \varepsilon$,
- f *is continuous at each point* $x \notin D$*, and consequently,* $U_{\overline{n}} f(x) \to f(x)$ *as* $\min(\overline{n}) \to \infty$.
- *the sequence* $\{U_{\overline{n}_r} f(x), r \geq 1\}$ *is divergent for all* $x \in D$.

We have formulated Theorems 1.1 and 1.2 for functions and operators on $[0,1]^d$. The same results can be obtained in the setting of $\mathbb{T}$ and $\mathbb{R}^d$. Remarks 1.1 and 1.2 below indicate the necessary changes.

Remark 1.1. Setting of $\mathbb{T}^d$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: conditions (k.1)-(k.5) should be formulated on $\mathbb{T}^d$. In particular, condition (k.5) should be formulated for the periodic distance on $\mathbb{T}^d$, i.e. $\operatorname{dist}_{\mathbb{T}^d}(\overline{x}, \overline{y}) = \sum_{i=1}^d \operatorname{dist}_{\mathbb{T}^1}(x_i, y_i)$, where $\operatorname{dist}_{\mathbb{T}^1}(t,u) = \min(|t-u|, 1-|t-u|)$ for $0 \leq t, u < 1$, and $\overline{x} = (x_1, \ldots, x_d)$, $\overline{y} = (y_1, \ldots, y_d)$. Then, in versions of Theorems 1.1 and 1.2 for $\mathbb{T}^d$, continuity of a function $f : \mathbb{T}^d \to \mathbb{R}$ at a point $\overline{x} \in \mathbb{T}^d$ should be understood in the sense of the distance on $\mathbb{T}^d$. With these adaptations, the conclusions of Theorems 1.1 and 1.2 are true in the $\mathbb{T}^d$ setting.

Remark 1.2. Setting of $\mathbb{R}^d$: conditions (k.1)-(k.5) should be formulated on $\mathbb{R}^d$. Moreover, in the statement of Theorem 1.2 for $\mathbb{R}^d$, we need the following assumption on function g:

$$\mu\{x \in \mathbb{R}^d : |g(x)| > \lambda\} \to 0 \text{ as } \lambda \to \infty. \tag{1.4}$$

With these adaptations, the conclusions of Theorems 1.1 and 1.2 are true in the $\mathbb{R}^d$ setting.

The proofs in the case of $\mathbb{T}^d$ or $\mathbb{R}^d$ follow by the same lines as in case of $[0,1]^d$, so we just comment on them at the end of section 3, but we do not include the details. However, we feel free to discuss examples not only on $[0,1]^d$, but on $\mathbb{T}^d$ or $\mathbb{R}^d$ as well.

2. Examples

Now, we give some examples of kernels satisfying (k.1)-(k.5).

2.1. **Sequences of orthogonal projections onto splines of fixed order.** Basic facts concerning spline spaces discussed below can be found e.g in L.L. Schumaker [25], or R.A. DeVore, G.G. Lorentz [26], Chapter 5.

Fix $k \geq 1$. For $\rho \geq k$, consider a sequence of points $\mathcal{T} = \{t_i, 1 \leq i \leq \rho+k\} \subset [0,1]$ such that

$$t_i \leq t_{i+1}, \quad t_i < t_{i+k}, \quad t_1 = \ldots = t_k = 0, \quad t_{\rho+1} = \ldots t_{\rho+k} = 1.$$

Denote $|\mathcal{T}| = \max\{t_{i+1} - t_i : 1 \leq i \leq \rho + k - 1\}$. We say that a point $\tau \in \mathcal{T}$ has multiplicity m, $1 \leq m \leq k$, if $t_{i-1} < \tau = t_i = \ldots = t_{i+m-1} < t_{i+m}$ for some i.

Let $S_k(\mathcal{T})$ be the space of splines of order k with knots $\mathcal{T}$. That is, if $f \in S_k(\mathcal{T})$, then f is polynomial of degree $\leq k-1$ on each interval (t_i, t_{i+1}), while for $\tau \in \mathcal{T}$ of multiplicity $m < k$, f is of class C^{k-m-1} at τ; in case $k = m$ both left- and right- limits $f(\tau-) = \lim_{t\to\tau-} f(t)$ and $f(\tau+) = \lim_{t\to\tau+} f(t)$ exist, and we put $f(\tau) = f(\tau+)$; the exception is the endpoint $\tau = 1$, where we put $f(1) = f(1-)$.

Recall that each function $f \in S_k(\mathcal{T})$ can be uniquely written as $f = \sum_{i=1}^{\rho} c_i N_i$, where $N_i = N_i^k$ are normalized B-splines on $[0,1]$ of order k with knots $\mathcal{T}$. In particular, $N_i \geq 0$, $\sum_{i=1}^{\rho} N_i = 1$ and $\mathrm{supp} N_i = (t_i, t_{i+k})$, with the exception of the case when t_i is of multiplicity k – in this case $N_i(t_i) = 1$, and $N_1(0) = 1$, $N_\rho(1) = 1$.

