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SOME BEST PROXIMITY POINT RESULTS IN THE ORTHOGONAL 0-COMPLETE b-METRIC-LIKE SPACES

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Abstract. Since Gordji et al. [Fixed Point Theory, 18(2), 2017, 569-578] suggested the concept of the orthogonal set, many authors investigated a uniqueness of best proximal point, but they used continuity of metric d. Here, we proved the uniqueness of the best proximal point under Banach and Hardy-Rodgers type of contraction with appropriate conditions in the orthogonal 0-complete b-metric-like spaces without continuity conditions.

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1. Introduction

Since it appeared in 1922, the Banach's contraction principle becomes a fundamental tool in pure and applied mathematics. As it is well known, in metric space (X, d) where X is a nonempty set and metric d is continuous, a self mapping T on X with contractive condition $d(Tx, Ty) \leq kd(x, y)$ for $k \in [0, 1)$ has a unique fixed point x, i.e. $Tx = x$. Two main directions in the generalization of Banach's contraction principle appeared - changing contraction condition with weaker (for example Kannan, Reich, Hardy-Rodgers etc. see [1]-[10], or with controlled and double controlled unlimited from above functions [11]-[13] and making generalization of metric space by modifying the axioms of metric d which gave many different types of generalized metric spaces (such as b-metric, partial, partial b-metric, metric-like, b-metric-like space etc. see [14]-[19]).

If we suppose that P and Q are two closed non-empty subsets of X such that $T : P \to Q$ and $P \cap Q = \emptyset$ then the equation $Tx = x$ has no solution. In that case one can find an approximate solution of $Tx = x$ choosing $x \in P$ the closest to $Tx \in Q$. So, if we denote distance between P and Q as $D(P, Q)$ then we are looking

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for $x \in P$ such that $D(P,Q) = d(x,Tx)$, and such x is a best proximity point of T. Firstly, best proximity points in the classical metric spaces and generalized metric spaces were investigated (see for example [20]-[25]).

Recently Gordji et al. [26, 27] suggested the concept of the orthogonal sets with some basic terms (such as O-sequence, O-Cauchy-sequence, O-continuity, O-contraction, $0-b$-complete, O-preserving, P-property etc.), and investigated a best proximal points under weaker conditions in the sense of contractive condition. Many authors investigate best proximal points under orthogonality [3, 24, 25, 28, 10]. In this paper we investigate best proximal point in b-metric-like space when metric is not continuous.

Paper is organized as follows. Firstly, necessary definitions and a few original examples are given. In the part Main result, definition of b-metric-like space as the most general type of metric space with original examples are presented and after that two best proximal point theorems in orthogonal 0-complete b-metric-like space are given, one with classical Banach contraction and the other with contraction of the Hardy-Rodgers type, where, under appropriate conditions, we avoid to use continuity of metric. In the part Fixed point results fixed point theorems are proven by applying the best proximal point theorems.

2. Preliminaries

Definition 2.1. [26] Let X be a nonempty set and $\perp \subset X \times X$ be a binary relation. If there exists an element $x_0 \in X$ such that for all $y \in X$ the following hold:

$$y \perp x_0 \quad or \quad x_0 \perp y,$$

then it is called an orthogonal set (briefly O-set) and x_0 is called an orthogonal element. We denote this O-set by $(X, \perp)$.

Example 2.1. Let X be a set of real numbers and define binary relation $\perp$ on X as for all $x, y \in X$ we say that $x \perp y$ if and only if $x + y = x$. Then $x + 0 = x$ is true for every $x \in X$. So, we have that $x \perp 0$ holds for every $x \in X$ and 0 is an orthogonal element.

Note that an orthogonal set does not need to have a unique orthogonal element.

Example 2.2. Let $X = [0,1]$ and define binary relation $\perp$ on X as for all $x, y \in X$ we say that $x \perp y$ if and only if $x \leq x^y$. Then $x \leq x^0 = 1$ and $x \leq x^1 = x$ is true for every $x \in X$. So, we have that $x \perp 0$ and $x \perp 1$ hold for every $x \in X$, that is 0 and 1 are orthogonal elements. Note that the binary relation in this example is not reflexive since 0^0 is not defined.

Definition 2.2. [26] Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n\in N}$ is called an orthogonal sequence (briefly O-sequence) if for all $n \in N$ the following holds:

$$x_n \perp x_{n+1} \quad \textit{or} \quad x_{n+1} \perp x_n.$$

Example 2.3. Let $X = N$ and let t be an arbitrary element in X. We define binary relation $\perp$ on X as for all $x, y \in X$ we say that $x \perp y$ if and only if $y = x + t$. Then the arithmetic sequence $\{x_n\}$, $x_{n+1} = x_n + t$ is an O-sequence.

Remark 2.1. Note that if we consider an O-sequence $\{x_n\}$ in an O-set $(X, \perp)$, then a subset P consists of the elements of the sequence $\{x_n\}$ is a partially ordered set.

Definition 2.3. [27] Let $(X, \perp)$ be an O-set. Then $f : X \to X$ is said to be orthogonal preserving (or $\perp$-preserving) if for all $x, y \in X$ such that $x \perp y$ yields $f(x) \perp f(y)$.

