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AN APPLICATION OF RICCERI THEOREM IN SOLVING BOUNDARY VALUE PROBLEMS

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Abstract. Professor Ricceri very recently in the interesting paper has obtained a global minima theorem. In this paper, we will provide an application of this theorem.

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1. Introduction

Authors in [1], considered a class of semilinear elliptic systems of the form

$$\begin{cases} -\operatorname{div}(h_1(x)\nabla u) = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(h_2(x)\nabla v) = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and λ is a positive parameter. Indeed, using the Mountain pass theorem, they proved if λ is large enough the above system has at least two nonnegative solutions. Such problems come from the consideration of standing waves in anisotropic Schrödinger systems (see [15]). These equations appear in many topics of applied physics, such as nuclear physics, field theory, solid waves and problems of false vacuum see ([2, 3]).

Ricceri in [4] established following theorem:

Theorem 1.1. *Let X be a topological space, $(Y, \langle\cdot,\cdot\rangle)$ a real Hilbert space, $T \subseteq Y$ a convex set dense in Y and $I : X \to \mathbb{R}$, $\varphi : X \to Y$ two functions such that, for each $y \in T$, the function $x \to I(x) + \langle\varphi(x), y\rangle$ is lower semicontinuous and* inf*-compact. Moreover, assume that there exists a point $x_0 \in X$, with $\varphi(x_0) \neq 0$ such that*

- (φ_1) *x_0 is a global minimum of both functions I and $\|\varphi(\cdot)\|$;*
- (φ_2) $\inf_{x\in X}\langle\varphi(x), \varphi(x_0)\rangle < \|\varphi(x_0)\|^2$.

Then, for each convex set $S \subseteq T$ dense in Y, there exists $\tilde{y} \in S$ such that the functional $x \to I(x) + \langle\varphi(x), \tilde{y}\rangle$ has at least two global minima in X.

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Then, he presented an application of this theorem in solving a system of elliptic equations. Indeed, he proved following theorem.
Let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary. We denote by $\mathcal{A}_1$ the class of all functions $\Phi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which are measurable in Ω, C^1 in $\mathbb{R}^2$ and satisfy

$$\sup_{(x,u,v)\in\Omega\times\mathbb{R}^2} \frac{|\Phi_u(x,u,v)| + |\Phi_v(x,u,v)|}{1 + |u|^m + |v|^m} < +\infty$$

where Φ_u (resp. Φ_v) denoting the derivative of Φ with respect to u (resp. v) and $m > 0$ with $m < \frac{n+2}{n-2}$ when $n > 2$.

Theorem 1.2. *Let $F_1, G_1, K_1 \in \mathcal{A}_1$, with $K_1(x, 0, 0) = 0$ for all $x \in \Omega$, satisfy the following conditions:*

(k_1) *there is $\eta \in (0, \frac{\lambda_1}{2})$ such that $K_1(x, s, t) \leq \eta(s^2 + t^2)$ for all $x \in \Omega$, $s, t \in \mathbb{R}$, where $\lambda_1 = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$;*

(f_1) $\lim_{s^2+t^2\to+\infty} \frac{\sup_{x\in\Omega}(|F_1(x,s,t)|+|G_1(x,s,t)|)}{s^2+t^2} = 0$;

(f_2) *one has* $\text{meas}(\{x \in \Omega : |F_1(x, 0, 0)|^2 + |G_1(x, 0, 0)|^2 > 0\}) > 0$ *and*

$$|F_1(x, 0, 0)|^2 + |G_1(x, 0, 0)|^2 \leq |F_1(x, s, t)|^2 + |G_1(x, s, t)|^2$$

for all $x \in \Omega$, $s, t \in \mathbb{R}$;

(f_3) *one has*

$$\text{meas}(\{x \in \Omega : \inf_{(s,t)\in\mathbb{R}^2}(|F_1(x, 0, 0)|F_1(x, s, t) + |G_1(x, 0, 0)|^2 G_1(x, s, t)) < |F_1(x, 0, 0)|^2 + |G_1(x, 0, 0)|^2\}) > 0.$$

Then, for every convex set $S_1 \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\sigma_1, \varsigma_1) \in S_1$ such that the problem

$$\begin{cases} -\Delta u = \sigma_1(x)F_{1u}(x, u, v) + \varsigma_1(x)G_{1u}(x, u, v) + K_{1u}(x, u, v) & \text{in } \Omega, \\ -\Delta v = \sigma_1(x)F_{1v}(x, u, v) + \varsigma_1(x)G_{1v}(x, u, v) + K_{1v}(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega) \times H_0^1(\Omega)$ of the functional

$$(u, v) \to \frac{1}{2}\left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\right) - \int_\Omega (\sigma_1(x)F_1(x, u, v) + \varsigma_1(x)G_1(x, u, v) + K_1(x, u(x), v(x)))dx.$$

Using Theorem 1.1, the author proved above theorem with the following choices: X is the space $H_0^1(\Omega) \times H_0^1(\Omega)$ and Y is $L^2(\Omega) \times L^2(\Omega)$.