Let $P_{\mathcal{T},k}$ be the orthogonal projection onto $S_k(\mathcal{T})$ with respect to the usual inner product in $L^2[0,1]$, i.e. $(f,g) = \int_0^1 f(x)g(x)dx$. Then $P_{\mathcal{T},k}$ is an integral operator, and its Dirichlet kernel $K_{\mathcal{T},k}$ can be written as follows:

$$K_{\mathcal{T},k}(x,y) = \sum_{i,j=1}^{\rho} a_{ij} N_i(x) N_j(y), \tag{2.1}$$

where the coefficients a_{ij} are the entries of the matrix inverse to the Gram matrix $G = [(N_i, N_j) : 1 \leq i,j \leq \rho]$, i.e. $G^{-1} = [a_{ij}, 1 \leq i,j \leq \rho]$. This formula implies in particular properties(k.1), (k.2) and (k.3), while (k.4) follows from the fact that $\chi_{[0,1]} \in S_k(\mathcal{T})$. Moreover, recall the constant $w(x)$ in (k.3) can be chosen as w_k, i.e.

a constant depending on the order of splines k, but not on a point $x \in [0,1]$, nor a sequence of points $\mathcal{T}$. This is a concequence of A.Yu. Shadrin [27], which states that

$$\|P_{\mathcal{T},k} : L^\infty[0,1] \to L^\infty[0,1]\| \le C_k, \tag{2.2}$$

with C_k depending on k, but not on $\mathcal{T}$ (cf. also M.v. Golitschek [28] for a simpified proof of this result).

To discuss condition (k.5), we need to recall some estimates for kernels $K_{\mathcal{T},k}$. First, note that – because of localization of B-splines – if $\tau = t_i = \ldots = t_{i+k-1} \in \mathcal{T}$, $\tau \neq 0,1$, is a point in $\mathcal{T}$ of multiplicity k, then for each $i_1 < i \le i_2$ there is $(N_{i_1}, N_{i_2}) = 0$, and consequently $a_{i_1,i_2} = 0$. It follows that if $x < \tau \le y$, then $K_{\mathcal{T},k}(x,y) = K_{\mathcal{T},k}(y,x) = 0$.

In general, we need to recall some estimates for kernels $K_{\mathcal{T},k}$ from M. Passenbrunner, A. Shadrin [29]. Let i,j be such that $t_i \le x < t_{i+1}$ and $t_j \le y < t_{j+1}$, and denote $I_{ij} = [t_{\min(i,j)}, t_{\max(i,j)+1})$. Then Lemma 2.1 of [29] states that there are $C = C_k$ and $\theta = \theta_k \in (0,1)$, depending on k, but not on $\mathcal{T}$, such that

$$|K_{\mathcal{T},k}(x,y)| \le C \frac{\theta^{|i-j|}}{\mu(I_{ij})}. \tag{2.3}$$

Formula (2.3) implies the following: for $x_1 < x_2$, denote $d_{\mathcal{T}}(x_1,x_2) = \#\{i : x_1 < t_i \le x_2\}$. Then there are a constant $C = C_k$ and $\vartheta = \vartheta_k \in (0,1)$, depending on k, but neither on $\mathcal{T}$ nor x and α, β with $\alpha < x < \beta$, such that

$$\int_{(\alpha,\beta)^c} |K_{\mathcal{T},k}(x,y)| dy \le C \vartheta^{\min\left(d_{\mathcal{T}}(\alpha,x), d_{\mathcal{T}}(x,\beta)\right)}. \tag{2.4}$$

In particular, if $\delta > 0$, and $\alpha = x - \delta$, $\beta = x + \delta$, then

$$\int_{y:|x-y|>\delta} |K_{\mathcal{T},k}(x,y)| dy \le C \vartheta^{\min\left(d_{\mathcal{T}}(x-\delta,x), d_{\mathcal{T}}(x,x+\delta)\right)}, \tag{2.5}$$

or more generally, if $\alpha < \alpha' \le \beta' < \beta$, then

$$\int_{(\alpha,\beta)^c} |K_{\mathcal{T},k}(x,y)| dy \le C \vartheta^{\min\left(d_{\mathcal{T}}(\alpha,\alpha'), d_{\mathcal{T}}(\beta',\beta)\right)} \quad \text{for} \quad x \in [\alpha', \beta']. \tag{2.6}$$

To summarize these considerations, we formulate the following:

Proposition 2.1. *Fix $k \ge 1$. Let $\{\mathcal{T}_n, n \ge 1\}$ be a sequence of partitions with points of multiplicity at most k and with $|\mathcal{T}_n| \to 0$ as $n \to \infty$. Then the kernels $K_{\mathcal{T}_n,k}$ satisfy conditions (k.1)-(k.5), and the sequence of operators $U_n = P_{\mathcal{T}_n,k}$ satisfies assumptions of Theorems 1.1 and 1.2.*

Moreover, the sequence of operators $U_n = P_{\mathcal{T}_n,k}$ satisfies conditions C1)-C4) of Theorem A.