Example 2.4. [27] Let $X = [0, 1)$ and define binary relation $\perp$ as $x \perp y$ if and only if $xy \leq \frac{x}{2}$. Then $0 \perp y$ for every $y \in X$. So, 0 is an orthogonal element and $(X, \perp)$ is an O-set.
Let $f : X \to X$ be a mapping defined as $f(x) = \frac{x}{2}$ if $x \leq \frac{1}{2}$ and $f(x) = 0$ if $x > \frac{1}{2}$. Let $x \perp y$ for $x, y \in X$. From $xy \leq \frac{x}{2}$ it follows that $x = 0$ or $y \leq \frac{1}{2}$. So, we have the following cases:

i) $x = 0$ and $y \leq \frac{1}{2}$ implies $f(x) \cdot f(y) = 0 \cdot \frac{y}{2} = 0 = \frac{f(x)}{2}$;
ii) $x = 0$ and $y > \frac{1}{2}$ implies $f(x) \cdot f(y) = 0 \cdot 0 = 0 = \frac{f(x)}{2}$;
iii) $x \leq \frac{1}{2}$ and $y \leq \frac{1}{2}$ implies $f(x) \cdot f(y) = \frac{x}{2} \cdot \frac{y}{2} \leq f(x) \cdot \frac{1}{4} < \frac{f(x)}{2}$;
iv) $x > \frac{1}{2}$ and $y \leq \frac{1}{2}$ implies $f(x) \cdot f(y) = 0 \cdot \frac{y}{2} = 0 = \frac{f(x)}{2}$.

These cases imply that $f(x) \cdot f(y) \leq \frac{f(x)}{2}$, that is f is $\perp$-preserving.

Example 2.5. Let $X = [0, 1] \times [0, 1]$ and for $x = (a_1, b_1)$, $y = (a_2, b_2)$ we define binary relation $\perp$ as $x \perp y$ if and only if $a_1 b_2 = 0$ or $a_2 b_1 = 0$. Then $(0, 0) \perp y$ for every $y \in X$, so $(0, 0)$ is an orthogonal element and $(X, \perp)$ is an O-set. Let $f : X \to X$ be a mapping defined as $f(a, b) = (a^2, b)$. Obviously, we get that if $x \perp y$ then $f(x) \perp f(y)$, i.e. f is $\perp$-preserving.

Definition 2.4. [26] Let $(X, \perp, d)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) be a metric space). Then $f : X \to X$ is said to be orthogonal continuous (or $\perp$-continuous) in $a \in X$ if for each O-sequence $\{a_n\}_{n\in N}$ in X with $a_n \to a$ as $n \to +\infty$, we have $f(a_n) \to f(a)$ as $n \to +\infty$.

Remark 2.2. Every continuous mapping is $\perp$-continuous mapping, but converse is not true. Such an example is given in [26].

Definition 2.5. [26] Let $(X, \perp, d)$ be an orthogonal metric space and $0 \leq \lambda < 1$. A mapping $f : X \to X$ is called an orthogonal contraction (briefly, $\perp$-contraction) with Lipshitz constant λ if, for all $x, y \in X$ with $x \perp y$, the following inequality holds

$$d(fx, fy) \leq \lambda d(x, y).$$

Remark 2.3. Every contraction mapping is $\perp$-contraction mapping but converse is not true, for example see [26].

Definition 2.6. [20] Let (X, d) be a metric space and $P, Q \subset X$ such that $P, Q \neq \emptyset$. We define distance between P and Q as $D(P, Q) = \inf\{d(p, q) | p \in P, q \in Q\}$. The point $p \in P$ is a best proximity point (or bpp) of a non-self mapping $T : P \to Q$ if $d(p, Tp) = D(P, Q)$.

Remark 2.4. Distance between two sets $P, Q \subset X$ as introduced in Definition 2.6 is not metric. For example, let (X, d) be a metric space, where $X = [0, 2]$ and $d(x, y) = |x - y|$. Let $P = [\frac{1}{2}, 1]$ and $Q = [\frac{1}{2}, \frac{3}{2}]$. Obviously, $D(P, Q) = 0$, but $P \neq Q$.

For arbitrary sets $P, Q \subset X$ such that $P, Q \neq \emptyset$ we define

$$P_0 = \{p \in P | (\exists q \in Q) d(p, q) = D(P, Q)\},$$
$$Q_0 = \{q \in Q | (\exists p \in P) d(p, q) = D(P, Q)\}.$$

Definition 2.7. [22] Let (X, d) be a metric space and $P, Q \subset X$ such that $P, Q \neq \emptyset$ and $P_0 \neq \emptyset$. Then(P, Q) has P-property if and only if

$$d(p_1, q_1) = D(P, Q) \text{ and } d(p_2, q_2) = D(P, Q) \text{ implies } d(p_1, p_2) = d(q_1, q_2),$$

for all $p_1, p_2 \in P_0$ and $q_1, q_2 \in Q_0$.