In this paper, motivated by [1, 4], we give an application of Theorem 1.1. Indeed, in Section 2, we will prove a new version of Theorem 1.2 with the choice $X =$

$H^1_0(\Omega, h_1) \times H^1_0(\Omega, h_2)$, where $h_i : \Omega \to [0, +\infty)$, $h_i \in L^1_{loc}$ $(i = 1, 2)$. We define the Hilbert spaces $H^1_0(\Omega, h_1)$ and $H^1_0(\Omega, h_2)$ as the closures of $C^\infty_0(\Omega)$ with respect to the norms

$$\|u\|_{h_1} = \left(\int_\Omega h_1(x)|\nabla u|^2 dx\right)^{\frac{1}{2}}$$

for all $u \in C^\infty_0(\Omega)$ and

$$\|v\|_{h_2} = \left(\int_\Omega h_2(x)|\nabla v|^2 dx\right)^{\frac{1}{2}}$$

for all $v \in C^\infty_0(\Omega)$, respectively. It is clear that $X = H^1_0(\Omega, h_1) \times H^1_0(\Omega, h_2)$ is a Hilbert space under the norm $\|w\|_X = \|u\|_{h_1} + \|v\|_{h_2}$ for all $w = (u, v) \in X$.

We denote by $\mathcal{A}_2$ the class of C^1 functions $A : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which possess the following properties:
there exist two positive constants c_1 and c_2 such that

$$|A_t(x, s, t)| \le c_1 s^\gamma t^{\delta+1}, \qquad |A_s(x, s, t)| \le c_2 s^{\gamma+1} t^\delta \tag{1.1}$$

for all $(t, s) \in \mathbb{R}^2$, a.e $x \in \Omega$ and some constants $\gamma, \delta > 1$ with $\frac{\gamma+1}{p} + \frac{\delta+1}{q} = 1$, $\frac{\gamma+1}{2^*_\alpha} + \frac{\delta+1}{2^*_\beta} < 1$ and, $\gamma + 1 < p < 2^*_\alpha = \frac{2n}{n-2+\alpha}$, $\delta + 1 < q < 2^*_\beta = \frac{2n}{n-2+\beta}$, where p, q, α, β are positive constants and $\alpha, \beta \in (0, 2)$.

Throughout this paper, we assume the functions h_1 and h_2 satisfying the following conditions:

- $(\mathbf{H_1})$ The function $h_1 : \Omega \to [0, +\infty)$ belongs to $L^1_{loc}(\Omega)$ and there exists a constant $\alpha \ge 0$ such that $\liminf_{x\to z} |x - z|^{-\alpha} h_1(x) > 0$ for all $z \in \overline{\Omega}$,
- $(\mathbf{H_2})$ The function $h_2 : \Omega \to [0, +\infty)$ belongs to $L^1_{loc}(\Omega)$ and there exists a constant $\beta \ge 0$ such that $\liminf_{x\to z} |x - z|^{-\beta} h_2(x) > 0$ for all $z \in \overline{\Omega}$.

2. Main results

Here, we will prove a new version of Theorem 1.2 with the choice $X = H^1_0(\Omega, h_1) \times H^1_0(\Omega, h_2)$, where $h_i : \Omega \to [0, +\infty)$, $h_i \in L^1_{loc}$ $(i = 1, 2)$.