Proof. In fact, we have already seen that conditions (k.1)-(k.4) are satisfied. To check (k.5), observe that if $|\mathcal{T}_n| \to 0$ as $n \to \infty$, then for each x, δ, $\min\big(d_{\mathcal{T}_n}(x-\delta, x), d_{\mathcal{T}_n}(x, x+\delta)\big) \to \infty$ as $n \to \infty$. Combining this observation with formula (2.5) we get (k.5).

Concerning conditions C1)-C4) of Theorem A, note that C1) coincides with (v.1), while C2) follows by the fact that $w(x)$ in (k.3) can be chosen independently of x and $\mathcal{T}$, as explained above. C3) follows by (k.4) and formula (2.6), while C4) is a consequence of Theorems 1.1 or 3.1 of [29]. □

Recall that in case $k = 1$, formula (2.1) can be written explicitly. That is, if $\mathcal{T} = \{t_i, 1 \le i \le \rho+1\}$, with $0 = t_1$, $t_i < t_{i+1}$, $t_{\rho+1} = 1$, and x, y are such that $t_i \le x, y < t_{i+1}$ (or $t_\rho \le x, y \le t_{\rho+1} = 1$ in case $i = \rho$), then $K_{\mathcal{T},1}(x, y) = \frac{1}{t_{i+1}-t_i}$, and $K_{\mathcal{T},1}(x, y) = 0$ otherwise.

Remark 2.1. In case each $\mathcal{T}_{n+1}$ is obtained from $\mathcal{T}_n$ by adding 1 knot, the sequence $P_{\mathcal{T}_n,k}$ is a sequence of partial sums with respect to the corresponding orthonormal spline systems of order k. The case of $k = 1$ corresponds to the general Haar systems, as introduced by A.Haar [30], Chapter III, section 4 (see also e.g. I.Novikov, E. Semenov [31], Chapter 10.b) and the case of $k = 2$ corresponds to general Franklin system, as studied e.g. in G.G. Gevorkyan, A. Kamont [32], [33]. The orthonormal spline systems with arbitrary knots and general $k \ge 1$ were studied in M. Passenbrunner [34]. In case when the sequence of knots is the sequence of dyadic points, we recover classical Haar and Franklin systems (cf. e.g. B.S. Kashin, A.A. Saakyan [35], Chapters III and VI) and orthonormal spline systems with dyadic knots, as studied e.g. by Z. Ciesielski, J. Domsta [36], [37].

Remark 2.2. Let us note that it is possible to consider also a periodic version of this example, i.e. sequences of orthogonal projections onto periodic splines with arbitrary knots. In order to verify that sequences of these operators satisfy the periodic version of conditions (k.1)-(k.5), the essential point is to get a periodic counterpart of the estimates (2.5) or (2.6). This is possible due to the estimate of the elements of the matrix inverse to the Gram matrix of the periodic B-splines, cf. M. Passenbrunner [38], Remark 3.2 (ii). Moreover, combination of these estimates and Theorem 1.3 of [38] implies that sequences of orthogonal projections onto periodic splines satisfy assumptions of the periodic version of Theorem A.

2.2. **Some other one-parameter examples.** Let us recall briefly some other classical examples of sequences of kernels satisfying conditions (k.1)-(k.5).

2.2.1. *Wavelet projections.* For terminlogy, existence of object discussed in this section and their properties we refer e.g. to Y. Meyer [39].

Let ϕ be an orthonormal scaling function on $\mathbb{R}^d$, associated with the dilation matrix $2\mathbb{I}$, where $\mathbb{I}$ denotes $d \times d$ identity matrix, and such that $|\phi(x)| \leq \frac{C}{(\|x\|+1)^{d+\epsilon}}$ with some $\epsilon > 0$. Then $K_0(x,y) = \sum_{k\in\mathbb{Z}^d} \phi(x-k)\overline{\phi}(y-k)$ is the Dirichlet kernel of the orthogonal projection onto $V_0 = \mathrm{span}_{L^2(\mathbb{R}^d)}\{\phi(\cdot - k), k \in \mathbb{Z}^d\}$, and it satisfies the estimate $|K(x,y)| \leq \frac{C}{(\|x-y\|+1)^{d+\epsilon}}$. Moreover, $\int_{\mathbb{R}^d} K_0(x,y)dy = 1$ for all $x \in \mathbb{R}^d$. Then $K_n(x,y) = 2^{nd}K_0(2^n x, 2^n y)$ is the Dirichlet kernel of the orthogonal projection onto $V_n = \mathrm{span}_{L^2(\mathbb{R}^d)}\{2^{nd/2}\phi(2^n \cdot -k), k \in \mathbb{Z}^d\}$. Then the sequence of kernels $\{K_n, n \in \mathbb{N}\}$ satisfies conditions (k.1)-(k.5).