Definition 2.8. [24] Let $(X, \preceq)$ be a partially ordered set and (P, Q) be a pair of nonempty subsets of X. A mapping $T : P \to Q$ is called order-preserving if for all $p_1, p_2, q_1, q_2 \in P$ the following holds

$$q_1 \preceq q_2 \text{ and } d(p_1, Tq_1) = d(p_2, Tq_2) = D(P, Q) \text{ implies } p_1 \preceq p_2.$$

Remark 2.5. Regarding the fact that $d(x, Tx) \geq D(P, Q)$ for all $x \in P$, it can be observed that the global minimum of the mapping $x \to d(x, Tx)$ is attained at best proximity point. Moreover, it is easy to see that best proximity point reduces to a fixed point if the underlying mapping T is a self-mapping.

Definition 2.9. [29] Let X be a nonempty set. A mapping $d^{ml} : X^2 \to [0, +\infty)$ is said to be a metric-like on X if for all $\bar{x}, \bar{y}, \bar{z} \in X$ the following three conditions hold:

(d^{ml} 1) $d^{ml}(\bar{x}, \bar{y}) = 0$ yields $\bar{x} = \bar{y}$;
(d^{ml} 2) $d^{ml}(\bar{x}, \bar{y}) = d^{ml}(\bar{y}, \bar{x})$;
(d^{ml} 3) $d^{ml}(\bar{x}, \bar{z}) \leq d^{ml}(\bar{x}, \bar{y}) + d^{ml}(\bar{y}, \bar{z})$.

The pair (X, d^{ml}) is called a metric-like space or dislocated metric space in some papers.

Examples of this kind of spaces can be seen in [29]. Note that metric-like space is not necessary a partial metric space [29].

Definition 2.10. [29] Let $\{x_n\}$ for $n \in N$ be a sequence in a metric-like space (X, d^{ml}).

(i) $\{x_n\}$ is said to converge to $\bar{x} \in X$ if $\lim\limits_{n \to +\infty} d^{ml}(x_n, \bar{x}) = d^{ml}(\bar{x}, \bar{x})$;
(ii) $\{x_n\}$ is said to be d^{ml}-Cauchy in (X, d^{ml}) if $\lim\limits_{n,p \to +\infty} d^{ml}(x_n, x_p)$ exists and is finite;
(iii) A metric-like space (X, d^{ml}) is d^{ml}- complete if for every d^{ml}-Cauchy sequence $\{x_n\}$ in X there exists an $x \in X$ such that

$$\lim_{n,p \to +\infty} d^{ml}(x_n, x_p) = d^{ml}(x, x) = \lim_{n \to +\infty} d^{ml}(x_n, x).$$

Remark 2.6. In [29] the authors noted that if the sequence $\{x_n\}$ is d^{ml}-Cauchy sequence such that $\lim_{n,p \to +\infty} d^{ml}(x_n, x_p) = 0$ and if X is d^{ml}-complete then sequence has a unique limit. In that case for (X, d^{ml}) we say that X is $0 - d^{ml}$-complete space, and $\{x_n\}$ is $0 - d^{ml}$-Cauchy sequence.

3. Main results

We recall definition of an one generalization of metric-like space.

Definition 3.1. [20] Let X be a nonempty set and $s \geq 1$. A mapping $d^{bml} : X^2 \to [0, +\infty)$ is said to be a b-metric-like on X if for all $\bar{x}, \bar{y}, \bar{z} \in X$ the following three conditions hold:

(d^{bml} 1) $d^{bml}(\bar{x}, \bar{y}) = 0$ yields $\bar{x} = \bar{y}$;
(d^{bml} 2) $d^{bml}(\bar{x}, \bar{y}) = d^{bml}(\bar{y}, \bar{x})$;
(d^{bml} 3) $d^{bml}(\bar{x}, \bar{z}) \leq s(d^{bml}(\bar{x}, \bar{y}) + d^{bml}(\bar{y}, \bar{z}))$.

The pair (X, d^{bml}, s) is called a b-metric-like space or b-dislocated metric space.

Convergent sequence, d^{bml}-Cauchy sequence and d^{bml}-completeness of the b-metric-like space are defined similarly as in metric-like space. We consider the case of $0 - d^{bml}$ sequences.

Example 3.1. Let $X = \{0, 1, 2\}$ and define d^{bml} as follows

$$d^{bml}(0,0) = 0,\ d^{bml}(1,1) = d^{bml}(2,2) = 3,$$

$$d^{bml}(0,1) = d^{bml}(1,0) = d^{bml}(0,2) = d^{bml}(2,0) = 3,\ d^{bml}(1,2) = d^{bml}(2,1) = 9.$$

Then $(d^{ml}\,1)$ and $(d^{ml}\,2)$ are obviously satisfied. Since inequality $d^{bml}(1,2) = 9 \leq d^{bml}(1,0) + d^{bml}(0,2) = 3 + 3 = 6$ is not true, we conclude that $(d^{ml}\,3)$ is not satisfied. If we choose $s \geq \frac{3}{2}$, then (X, d^{bml}, s) is b-metric-like space.

Example 3.2. Let $X = [0, 1)$ and define d^{bml} as follows.

$$d^{bml}(x,y) = \begin{cases} 3 \cdot \max\{x, y\}, & for \quad x, y \leq \frac{1}{2}, \\ \frac{1}{3} \cdot \max\{x, y\}, & otherwise. \end{cases}$$

Then (X, d^{bml}) is not metric space since, for example, $d^{bml}(\frac{1}{4}, \frac{1}{4}) = \frac{3}{4} \neq 0$, and it is not metric-like space since, for example, the inequality $d^{bml}(\frac{1}{8}, \frac{1}{4}) = \frac{3}{4} \leq d^{bml}(\frac{1}{8}, 1) + d^{bml}(1, \frac{1}{4}) = \frac{2}{3}$ is not true. (X, d^{bml}) is b-metric-like space with $s \geq \frac{9}{2}$.