Theorem 2.1. *Assume that the hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *are satisfied. Also, let* $F_2, G_2, K_2 \in \mathcal{A}_2$, *with* $K_2(x, 0, 0) = 0$ *for all* $x \in \Omega$, *satisfy the following conditions:*

- (k_2) *there is* $a \in (0, \frac{\lambda_1}{2}\theta)$ *such that* $K_2(x, s, t) \le a|s|^{\gamma+1}|t|^{\delta+1}$ *for all* $x \in \Omega$, $s, t \in \mathbb{R}$, *where* $\lambda_1 = \inf_{w=(u,v)\in X\setminus\{(0,0)\}} \frac{\int_\Omega\left(\frac{\gamma+1}{p}h_1(x)|\nabla u|^2+\frac{\delta+1}{q}h_2(x)|\nabla v|^2\right)dx}{\int_\Omega |u|^{\gamma+1}|v|^{\delta+1}dx}$ *and* $\theta = \frac{1}{2\max\left\{\frac{\gamma+1}{p}, \frac{\delta+1}{q}\right\}}$,
- (f_4)
$$\lim_{s^2+t^2\to+\infty} \frac{\sup_{x\in\Omega}(|F_2(x, s, t)| + |G_2(x, s, t)|)}{s^2 + t^2} = 0;$$

(f_5)

$$\text{meas}\left(\left\{x \in \Omega : \left|F_2(x,0,0)\right|^2 + \left|G_2(x,0,0)\right|^2 > 0\right\}\right) > 0$$

and

$$\left|F_2(x,0,0)\right|^2 + \left|G_2(x,0,0)\right|^2 \leq \left|F_2(x,s,t)\right|^2 + \left|G_2(x,s,t)\right|^2$$

for all $s, t \in \mathbb{R}$, $x \in \Omega$;

(f_6) *the set of all* $x \in \Omega$ *that satisfy following condition*

$$\inf_{(s,t)\in\mathbb{R}^2} \Big(F_2(x,0,0)F_2(x,s,t) + G_2(x,0,0)G_2(x,s,t))\Big) < \left|F_2(x,0,0)\right|^2 + \left|G_2(x,0,0)\right|^2$$

has positive measure.

Then, for every convex set $S_2 \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ *dense in* $L^2(\Omega) \times L^2(\Omega)$, *there exists* $(\sigma_2, \varsigma_2) \in S_2$ *such that the problem*

(2.1)
$$\begin{cases} -\text{div}(h_1(x)\nabla u) = \sigma_2(x)F_{2u}(x,u,v) + \varsigma_2(x)G_{2u}(x,u,v) + K_{2u}(x,u,v) & \text{in } \Omega, \\ -\text{div}(h_2(x)\nabla v) = \sigma_2(x)F_{2v}(x,u,v) + \varsigma_2(x)G_{2v}(x,u,v) + K_{2v}(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in X *of the functional*

$$\begin{aligned}(u,v) \to &\frac{1}{2}\left(\int_\Omega h_1(x)|\nabla u|^2 dx + \int_\Omega h_2(x)|\nabla v|^2 dx\right) \\ &- \int_\Omega (\sigma_2(x)F_2(x,u,v) + \varsigma_2(x)G_2(x,u,v) + K_2(x,u(x),v(x)))dx.\end{aligned}$$

Proof. In order to apply Theorem 1.1 to our problem, we take $X = H_0^1(\Omega, h_1) \times H_0^1(\Omega, h_2)$, where endowed with the weak topology induced by the scalar product

$$\langle (u,v), (\zeta,\tau)\rangle_X = \int_\Omega (h_1(x)\nabla u(x)\nabla\zeta(x) + h_2(x)\nabla v(x)\nabla\tau(x))dx;$$

also, Y is the space $L^2(\Omega) \times L^2(\Omega)$ with the scalar product

$$\langle (\kappa,\omega), (h,k)\rangle_Y = \int_\Omega \kappa(x)h(x)dx + \int_\Omega \omega(x)k(x)dx$$

and T is $L^\infty(\Omega) \times L^\infty(\Omega)$. We define the functional $I : X \to \mathbb{R}$

$$I(u,v) = \frac{1}{2}\left(\int_\Omega h_1(x)|\nabla u|^2 dx + \int_\Omega h_2(x)|\nabla v|^2 dx\right) - \int_\Omega K_2(x,u(x),v(x))dx.$$

for all $(u,v) \in X$. Also, let x_0 is the zero of X and φ be the function defined by

$$\varphi(u,v) = \Big(F_2(\cdot,u(\cdot),v(\cdot)), G_2(\cdot,u(\cdot),v(\cdot))\Big)$$

for all $(u,v) \in X$. We check that the assumptions of Theorem 1.1 are satisfied. At first, from (f_5), we observe that