2.2.2. *Cesaro (C,α), $\alpha > 0$, means of partial sums with respect to the trigonometric system.* This is a classical example. For precise formulae, we refer e.g. to [1], vol. I, Chapter III, section 5. Conditions (k.1) and (k.4) are clear, while (k.2), (k.3) and (k.5) can be seen as a consequence of formula (5.5) in [1], vol. I, Chapter III, section 5.

2.2.3. *Cesaro $(C,1)$ means of partial sums with respect to the Walsh system.* For definition and properties of the Walsh system we refer e.g. to F. Schipp, W.R. Wade, P. Simon, J. Pál [40], or B. Golubov, A. Efimov, V, Skvortsov [41]. Here, we refer to the Walsh system in the Walsh-Paley order. Again, properties (k.1), (k.2) and (k.4) are clear, and (k.3) and (k.5) are consequences e.g. of [40], Chapter 1, Theorem 16: uniform version of (k.3) is a content Theorem 16 v), while (k.5) is a consequence of the pointwise estimate for the Walsh-Fejer kernel as in Theorem 16 iv), see also [41], Chapter 4, section 4.2, from the formulas 4.2.6 and 4.2.10 of this section follows that the Walsh-Fejer kernel satisfies the conditions (k.1)-(k.5).

2.3. **Examples via tensor product procedure.** Observe that tensor product of kernels satisfying conditions (k.1)-(k.5) gives again a kernel – with more variables and more parameters – satisfying conditions (k.1)-(k.5). For the sake of completeness, we formulate this observation as Fact 2.1 below.

Fix $k \geq 1$. For each $1 \leq j \leq k$, we are given kernels

$$K_{\overline{n}_j} : [0,1]^{d_j} \times [0,1]^{d_j} \to \mathbb{R}, \quad \overline{n}_j \in \mathbb{N}^{p_j}.$$

Let $d = d_1 + \ldots + d_k$ and $p = p_1 + \ldots + p_k$. For $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_k)$, $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_k)$ with $\overline{x}_j, \overline{y}_j \in [0,1]^{d_j}$ and $\overline{n} = (\overline{n}_1, \ldots, \overline{n}_k)$ with $\overline{n}_j \in \mathbb{N}^{p_j}$, define

$$K_{\overline{n}}(\overline{x},\overline{y}) = \prod_{j=1}^{k} K_{\overline{n}_j}(\overline{x}_j,\overline{y}_j). \tag{2.7}$$

Fact 2.1. Let the kernels $\{K_{\overline{n}_j}, \overline{n}_j \in \mathbb{N}^{p_j}\}$ satisfy conditions (k.1)-(k.5) on $[0,1]^{d_j}$, $j = 1, \ldots, k$. Then the kernels $\{K_{\overline{n}}, \overline{n} \in \mathbb{N}^p\}$ satisfy conditions (k.1)-(k.5) on $[0,1]^d$

Proof. Indeed, (k.1)-(k.4) should be clear. To see (k.5), note that

$$\{\overline{y} \in [0,1]^d : \|\overline{y} - \overline{x}\| > \delta\} \subset \bigcup_{j=1}^{k} \{\overline{y} \in [0,1]^d : \|\overline{y}_j - \overline{x}_j\| > \delta/\sqrt{k}\}.$$

Therefore, (k.5) for $K_{\overline{n}}$ follows by combining (k.3) and (k.5) for $K_{\overline{n}_j}$, $j = 1, \ldots, k$. $\square$

Now, we can combine one-parameter examples from sections 2.1 or 2.2 with Fact 2.1 to get some multivariate and multiparameter examples. For example, combining Proposition 2.1 and Fact 2.1 we get the following:

Corollary 2.1. *Let $d \geq 1$ and $\overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$. For each $1 \leq j \leq d$, consider a sequence of points $\{\mathcal{T}_{n_j}, n_j \geq 1\}$ of multiplicity at most k_j and with $|\mathcal{T}_{n_j}| \to 0$ as $n_j \to \infty$. For $\overline{n} = (n_1, \ldots, n_d)$, put*

$$K_{\overline{n},\overline{k}}(\overline{x}, \overline{y}) = \prod_{j=1}^{d} K_{\mathcal{T}_{n_j},k_j}(x_j, y_j),$$

where $\overline{x} = (x_1, \ldots, x_d)$, $\overline{y} = (y_1, \ldots, y_d)$. Then the kernels $K_{\overline{n},\overline{k}}$ satisfy conditions (k.1)-(k.5), and the sequence of operators $P_{\overline{n},\overline{k}} = P_{\mathcal{T}_{n_1},k_1} \otimes \ldots \otimes P_{\mathcal{T}_{n_d},k_d}$, $\overline{n} \in \mathbb{N}^d$, satisfies assumptions of Theorems 1.2 and 1.1.