Remark 3.1. If (X, d) is an orthogonal b-metric-like space then terms such as $\perp$-continuity, $\perp$-preserving etc. are defined in the same way as in the previous section.

Example 3.3. Let $X = [0, 1]$ and let us define d as

$$d(x,y) = \begin{cases} 3 \cdot \max\{x, y\} & for \quad x, y \leq \frac{1}{3}, \\ \frac{1}{3} \cdot \max\{x, y\} & otherwise. \end{cases}$$

Then (X, d) is a b-metric-like space. Let us define a mapping $T : X \to X$ as

$$T(x) = \begin{cases} \frac{2}{9} & for \quad 0 < x \leq \frac{1}{3}, \\ 0 & otherwise. \end{cases}$$

We define binary relation $\perp$ on X as $x \perp y$ if and only if $xy \leq \frac{x}{3}$. It is obvious that $x = 0$ is the orthogonal element, so $(X, \perp)$ is an O-set.

Let $x \perp y$. Then, $x = 0$ or $y \leq \frac{1}{3}$. We consider the following cases:

i) $x = y = 0$ implies $T(x) = T(y) = 0$, so $T(x) \cdot T(y) = 0 \leq \frac{T(x)}{3}$;

ii) $x = 0$ and $0 < y \leq \frac{1}{3}$ implies $T(x) = 0$, $T(y) = \frac{2}{9}$, so $T(x) \cdot T(y) = 0 \leq \frac{T(x)}{3}$;

iii) $x = 0$ and $y > \frac{1}{3}$ implies $T(x) = T(y) = 0$, so $T(x) \cdot T(y) = 0 \leq \frac{T(x)}{3}$;

iv) $y = 0$ and $0 < x \leq \frac{1}{3}$ implies $T(y) = 0$, $T(x) = \frac{2}{9}$, so $T(x) \cdot T(y) = 0 \leq \frac{T(x)}{3}$;

v) $y = 0$ and $x > \frac{1}{3}$ implies $T(x) = T(y) = 0$, so $T(x) \cdot T(y) = 0 \leq \frac{T(x)}{3}$;

vi) $0 < y \leq \frac{1}{3}$ and $0 < x \leq \frac{1}{3}$ implies $T(x) = T(y) = \frac{2}{9}$, so $T(x) \cdot T(y) = \frac{4}{81} \leq \frac{T(x)}{3} = \frac{2}{27}$;

vii) $0 < y \leq \frac{1}{3}$ and $x > \frac{1}{3}$ implies $T(y) = \frac{2}{9}$, $T(x) = 0$, so $T(x) \cdot T(y) = 0 \leq \frac{T(x)}{3} = 0$.

Therefore, T is $\perp$-preserving.

Now, we formulate and prove the theorem for existence and uniqueness of best proximal point of an $\perp$-contraction self mapping in orthogonal $0-d^{bml}$-complete b-metric-like space without assumption of continuity of metric.

Theorem 3.1. *Let $(X, \perp, d^{bml}, s)$ be an orthogonal $0-d^{bml}$-complete b-metric-like space with $s \geq 1$, (P, Q) be a pair of two non-empty closed subsets of X having P-property and $P_0 \neq \emptyset$. Suppose that a mapping $T : P \to Q$ satisfies the following three conditions:*

i) T is a $\perp$-order-preserving and $T(P_0) \subset Q_0$;

ii) There exist $a_0, a_1 \in P_0$ such that $a_0 \perp a_1$ and $d^{bml}(a_1, Ta_0) = D^{bml}(P, Q)$, where $D^{bml}(P, Q) = \inf\{d^{bml}(x, y)|x \in P, y \in Q\}$;

iii) T is $\perp$-contraction and O-continuous mapping on P with Lipshitz constant $k \in [0, \frac{1}{s})$.

Then T has an unique bpp $a' \in P$, i.e. $d^{bml}(a', Ta') = D^{bml}(P, Q)$.

Proof. Similarly as those one in [25] we will construct an one O-sequence. From ii), we obtain that there exist $a_0, a_1 \in P_0$ such that

$$a_0 \perp a_1 \quad and \quad d^{bml}(a_1, Ta_0) = D^{bml}(P, Q).$$

Since, $Ta_1 \in T(P_0) \subset Q_0$, there exists $a_2 \in P_0$ such that

$$d^{bml}(a_2, Ta_1) = D^{bml}(P, Q).$$

According to the assumption T is a $\perp$-order-preserving we obtain $a_1 \perp a_2$. Continuing this process we obtain a sequence $\{a_n\}$ in P_0 such that for all $n \in N$ the following hold

$$d^{bml}(a_{n+1}, Ta_n) = D^{bml}(P, Q) \quad and \quad a_n \perp a_{n+1}, \tag{3.1}$$

i.e. $\{a_n\}$ is an O-sequence.