$$\|\varphi(0,0)\|_Y^2 = \int_\Omega \left|F_2(x,0,0)\right|^2 dx + \int_\Omega \left|G_2(x,0,0)\right|^2 dx > 0$$

and

$$\|\varphi(0,0)\|_Y^2 \leq \|\varphi(u,v)\|_Y^2$$

for all $(u,v) \in X$. Moreover, from (k_2), we obtain that

$$\int_\Omega K_2(x,u(x),v(x))dx \leq a \int_\Omega (|u(x)|^{\gamma+1}|v(x)|^{\delta+1})dx$$
$$\leq \frac{a}{\lambda_1} \int_\Omega \left(\frac{\gamma+1}{p} h_1(x)|\nabla u(x)|^2 + \frac{\delta+1}{q} h_2(x)|\nabla v(x)|^2 \right) dx$$

for all $(u,v) \in X$ and so, one has

$$I(u,v) \geq \frac{\theta}{2} \int_\Omega \left(\frac{\gamma+1}{p} h_1(x)|\nabla u(x)|^2 + \frac{\delta+1}{q} h_2(x)|\nabla v(x)|^2 \right) dx$$
$$\geq \frac{\theta}{2} \frac{\gamma+1}{p} \|u\|_{h_1}^2 + \frac{\theta}{2} \frac{\delta+1}{q} \|v\|_{h_2}^2 \tag{2.2}$$

for all $(u,v) \in X$. Take $K_2(x,0,0) = 0$ and $\frac{a}{\lambda_1} < \frac{\theta}{2}$ into account, from (2.2) we obtain $(0,0)$ is a global minimum of I in X. So, condition (φ_1) of Theorem 1.1 is satisfied. Also, by condition (f_3), we find (φ_2) is satisfied. Finally, fix $\sigma_2, \varsigma_2 \in \mathbb{R}$. Clearly, the function

$$(x,s,t) \to \sigma_2(x)F(x,s,t) + \varsigma_2(x)G(x,s,t) + K(x,s,t)$$

belongs to $\mathcal{A}_2$, and so the functional

$$(u,v) \to I(u,v) + \langle \varphi(u,v), (\sigma_2, \varsigma_2) \rangle_Y$$

is sequentially weakly lower semicontinuous in X. Indeed, let $\{w_m\} = \{(u_m, v_m)\}$ be a sequence that converges weakly to $w = (u,v)$ in X. We have

$$\int_\Omega \sigma_2(x)[F(x,u_m,v_m) - F(x,u,v)]dx + \int_\Omega \varsigma_2(x)[G(x,u_m,v_m) - G(x,u,v)]dx$$
$$= \int_\Omega \sigma_2(x)\nabla F(x, \theta_m(w_m - w))(w_m - w)dx +$$
$$\int_\Omega \varsigma_2(x)\nabla G(x, \rho_m(w_m - w))(w_m - w)dx$$
$$= \int_\Omega \sigma_2(x)F_u(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v))(u_m - u)dx$$
$$+ \int_\Omega \sigma_2(x)F_v(x, u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v))(v_m - v)dx$$
$$+ \int_\Omega \varsigma_2(x)G_u(x, u + \rho_{1,m}(u_m - u), v + \rho_{2,m}(v_m - v))(u_m - u)dx$$
$$+ \int_\Omega \varsigma_2(x)G_v(x, u + \rho_{1,m}(u_m - u), v + \rho_{2,m}(v_m - v))(v_m - v)dx,$$

where $\theta_m = (\theta_{1,m}, \theta_{2,m})$, $\rho_m = (\rho_{1,m}, \rho_{2,m})$ and $0 \leq \theta_{1,m}(x), \theta_{2,m}(x), \rho_{1,m}(x), \rho_{2,m}(x) \leq 1$ for all $x \in \Omega$. Now, take (1.1) and Hölder's inequality into account, we deduce that there exist $c_1, c_2, c_3, c_4 > 0$ such that