Remark 2.3. In case each $\mathcal{T}_{n_j+1}$ is obtained from $\mathcal{T}_{n_j}$ by adding 1 knot, the sequence $P_{\overline{n},\overline{k}}$ is a sequence of rectangular partial sums with respect to orthonormal system on $[0,1]^d$, which is tensor product of respective univariate orthonormal spline systems from Remark 2.1.

Other examples obtained by this procedure include in particular:

- d-variate Cesaro $(C, \overline{\alpha})$, $\overline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ means of partial sums respect to the d-variate trigonometric series on $\mathbb{T}^d$.
- d-variate Cesaro $(C, \overline{1})$, $\overline{1} = (1, \ldots, 1)$ means of partial sums with respect to the tensor product Walsh system on $[0,1]^d$.

3. Proof of Theorems 1.1 and 1.2

In both Lemmas 3.1 and 3.2 below, $\{K_{\overline{n}}, \overline{n} \in \mathbb{N}^p\}$ is a sequence of kernels as in (1.2), satisfying conditions (k.1)-(k.5), and $\{U_{\overline{n}}, \overline{n} \in \mathbb{N}^p\}$ is a sequence of corresponding operators given by formula (1.3).

Denote $\nu = \mu\{x \in \mathbb{R}^d : \|x\| < 1\}$. We begin with a technical lemma:

Lemma 3.1. *Fix $s \in [0,1]^d$, $0 < \eta < 1$ and a sequence $\{\overline{n}_r, r \geq 1\}$ of elements of $\mathbb{N}^p$ with $\lim_{r\to\infty} \min(\overline{n}_m) = \infty$. Moreover, let $\delta > 0$ and $m \in \mathbb{N}$. Then there are $j > m$ and $0 < \kappa < \delta$ such that:*

(i) For every $f \in M[0,1]^d$ satisfying $0 \leq f(x) \leq 1$ for all $x \in [0,1]^d$ and $\mu\{x : \|x - s\| < \delta, f(x) \neq 1\} \leq \nu\kappa^d$ there is $U_{\overline{n}_j} f(s) \geq 1 - \eta$.

(ii) For every $f \in M[0,1]^d$ satisfying $0 \leq f(x) \leq 1$ for all $x \in [0,1]^d$ and $f(x) = 0$ for x such that $\kappa < \|x - s\| < \delta$, there is $U_{\overline{n}_j} f(s) \leq \eta$.

Proof. Take any $j > m$ such that

$$\int_{\{y\in[0,1]^d : \|s-y\|>\delta\}} |K_{\overline{n}_j}(s, y)| dy \leq \eta/2. \tag{3.1}$$

It exists by assumption (k.5). Having chosen j, put $\kappa = \left(\frac{\eta}{2\nu\gamma_{\overline{n}_j}}\right)^{1/d}$.

Let us check (i). Let f be as in (i). Using $\int_{[0,1]^d} K(s, y)dy = 1$ we can write

$$U_{\overline{n}_j} f(s) = 1 - \int_{[0,1]^d} (1 - f(y))K_{\overline{n}_j}(s, y)dy. \tag{3.2}$$

Recall that $0 \leq f(y) \leq 1$, so $0 \leq 1 - f(y) \leq 1$. Using this fact and (3.1) we find

$$\int_{\{y\in[0,1]^d : \|s-y\|>\delta\}} |1 - f(y)| \cdot |K_{\overline{n}_j}(s, y)| dy \leq \eta/2.$$

On the other hand, by the choice of κ we have

$$\int_{\{y\in[0,1]^d : \|s-y\|\leq\delta\}} |1 - f(y)| \cdot |K_{\overline{n}_j}(s, y)| dy \leq \nu\kappa^d \cdot \gamma_{\overline{n}_j} = \eta/2.$$

Putting these estimates to (3.2), we get $U_{\overline{n}_j} f(s) \geq 1 - \eta$.

Next, we check (ii), so let f be as in (ii). Using these assumptions and (3.1) we find

$$\begin{aligned} |U_{\overline{n}_j} f(s)| &= \left| \int_{\{y\in[0,1]^d : \|s-y\|>\delta\}} f(y) \cdot K_{\overline{n}_j}(s, y)dy + \right. \\ &\quad + \left. \int_{\{y\in[0,1]^{\cdot} : \|s-y\|<\kappa\}} f(y) \cdot K_{\overline{n}_j}(s, y)dy \right| \leq \eta/2 + \nu\kappa^d\gamma_{\overline{n}_j} = \eta. \quad \square \end{aligned}$$

With Lemma 3.1 at hand, we get the following:

Lemma 3.2. *Fix $s \in [0,1]^d$, $\varepsilon > 0$, $0 < \eta < 1/2$ and a sequence $\{\overline{n}_r, r \geq 1\}$ of elements of $\mathbb{N}^p$ with $\lim_{r\to\infty} \min(\overline{n}_r) = \infty$. Then there is a real-valued function $f \in M[0,1]^d$ such that*

- *$\mu\{x \in [0,1]^d : f(x) \neq 0\} < \varepsilon$,*
- *f is continuous at each point $x \neq s$, and consequently, $U_{\overline{n}} f(x) \to f(x)$ as $\min(\overline{n}) \to \infty$,*
- *the sequence $\{U_{\overline{n}_r} f(s), r \geq 1\}$ is divergent.*

Proof. First, given $0 < \eta < 1/2$, we choose inductively

- two increasing sequences of natural numbers $\{j_k, k \geq 1\}$, $\{l_k, k \geq 1\}$, with $j_k < l_k < j_{k+1}$,
- two decreasing sequences of positive numbers, $\{\delta_k, k \geq 1\}$ and $\{\xi_k, k \geq 1\}$, with $\delta_k > \xi_k > \delta_{k+1}$,
- a sequence of real-valued functions $h_k \in C[0, 1^d]$, $k \geq 1$,

with the following properties:

(h.1) $\text{supp} h_k \subset \{x : \xi_k < \|x - s\| < \delta_k\}$ and $0 \leq h_k \leq 1$ on $[0, 1]^d$,

(h.2) if $\phi \in M[0, 1]^d$, $0 \leq \phi \leq 1$ and $\phi = h_k$ on $\{x : \xi_k < \|x - s\| < \delta_k\}$, then $U_{\overline{n}_{j_k}} \phi(s) \geq 1 - \eta$,

(h.3) if $\phi \in M[0, 1]^d$, $0 \leq \phi \leq 1$ and $\phi = 0$ on $\{x : \delta_{k+1} < \|x - s\| < \xi_k\}$, then $U_{\overline{n}_{l_k}} \phi(s) \leq \eta$.

Each inductive step has two stages, which we call *the filling step* and *the gap step*, respectively.

Initialization of induction. Take $\delta_1 = \left(\varepsilon/\nu\right)^{1/d}$ and $m = 1$.

Step 1.a – "the filling step". Apply Lemma 3.1 with $\delta = \delta_1$ and $m = 1$, to get $j_1 > 1$ and $\kappa_1 < \delta_1$. Put $\xi_1 = \kappa_1/2^{1/d}$ – note that $\xi_1 < \delta_1$. Fix a real-valued function $h_1 \in C[0, 1]^d$, with $\text{supp} h_1 \subset \{x : \xi_1 < \|x - s\| < \delta_1\}$, $0 \leq h_1 \leq 1$ and $\mu\{x : h_1(x) \neq 1, \xi_1 < |x - s| < \delta_1\} \leq \nu\kappa_1^d/2$. Therefore, if $\phi \in M[0, 1]^d$ and $\phi = h_1$ on $\{x : \xi_1 < |x - s| < \delta_1\}$, then $\mu\{x : |x - s| < \delta_1, \phi(x) \neq 1\} \leq \nu\kappa_1^d$. By Lemma 3.1 (i), it follows that $U_{\overline{n}_{j_1}} \phi(s) \geq 1 - \eta$.

Step 1.b – "the gap step". Apply Lemma 3.1 with $\delta = \xi_1$ and $n = j_1$, to get corresponding $l_1 > j_1$ and $\kappa_1' < \xi_1$. Put $\delta_2 = \kappa_1'$. Lemma 3.1 (ii) guarantees that if $\phi \in M[0, 1]^d$, $0 \leq \phi \leq 1$ and $\phi = 0$ on $\{x : \delta_2 < \|x - s\| < \xi_1\}$, then $U_{\overline{n}_{l_1}} \phi(s) \leq \eta$.

Thus, after step 1, we have fixed $\delta_1, \delta_2, \xi_1$, j_1, l_1 and function h_1, see figure 1 below.

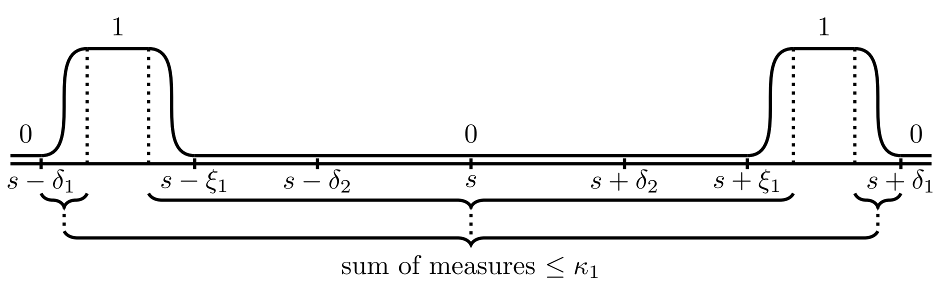

Рис. 1. Illustration of the function h_1 after the first step, in $d = 1$ case.