Now, we will prove that $\{a_n\}$ is an $0-d^{bml}$-Cauchy sequence.

Since T has P-property, we obtain

(3.2)
$$\left.\begin{array}{l} d^{bml}(a_{n+1}, Ta_n) = D^{bml}(P, Q) \\ d^{bml}(a_n, Ta_{n-1}) = D^{bml}(P, Q) \end{array}\right\} \text{implies } d^{bml}(a_{n+1}, a_n) = d^{bml}(Ta_n, Ta_{n-1}).$$

Hence, since T is $\perp$-contraction with Lipshitz constant $k \in [0, \frac{1}{s})$ we have

$$d^{bml}(a_{n+1}, a_n) = d^{bml}(Ta_n, Ta_{n-1}) \leq kd^{bml}(a_n, a_{n-1}). \tag{3.3}$$

Since (2) and (3) hold for every $n \in N$ we get

$$d^{bml}(a_{n+1}, a_n) \leq k^n d^{bml}(a_0, a_1). \tag{3.4}$$

Let $m, n \in N$, $m > n$. Then we have

$$\begin{aligned} d^{bml}(a_n, a_m) &\leq s(d^{bml}(a_n, a_{n+1}) + d^{bml}(a_{n+1}, a_m)) \\ &\leq sk^n d^{bml}(a_0, a_1) + s d^{bml}(a_{n+1}, a_m) \\ &\leq sk^n d^{bml}(a_0, a_1) + s^2(d^{bml}(a_{n+1}, a_{n+2}) + d^{bml}(a_{n+2}, a_m)) \\ &\leq sk^n d^{bml}(a_0, a_1) + s^2 k^{n+1} d^{bml}(a_0, a_1) + s^2 d^{bml}(a_{n+2}, a_m) \\ &\vdots \\ &\leq d^{bml}(a_0, a_1) \left(sk^n + s^2 k^{n+1} + \cdots + s^{m-n-2} k^{m-2} + s^{m-n-2} k^{m-1}\right) \\ &= sk^n d^{bml}(a_0, a_1) \left(1 + sk + \cdots + (sk)^{m-n-1} + k(sk)^{m-n-1}\right) \\ &= sk^n d^{bml}(a_0, a_1) \left(\frac{1 - (sk)^{m-n-1}}{1 - sk} + k(sk)^{m-n-1}\right). \end{aligned}$$

Having in mind that the sum of a finite number of elements in the geometric sequence is less than the sum of the whole sequence and $k \in [0, \frac{1}{s})$, $0 < sk < 1$, we get

$$d^{bml}(a_n, a_m) \leq sk^n d^{bml}(a_0, a_1) \left(\frac{1}{1 - sk} + 1\right) < k^n \cdot \frac{2s d^{bml}(a_0, a_1)}{1 - sk}.$$

Hence, we get that $d^{bml}(a_n, a_m) \to 0$ as $m, n \to +\infty$, so $\{a_n\}$ is an $0 - d^{bml}$-Cauchy sequence. Since the space is $0 - d^{bml}$-complete there exists a unique point $a^{'} \in P$ (P is closed set by assumption) such that

$$\lim_{n,m \to +\infty} d^{bml}(a_n, a_m) = \lim_{n \to +\infty} d^{bml}(a_n, a^{'}) = 0.$$

Since T is $\perp$-continuity we have that $\lim_{n \to +\infty} d^{bml}(Ta_n, Ta^{'}) = 0$ and $Ta^{'} \in Q$ (Q is closed set by assumption).

We will prove that $d^{bml}(a^{'}, Ta^{'}) = D^{bml}(P, Q)$, or $a^{'}$ is bpp for T.

Since $a^{'} \in P$ and $Ta^{'} \in Q$ (by assumption P and Q are clesed sets), it is obvious that $d^{bml}(a^{'}, Ta^{'}) \geq D^{bml}(P, Q)$.

If $d^{bml}(a^{'}, Ta^{'}) > D^{bml}(P, Q)$, then since $Ta^{'} \in Q$ and $P_0 \neq \emptyset$, there exists $a^* \in P$ such that

$$d^{bml}(a^{'}, Ta^{'}) > D^{bml}(P, Q) = d^{bml}(a^*, Ta^{'}).$$

By P-property of T we have

$$\left.\begin{array}{r} d^{bml}(a^*, Ta^{'}) = D^{bml}(P, Q) \\ d^{bml}(a_n, Ta_{n-1}) = D^{bml}(P, Q) \end{array}\right\} \text{implies } d^{bml}(a^*, a_n) = d^{bml}(Ta^{'}, Ta_{n-1}).$$

Hence

$$\lim_{n \to +\infty} d^{bml}(a^*, a_n) = \lim_{n \to +\infty} d^{bml}(Ta^{'}, Ta_{n-1}).$$