$$
\begin{aligned}
&\left|\int_\Omega \sigma_2(x)[F(x,u_m,v_m) - F(x,u,v)]dx + \int_\Omega \varsigma_2(x)[G(x,u_m,v_m) - G(x,u,v)]dx\right| \\
&\le \|\sigma_2\|_\infty \int_\Omega \big|F_u(x, u+\theta_{1,m}(u_m-u), v+\theta_{2,m}(v_m-v))\big||u_m-u|dx \\
&+ \|\sigma_2\|_\infty \int_\Omega \big|F_v(x, u+\theta_{1,m}(u_m-u), v+\theta_{2,m}(v_m-v))\big||v_m-v|dx \\
&+ \|\varsigma_2\|_\infty \int_\Omega \big|G_u(x, u+\rho_{1,m}(u_m-u), v+\rho_{2,m}(v_m-v))\big||u_m-u|dx \\
&+ \|\varsigma_2\|_\infty \int_\Omega \big|G_v(x, u+\rho_{1,m}(u_m-u), v+\rho_{2,m}(v_m-v))\big||v_m-v|dx \\
&\le c_1\|\sigma_2\|_\infty \int_\Omega \big|u+\theta_{1,m}(u_m-u)\big|^\gamma \big|v+\theta_{2,m}(v_m-v))\big|^{\delta+1}|u_m-u|dx \\
&+ c_2\|\sigma_2\|_\infty \int_\Omega \big|u+\theta_{1,m}(u_m-u)\big|^{\gamma+1} \big|v+\theta_{2,m}(v_m-v))\big|^{\delta}|v_m-v|dx \\
&+ c_3\|\varsigma_2\|_\infty \int_\Omega \big|u+\rho_{1,m}(u_m-u)\big|^\gamma \big|v+\rho_{2,m}(v_m-v))\big|^{\delta+1}|u_m-u|dx \\
&+ c_4\|\varsigma_2\|_\infty \int_\Omega \big|u+\rho_{1,m}(u_m-u)\big|^{\gamma+1} \big|v+\rho_{2,m}(v_m-v))\big|^{\delta}|v_m-v|dx.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
&\left|\int_\Omega \sigma_2(x)[F(x,u_m,v_m) - F(x,u,v)]dx + \int_\Omega \varsigma_2(x)[G(x,u_m,v_m) - G(x,u,v)]dx\right| \\
&\le c_1\|\sigma_2\|_\infty \big\|u+\theta_{1,m}(u_m-u)\big\|^\gamma_{L^p(\Omega)} \big\|v+\theta_{2,m}(v_m-v))\big\|^{\delta+1}_{L^q(\Omega)}\|u_m-u\|_{L^p(\Omega)} \\
&+ c_2\|\sigma_2\|_\infty \big\|u+\theta_{1,m}(u_m-u)\big\|^{\gamma+1}_{L^p(\Omega)} \big\|v+\theta_{2,m}(v_m-v))\big\|^{\delta}_{L^q(\Omega)}\|v_m-v\|_{L^q(\Omega)} \\
&+ c_3\|\varsigma_2\|_\infty \big\|u+\rho_{1,m}(u_m-u)\big\|^\gamma_{L^p(\Omega)} \big\|v+\rho_{2,m}(v_m-v))\big\|^{\delta+1}_{L^q(\Omega)}\|u_m-u\|_{L^p(\Omega)} \\
&+ c_4\|\varsigma_2\|_\infty \big\|u+\rho_{1,m}(u_m-u)\big\|^{\gamma+1}_{L^p(\Omega)} \big\|v+\rho_{2,m}(v_m-v))\big\|^{\delta}_{L^q(\Omega)}\|v_m-v\|_{L^q(\Omega)}.
\end{aligned}
$$

Since $2 < \gamma+1 < p < 2^*_\alpha$ and $2 < \gamma+1 < q < 2^*_\beta$, by the compact embedding $X \hookrightarrow L^p(\Omega)\times L^q(\Omega)$, the sequence $\{w_m\}$ convergence strongly to $w=(u,v)$ in the space $L^p(\Omega)\times L^q(\Omega)$, i.e., the sequence $\{u_m\}$ converges strongly to u in $L^p(\Omega)$ and $\{v_m\}$ converges strongly to v in $L^q(\Omega)$. Hence, it is easy to see that the sequences $\{\big\|u+\theta_{1,m}(u_m-u)\big\|_{L^p(\Omega)}\}$, $\{\big\|v+\theta_{2,m}(v_m-v)\big\|_{L^q(\Omega)}\}$, $\{\big\|u+\rho_{1,m}(u_m-u)\big\|_{L^p(\Omega)}\}$ and $\{\big\|v+\rho_{2,m}(v_m-v)\big\|_{L^q(\Omega)}\}$ are bounded. Thus, it follows

$$
\begin{aligned}
&\lim_{m\to\infty}\int_\Omega [\sigma_2(x)F(x,u_m,v_m) + \varsigma_2(x)G(x,u_m,v_m)]dx \\
&= \int_\Omega [\sigma_2 F(x,u,v) + \varsigma_2 G(x,u,v)]dx. \qquad (2.3)
\end{aligned}
$$