Inductive step. Assume that our sequences are fixed for $i = 1, \dots, k$, with the exception that δ_{k+1} is fixed as well. We shall fix $\xi_{k+1}, j_{k+1}, l_{k+1}, \delta_{k+2}$ and function h_{k+1}.

Step $(k+1).a$ *– "the filling step".* Apply Lemma 3.1 with $\delta = \delta_{k+1}$ and $n = l_k$, to get $j_{k+1} > l_k$ and $\kappa_{k+1} < \delta_{k+1}$. Put $\xi_{k+1} = \kappa_{k+1}/2^{1/d}$ – note that $\xi_{k+1} < \delta_{k+1}$. Fix a real-valued function $h_{k+1} \in C[0,1]^d$, with $\mathrm{supp} h_{k+1} \subset \{x : \xi_{k+1} < \|x - s\| < \delta_{k+1}\}$, $0 \le h_{k+1} \le 1$ and $\mu\{x : h_{k+1}(x) \ne 1, \xi_{k+1} < \|x - s\| < \delta_{k+1}\} \le \nu\kappa_{k+1}^d/2$. Therefore, if $\phi \in M[0,1]^d$ and $\phi = h_{k+1}$ on $\{x : \xi_{k+1} < \|x - s\| < \delta_{k+1}\}$, then $\mu\{x : \|x - s\| < \delta_{k+1}, \phi(x) \ne 1\} \le \nu\kappa_{k+1}^d$. By Lemma 3.1 (i), it follows that $U_{\overline{n}_{j_{k+1}}}\phi(s) \ge 1 - \eta$.

Step $(k+1).b$ *– "the gap step".* Apply Lemma 3.1 with $\delta = \xi_{k+1}$ and $n = j_{k+1}$, to get corresponding $l_{k+1} > j_{k+1}$ and $\kappa'_{k+1} < \xi_{k+1}$. Put $\delta_{k+2} = \kappa'_{k+1}$. Lemma 3.1 (ii) guarantees that if $\phi \in M[0,1]^d$, $0 \le \phi \le 1$ and $\phi = 0$ on $\{x : \delta_{k+2} < \|x-s\| < \xi_{k+1}\}$, then $U_{\overline{n}_{l_{k+1}}}\phi(s) \le \eta$.

Now, put $f(x) = \sum_{k=1}^{\infty} h_k(x)$, see figure 2 for the illustration in $d = 1$ case. The

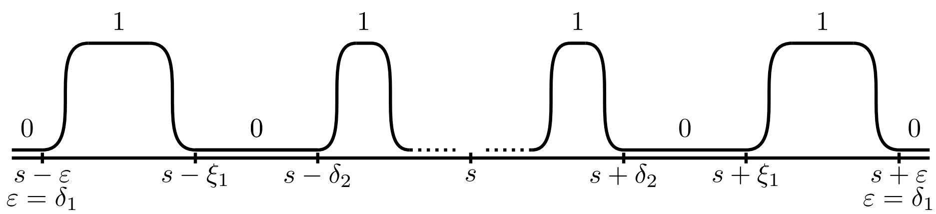

Рис. 2. Illustration of the function f in $d = 1$ case.

supports of functions h_k are disjoint, so for each x, the series defining $f(x)$ has at most 1 non-zero term. This guarantees that f is well defined and $0 \le f \le 1$. Moreover, by the construction, the supports of functions h_k are well-separated, or more precisely $\mathrm{dist}(\mathrm{supp} h_1, \bigcup_{i \ge 2} \mathrm{supp} h_i) > \xi_1 - \delta_2 > 0$, while for $k \ge 2$ there is , $\mathrm{dist}(\mathrm{supp} h_k, \bigcup_{i \ne k} \mathrm{supp} h_i) > \min(\xi_{k-1} - \delta_k, \xi_k - \delta_{k+1}) > 0$. It follows that for each point $x \ne s$ there is $\zeta > 0$ such that on $\{y : \|x - y\| < \zeta\}$ either $f = 0$ or $f = h_k$ for some $k \in \mathbb{N}$. This implies continuity of f at all points $x \ne s$. In addition, note that f satisfies conditions of (h.2) and (h.3). Consequently, there is

$$U_{\overline{n}_{j_k}} f(s) \ge 1 - \eta > \eta \ge U_{\overline{n}_{l_k}} f(s).$$

It follows that the sequence $\{U_{\overline{n}_r} f(s), r \ge 1\}$ is divergent.