From

$$Ta_n \to Ta^{'},\ n \to +\infty$$

we have $\lim_{n\to+\infty} d^{bml}(Ta^{'}, Ta_{n-1}) = \lim_{n\to\infty} d^{bml}(a^*, a_n) = 0$, that is $0 - d^{bml}$-Cauchy sequence $\{a_n\}$ has two limits $a^{'}$ and a^*. But $0 - d^{bml}$-Cauchy sequence has a unique limit point, so $a^{'} = a^*$, that is $a^{'}$ is bpp for T.
Suppose that $a^{''} \in P$ is bpp of T such that $a' \neq a^{''}$. By P-property of T we have

$$\left.\begin{array}{r} d^{bml}(a^{'}, Ta^{'}) = d^{bml}(P,Q) \\ d^{bml}(a^{''}, Ta^{''}) = d^{bml}(P,Q) \end{array}\right\} \text{implies } d^{bml}(a^{'}, a^{''}) = d^{bml}(Ta^{'}, Ta^{''}).$$

Since T is $\perp$-contraction, we get

$$d^{bml}(a^{'}, a^{''}) = d^{bml}(Ta^{'}, Ta^{''}) \leq kd^{bml}(a^{'}, a^{''}),$$

or

$$(1-k)d^{bml}(a^{'}, a^{''}) \leq 0 \wedge 1-k > 0, \text{ so, } d^{bml}(a^{'}, a^{''}) = 0 \text{ i.e } a^{'} = a^{''}. \qquad \square$$

Example 3.4. Let $X = [0,1]$ and let us define $d : X \times X \to [0, +\infty)$ as

$$d(x,y) = \begin{cases} 4 \cdot \max\{x,y\} & for \quad x, y \leq \frac{1}{9}, \\ \frac{1}{4} \cdot \max\{x,y\} & otherwise. \end{cases}$$

Then $(X, d, s = 8)$ is a b-metric-like space (see Example 3.2).
Let $P = \{x | \frac{8}{81} \leq x \leq \frac{1}{9}\}$ and $Q = \{y | \frac{4}{27} \leq y \leq \frac{1}{6}\}$. Then P and Q are closed subsets in X. Let us compute $D(P,Q)$:

$$\begin{aligned} D(P,Q) &= \inf\{d(x,y) | x \in P, y \in Q\} \\ &= \inf\left\{d(x,y) | \frac{8}{81} \leq x \leq \frac{1}{9} \text{ and } \frac{4}{27} \leq y \leq \frac{1}{6}\right\} \\ &= \inf\left\{\frac{1}{4}\max\{x,y\} | \frac{8}{81} \leq x \leq \frac{1}{9} \text{ and } \frac{4}{27} \leq y \leq \frac{1}{6}\right\} = \frac{1}{27}. \end{aligned}$$

It is obviously that $P_0 = P$, $Q_0 = Q$. Next, let us define mapping $T : X \to X$ as

$$T(x) = \begin{cases} \frac{3}{2}x, & for \quad \frac{8}{81} \leq x \leq \frac{1}{9}, \\ 0, & otherwise. \end{cases}$$

Then $T(P_0) \subset Q_0$. Let $k = \frac{3}{32}$. We define binary relation $\perp$ on X as $x \perp y$ if and only if $xy \leq \frac{x}{6}$. It is obvious that $x = 0$ is the orthogonal element, so $(X, \perp)$ is an O-set. Let $x \perp y$. Then, $x = 0$ or $y \leq \frac{1}{6}$. We will prove that T is $\perp$-preserving and $\perp$-contraction on P for $k = \frac{3}{32}$. If $x, y \in P$ such that $x \perp y$, i.e. $\frac{8}{81} \leq x, y \leq \frac{1}{9}$, then $T(x) = \frac{3}{2}x$, $T(y) = \frac{3}{2}y$, and since $\frac{4}{27} \leq \frac{3}{2}y \leq \frac{1}{6}$ we have that

$$T(x) \cdot T(y) = T(x) \cdot \frac{3}{2}y \leq \frac{T(x)}{6},$$

and

$$d(T(x), T(y)) = d(\frac{3}{2}x, \frac{3}{2}y) = \frac{1}{4} \cdot \max\{\frac{3}{2}x, \frac{3}{2}y\} = \frac{3}{8}\max\{x,y\} = \frac{3}{32} \cdot d(x,y).$$

So, T is $\perp$-preserving and $\perp$-contraction on P for $k = \frac{3}{32}$. T is obviously O-continuous. Therefore, the conditions of the previous theorem are satisfied and T has unique proximal point $x = \frac{8}{81}$.

Now, we formulate and prove the theorem for existence and uniqueness of a best proximal point of a self mapping under weaker $\perp$-contraction in orthogonal $0-d^{ml}$-complete b-metric-like space without assumption of continuity of metric.

Theorem 3.2. *Let $(X, \perp, d^{bml}, s)$ be an orthogonal $0-d^{ml}$-complete b-metric-like space with $s \geq 1$, (P, Q) be a pair of two non-empty closed subsets of X having P-property and $P_0 \neq \emptyset$. Suppose that a mapping $T : P \to Q$ satisfies the following three conditions:*

i) T is a $\perp$-order-preserving and $T(P_0) \subset Q_0$;
ii) There exist $a_0, a_1 \in P_0$ such that $a_0 \perp a_1$ and $d^{bml}(a_1, Ta_0) = D^{bml}(P, Q)$, where $D^{bml}(P, Q) = \inf\{d^{bml}(x, y) | x \in P, y \in Q\}$;
iii) T is O-continuous mapping on P such that

$$d^{bml}(Ta, Tb) \leq H(a, b), \tag{3.5}$$

where,

$$\begin{aligned} H(a,b) &= \alpha_1 d^{bml}(a,b) + \alpha_2 d^{bml}(a,Ta) + \alpha_3 d^{bml}(b,Tb) + \alpha_4 d^{bml}(a,Tb) \\ &+ \alpha_5 d^{bml}(b,Ta) - C, \end{aligned}$$

and $C = \left(\alpha_4 + (\alpha_2 + \alpha_3)s + \alpha_5 s^2\right) D^{bml}(P, Q)$, for all $a, b \in P$ such that $a \perp b$, $\alpha_i > 0, i = 1, 2, 3, 4, 5$, $\alpha_2 + s\alpha_5 < \frac{1}{s}$ and $s(\alpha_1 + \alpha_2) + s^2(\alpha_3 + 2\alpha_5) < 1$.