Using similar arguments as those used above, we obtain that

$$
\lim_{m\to\infty}\int_\Omega K(x,u_m,v_m)dx = \int_\Omega K(x,u,v)dx. \qquad (2.4)
$$

By the weak lower semicontinuity of the norms in the spaces $H_0^1(\Omega,h_1)$ and $H_0^1(\Omega,h_2)$ we deduce that

(2.5)
$$\lim_{m\to\infty}\inf\int_\Omega [h_1(x)|\nabla u_m|^2+h_2(x)|\nabla v_m|^2]dx \geq \int_\Omega [h_1(x)|\nabla u|^2+h_2(x)|\nabla v|^2]dx.$$

Hence, relations (2.3), (2.4) and (2.5) imply that the function

$$(u,v)\to I(u,v)+\langle\varphi(u,v),(\sigma_2,\varsigma_2)\rangle_Y$$

is sequentially weakly lower semicontinuous in X.

We now prove that the function

$$(u,v)\to I(u,v)+\langle\varphi(u,v),(\sigma_2,\varsigma_2)\rangle_Y$$

is coercive. Put

$$b=\max\{\|\sigma_2\|_{L^\infty(\Omega)},\|\varsigma_2\|_{L^\infty(\Omega)}\}$$

and fix $\epsilon>0$ such that

$$\epsilon<\min\left\{\frac{\lambda_1(h_1)}{b}\frac{\theta(\gamma+1)}{2p},\frac{\lambda_1(h_2)}{b}\frac{\theta(\delta+1)}{2q}\right\}, \tag{2.6}$$

where

$$\lambda_1(h_i):=\inf_{\phi\in H_0^1(\Omega,h_i)\setminus\{0\}}\frac{\int_\Omega h_i(x)|\nabla\phi|^2}{\int_\Omega|\phi|^2dx},\qquad i=1,2.$$

From (f_4), there is $c_\epsilon>0$ such that

$$|F(x,s,t)|+|G(x,s,t)|\leq\epsilon(|s|^2+|t|^2)+c_\epsilon$$

for all $(x,s,t)\in\Omega\times\mathbb{R}^2$. Hence, in view of (2.2), for each $(u,v)\in H_0^1(\Omega,h_1)\times H_0^1(\Omega,h_2)$, we conclude that

$$\begin{aligned}
&I(u,v)+\langle\varphi(u,v),(\sigma_2,\varsigma_2)\rangle_Y\geq\frac{\theta}{2}\frac{\gamma+1}{p}\|u\|_{h_1}^2+\frac{\theta}{2}\frac{\delta+1}{q}\|v\|_{h_2}^2\\
&-\int_\Omega|\sigma_2(x)F(x,u(x),v(x))+\varsigma_2(x)G(x,u(x),v(x))|dx\\
&\geq\frac{\theta}{2}\frac{\gamma+1}{p}\|u\|_{h_1}^2+\frac{\theta}{2}\frac{\delta+1}{q}\|v\|_{h_2}^2-b\epsilon\int_\Omega(|u(x)|^2+|v(x)|^2)dx-bc_\epsilon\mathrm{meas}(\Omega)\\
&\geq\frac{\theta}{2}\frac{\gamma+1}{p}\|u\|_{h_1}^2+\frac{\theta}{2}\frac{\delta+1}{q}\|v\|_{h_2}^2-b\epsilon\bigg(\frac{1}{\lambda_1(h_1)}\int_\Omega h_1(x)|\nabla u(x)|^2dx\\
&+\frac{1}{\lambda_1(h_2)}\int_\Omega h_2(x)|\nabla v(x)|^2dx\bigg)-bc_\epsilon\mathrm{meas}(\Omega)\\
&\geq\left(\frac{\theta(\gamma+1)}{2p}-\frac{b\epsilon}{\lambda_1(h_1)}\right)\int_\Omega h_1(x)|\nabla u|^2dx\\
&+\left(\frac{\theta(\delta+1)}{2q}-\frac{b\epsilon}{\lambda_1(h_2)}\right)\int_\Omega h_2(x)|\nabla v|^2dx-bc_\epsilon\mathrm{meas}(\Omega).
\end{aligned}$$