Finally, by the choice of δ_1, there is $\mathrm{supp} f \subset \{x : \|x - s\| \le \Big(\varepsilon/\nu\Big)^{1/d}\}$, and consequently $\mu(\mathrm{supp} f) \le \varepsilon$. $\square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Once we have Lemma 3.2, the proof of Theorem 1.1 follows by the line of arguments of the proof of the analogous result in the trigonometric case, cf. e.g. [2], chapter IV, section 21, or [1], vol 1, chapter VIII, a remark following the proof Theorem 1.16. However, we present the details for the reader's convenience. The argument can be simplified a little because of property (v.2).

Let $D = \{x_1, x_2, .., x_k, ..\}$ and $\varepsilon > 0$. Fix a sequence $\{\overline{n}_r, r \geq 1\}$ of elements of $\mathbb{N}^p$ with $\lim_{r\to\infty} \min(\overline{n}_r) = \infty$. Let $\{a_k, k \geq 1\}$ be a sequence of real numbers such that

$$\sum_{k=1}^{\infty} |a_k| < \infty \quad \text{and} \quad a_k \neq 0 \quad \text{for all} \quad k \geq 1. \tag{3.3}$$

Successively applying Lemma 3.2 for each point $x_k \in D$ and $\varepsilon_k = \varepsilon/2^k$, we obtain a sequence of functions f_k, $k \in \mathbb{N}$ such that

- $0 \leq f_k \leq 1$, $\mu\{x : f_k(x) \neq 0\} < \varepsilon/2^k$,
- For each k and $x \neq x_k$, f_k is continuous at x.
- For all $k \in \mathbb{N}$

$$\limsup_{r\to\infty} U_{\overline{n}_r} f_k(x_k) > \liminf_{r\to\infty} U_{\overline{n}_r} f_k(x_k). \tag{3.4}$$

Define $f = \sum\limits_{k=1}^{\infty} a_k f_k$. Condition (3.3) guarantees that the sum defining f converges in $M[0,1]^d$. Moreover, it implies that f is continuous at each point $x \in D^c$.

Next, observe that $\mu\left(\text{supp}(f)\right) \leq \sum\limits_{k=1}^{\infty} \mu\left(\text{supp}(f_k)\right) < \varepsilon$.

It remains to check that the sequence $\{U_{\overline{n}_r} f(x), r \geq 1\}$ is divergent for each $x \in D$. Fix k and consider the function

$$w_k(x) = \sum_{j\neq k} a_j f_j(x) = f(x) - a_k f_k(x).$$

Denote $D_k = D \setminus \{x_k\}$. By the first part of the proof we know that w_k is continuous at each point in D_k^c, in particular it is continuous at x_k. Consequently, by (v.2), we have $U_{\overline{n}} w_k(x_k) \to w_k(x_k)$ as $\min(\overline{n}) \to \infty$. So in particular $\lim_{r\to\infty} U_{\overline{n}_r} w_k(x_k) = w_k(x_k)$. Note that $U_{\overline{n}_r}(f; x) = U_{\overline{n}_r}(w_k; x) + a_k U_{\overline{n}_r}(f_k; x)$. As $a_k \neq 0$, combining $U_{\overline{n}_r}(w_k; x_k) \to w_k(x_k)$ with (3.4) we see that the sequence $\{U_{\overline{n}_r}(f; x_k), r \geq 1\}$ cannot be a convergent sequence. □

Proof of Theorem 1.2. Let g be a measurable, almost everywhere finite function. Fix $\varepsilon > 0$. By Luzin's Theorem (see [42] and Theorem 2.24 in [43]), there is a function $g_\varepsilon \in C[0,1]^d$ such that

$$\mu\{x \in [0,1]^d \colon g(x) \neq g_\varepsilon(x)\} < \frac{\varepsilon}{2}.$$

Next, applying Theorem 1.1 for $\varepsilon/2$, we find a function $f_\epsilon \in M[0,1]^d$ such that $\mu\Big(\text{supp}(f_\varepsilon)\Big) < \varepsilon/2$, f_ε is continuous at each point $x \in D^c$ and the sequence $\{U_{\overline{n}_r} f_\varepsilon(x), r \geq 1\}$ diverges for $x \in D$. Note that $f = g_\varepsilon + f_\epsilon$ satisfies conditions required by Theorem 1.2. □

Comments on the proofs of Remarks 1.1 and 1.2. First, we need a version of Theorem 1.1 in the setting of $\mathbb{T}^d$ or $\mathbb{R}^d$. This is proved by the same arguments as in the case of $[0,1]^d$. Next, in the setting of Remark 1.2, we need to observe that Luzin's theorem implies that if g satisfies condition (1.4), then there is a function $g_\varepsilon \in C(\mathbb{R}^d) \cap M(\mathbb{R}^d)$ such that

$$\mu\{x \in \mathbb{R}^d \colon g(x) \neq g_\varepsilon(x)\} < \frac{\varepsilon}{2}.$$

With this observation at hand, the proofs of the variants of Theorem 1.2 in the setting of $\mathbb{T}^d$ or $\mathbb{R}^d$, are the same as in the setting of $[0,1]^d$. □

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