Then T has an unique bpp $a' \in P$, i.e. $d^{bml}(a', Ta') = D^{bml}(P, Q)$.

Proof. Similarly as in the proof of Theorem 3.1 we make an O-sequence $\{a_n\}$ in P such that for all $n \in N$ we have that $a_n \perp a_{n+1}$ and $d^{bml}(a_{n+1}, Ta_n) = D^{bml}(P, Q)$ and, by P-property, we obtain

$$\left.\begin{array}{l} d^{bml}(a_{n+1}, Ta_n) = D^{bml}(P, Q) \\ d^{bml}(a_n, Ta_{n-1}) = D^{bml}(P, Q) \end{array}\right\} \text{ implies } d^{bml}(a_{n+1}, a_n) = d^{bml}(Ta_n, Ta_{n-1}). \tag{3.6}$$

Now, we will prove that $\{a_n\}$ is an $0-d^{bml}$-Cauchy sequence.
Hence, since (5) and (6) hold, we have

$$
\begin{aligned}
& d^{bml}(a_2,a_1) = d^{bml}(Ta_1,Ta_0) \le \alpha_1 d^{bml}(a_1,a_0) + \alpha_2 d^{bml}(a_1,Ta_1) + \alpha_3 d^{bml}(a_0,Ta_0) \\
+ \quad & \alpha_4 d^{bml}(a_1,Ta_0) + \alpha_5 d^{bml}(a_0,Ta_1) - \left(\alpha_4 + (\alpha_2+\alpha_3)s + \alpha_5 s^2\right) D^{bml}(P,Q) \\
\le \quad & \alpha_1 d^{bml}(a_1,a_0) + \alpha_2 s(d^{bml}(a_1,a_2) + d^{bml}(a_2,Ta_1)) \\
+ \quad & \alpha_3 s(d^{bml}(a_0,a_1) + d^{bml}(a_1,Ta_0)) + \alpha_4 D^{bml}(P,Q) + \alpha_5 d^{bml}(a_0,Ta_1) \\
- \quad & \left(\alpha_4 + (\alpha_2+\alpha_3)s + \alpha_5 s^2\right) D^{bml}(P,Q) \\
= \quad & \alpha_1 d^{bml}(a_1,a_0) + \alpha_2 s(d^{bml}(a_1,a_2) + D^{bml}(P,Q)) \\
+ \quad & \alpha_3 s(d^{bml}(a_0,a_1) + D^{bml}(P,Q)) + \alpha_5 d^{bml}(a_0,Ta_1) \\
- \quad & \left((\alpha_2+\alpha_3)s + \alpha_5 s^2\right) D^{bml}(P,Q) \\
\le \quad & (\alpha_1 + \alpha_3 s) d^{bml}(a_1,a_0) + \alpha_2 s d^{bml}(a_1,a_2) + \alpha_5 s(d^{bml}(a_0,a_1) + d^{bml}(a_1,Ta_1)) \\
- \quad & \left(\alpha_5 s^2\right) D^{bml}(P,Q) \\
\le \quad & (\alpha_1 + (\alpha_3+\alpha_5)s) d^{bml}(a_1,a_0) + \alpha_2 s d^{bml}(a_1,a_2) + \alpha_5 s^2 (d^{bml}(a_1,a_2) \\
+ \quad & d^{bml}(a_2,Ta_1)) - \left(\alpha_5 s^2\right) D^{bml}(P,Q) \\
= \quad & (\alpha_1 + (\alpha_3+\alpha_5)s) d^{bml}(a_1,a_0) + \alpha_2 s d^{bml}(a_1,a_2) + \alpha_5 s^2 d^{bml}(a_1,a_2) \\
+ \quad & \alpha_5 s^2 D^{bml}(P,Q)) - \left(\alpha_5 s^2\right) D^{bml}(P,Q) \\
= \quad & (\alpha_1 + (\alpha_3+\alpha_5)s) d^{bml}(a_1,a_0) + \alpha_2 s d^{bml}(a_1,a_2) + \alpha_5 s^2 d^{bml}(a_1,a_2)
\end{aligned}
$$

So, we get:

$$(1-\alpha_2 s - \alpha_5 s^2) d^{bml}(a_1,a_2) \le (\alpha_1 + (\alpha_3+\alpha_5)s) d^{bml}(a_1,a_0). \tag{3.7}$$