bearing in mind the relation (2.6), we have $\frac{\theta(\gamma+1)}{2p} - \frac{b\epsilon}{\lambda_1(h_1)}, \frac{\theta(\delta+1)}{2q} - \frac{b\epsilon}{\lambda_1(h_2)} > 0$, and so

$$\lim_{\|(u,v)\|_X \to +\infty} (I(u,v) + \langle \varphi(u,v), (\sigma_2, \varsigma_2)\rangle_Y) = +\infty,$$

as claimed. In virtue of Eberlein-Smulyan theorem, this also follows that the functional $I(u,v) + \langle \varphi(u,v), (\sigma_2, \varsigma_2)\rangle_Y$ is weakly lower semicontinuous. Thus, the assumptions of Theorem 1.1 are verified. Therefore, for each convex set $S_2 \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$, there exists $(\sigma_2, \varsigma_2) \in S_2$, such that the functional

$$\begin{aligned}(u,v) \to & \frac{1}{2}\left(\int_\Omega h_1(x)|\nabla u|^2 dx + \int_\Omega h_2(x)|\nabla v|^2 dx\right) \\ & - \int_\Omega (\sigma_2(x) F_2(x,u,v) + \varsigma_2(x) G_2(x,u,v) + K_2(x,u(x),v(x)))dx.\end{aligned}$$

has at least two global minima in X and by Example 38.25 of [5] it admits at least three critical points. Hence, the conclusion is achieved. □

Remark 2.1. Assume that $w = (u,v) \in X$ is a weak solution of problem (2.1), then $u \geq 0$ and $v \geq 0$ in Ω. Indeed, from our assumptions in Theorem 2.1, we have

$$\begin{aligned}0 = & \int_\Omega (h_1(x)\nabla u \cdot \nabla u^- + h_2(x)\nabla v \cdot \nabla v^-)dx \\ & - \int_\Omega \sigma_2(x)(F_{2u}(x,u,v)u^- + F_{2v}(x,u,v)v^-)dx \\ & - \int_\Omega \varsigma_2(x)(G_{2u}(x,u,v)u^- + G_{2v}(x,u,v)v^-)dx \\ & + \int_\Omega (K_{2u}(x,u,v)u^- + K_{2v}(x,u,v)v^-)dx \\ & = \|u^-\|^2_{h_1} + \|v^-\|^2_{h_2} \geq \lambda_1(h_1)\int_\Omega |u^-|^2 dx + \lambda_1(h_2)\int_\Omega |v^-|^2 dx,\end{aligned}$$

which implies that $u(x) \geq 0$ and $v(x) \geq 0$ for a.e. $x \in X$.

A special case of our main result is the following theorem.

Theorem 2.2. *Let $K_3 \in \mathcal{A}_2$, with $K_3(x,0,0) = 0$ for all $x \in \Omega$, satisfies (k_2). Then, for every convex set $S_3 \subseteq \mathbb{R}^2$, there exists $(\sigma_3, \varsigma_3) \in S_3$ such that the problem*

$$\begin{cases} -\mathrm{div}\big(|x|\nabla u\big) & \\ = (\sigma_3(x)\cos(uv\sqrt{uv}) - \varsigma_3(x)\sin(uv\sqrt{uv}))\frac{3\sqrt{uv^3}}{2} + K_{3u}(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}\big(|x|\nabla v\big) & \\ = (\sigma_3(x)\cos(uv\sqrt{uv}) - \varsigma_3(x)\sin(uv\sqrt{uv}))\frac{3\sqrt{u^3v}}{2} + K_{3v}(x,u,v) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega, |x|) \times H_0^1(\Omega, |x|)$ *of the functional*

$$
\begin{aligned}
(u, v) \to & \frac{1}{2}\left(\int_\Omega |x||\nabla u|^2 dx + \int_\Omega |x||\nabla v|^2 dx \right) \\
& - \int_\Omega (\sigma_3(x) sin(uv\sqrt{uv}) + \varsigma_3(x) cos(uv\sqrt{uv}) + K_3(x, u(x), v(x)))dx.
\end{aligned}
$$

Proof. Apply Theorem 2.1 to the functions $F_2, G_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $F_2(s,t) = \sin(st\sqrt{st})$ and $G_2(s,t) = \cos(st\sqrt{st})$ for all $(s,t) \in \mathbb{R}^2$. □

Conclusion. Due to the generality of the Theorem 1.1, it can be applied to many different situations.

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