By assumption $1-\alpha_2 s - \alpha_5 s^2 > 0$, so, from (7) we have

$$d^{bml}(a_1,a_2) \le \beta \cdot d^{bml}(a_1,a_0). \tag{3.8}$$

where $\beta = \frac{\alpha_1 + (\alpha_3+\alpha_5)s}{1-\alpha_2 s - \alpha_5 s^2} < \frac{1}{s} < 1$. From (8), we obtain

$$d^{bml}(a_n,a_{n+1}) \le \beta^n \cdot d^{bml}(a_1,a_0). \tag{3.9}$$

Let $m, n \in N$, $m > n$. Then we have

$$\begin{aligned}
d^{bml}(a_n, a_m) &\leq s(d^{bml}(a_n, a_{n+1}) + d^{bml}(a_{n+1}, a_m)) \\
&\leq s\beta^n d^{bml}(a_0, a_1) + sd^{bml}(a_{n+1}, a_m) \\
&\leq s\beta^n d^{bml}(a_0, a_1) + s^2(d^{bml}(a_{n+1}, a_{n+2}) + d^{bml}(a_{n+2}, a_m)) \\
&\leq s\beta^n d^{bml}(a_0, a_1) + s^2\beta^{n+1} d^{bml}(a_0, a_1) + s^2 d^{bml}(a_{n+2}, a_m) \\
&\vdots \\
&\leq d^{bml}(a_0, a_1)\left(s\beta^n + s^2\beta^{n+1} + \cdots + s^{m-n-2}\beta^{m-2} + s^{m-n-2}\beta^{m-1}\right) \\
&= s\beta^n d^{bml}(a_0, a_1)\left(1 + s\beta + \cdots + (s\beta)^{m-n-1} + \beta(s\beta)^{m-n-1}\right) \\
&= s\beta^n d^{bml}(a_0, a_1)\left(\frac{1 - (s\beta)^{m-n-1}}{1 - s\beta} + \beta(s\beta)^{m-n-1}\right).
\end{aligned}$$

Having in mind that the sum of a finite number of elements in the geometric sequence is less than the sum of the whole sequence and $0 < \beta < \frac{1}{s}$, $0 < s\beta < 1$, we get

$$d^{bml}(a_n, a_m) \leq s\beta^n d^{bml}(a_0, a_1)\left(\frac{1}{1 - s\beta} + 1\right) < \beta^n \cdot \frac{2sd^{bml}(a_0, a_1)}{1 - s\beta}.$$

Hence, we get that $d^{bml}(a_n, a_m) \to 0$ as $m, n \to +\infty$, so $\{a_n\}$ is an $0 - d^{bml}$-Cauchy sequence.

The rest of the proof is as the proof of the Theorem 3.1. □

4. Fixed point results

We will apply the best proximal point result given in the previous section to prove fixed point theorems.

Theorem 4.1. *Let $(X, \perp, d^{bml}, s)$ be an orthogonal $0 - d^{ml}$-complete b-metric-like space with $s \geq 1$. Suppose that a mapping $T : X \to X$ is $\perp$-order-preserving, O-continuous and $\perp$-contraction mapping on X with Lipshitz constant $k \in [0, \frac{1}{s})$. Then T has an unique fixed point in X.*

Proof. Suppose that a_0 is an orthogonal element in X and define sequence $\{a_n\}$ as $a_{n+1} = Ta_n$. Since T is a $\perp$-order-preserving we have that $\{a_n\}$ is an O-sequence. Let $P = \{a_n | n \in N\}$ and $Q = \{Ta_n | n \in N\}$. Then $D(P, Q) = 0$ and P_0 is non-empty. So, all conditions of Theorem 3.1 is satisfied. Consequently there exists a unique bpp of T and this bpp is a fixed point of T. □

Theorem 4.2. *Let $(X, \perp, d^{bml}, s)$ be an orthogonal $0 - d^{ml}$-complete b-metric-like space with $s \geq 1$. Suppose that a mapping $T : X \to X$ is T is a $\perp$-order-preserving* *and O-continuous mapping on P such that*

$$d^{bml}(Ta,Tb) \leq H(a,b) \tag{4.1}$$

whrere

$$H(a,b) = \alpha_1 d^{bml}(a,b)+\alpha_2 d^{bml}(a,Ta)+\alpha_3 d^{bml}(b,Tb)+\alpha_4 d^{bml}(a,Tb)+\alpha_5 d^{bml}(b,Ta)$$

for all $a, b \in P$ such that $a \perp b$ and $\alpha_i > 0, i = 1,2,3,4,5$ and $\alpha_2 + s\alpha_5 < \frac{1}{s}$ and $s(\alpha_1+\alpha_2)+s^2(\alpha_3+2\alpha_5) < 1$. Then T has an unique fixed point.

Proof. Obviously from Theorem 3.2.

Open problem. Whether result presented in Theorem 3.1 holds for $k \in [0,1)$?

Conclusion

The significance of these results is reflected in the fact that the continuity of the metric was not used, and the result is valid both in spaces where the metric is continuous and in spaces where the metric is not continuous.

Since b-metric-like space is the widest class of the metric spaces, in the sense that every metric space contained in partially metric space, partially metric space contained in metric-like space, and metric space contained in b-metric space, partially metric space contained in partially b-metric space and metric-like space contained in b-metric-like space, obtained results hold for every aforesaid metric space.

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