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A NOTE ON LAGRANGE INTERPOLATION AT PRINCIPAL LATTICES

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Abstract. We give a geometric condition on principal lattices in $\mathbb{R}^n$ that ensures that the corresponding Lagrange interpolation polynomials of any sufficient smooth function converges to a Taylor polynomial.

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1. Introduction

Let $\mathcal{P}(\mathbb{R}^n)$ be the vector space of all polynomials (of real coefficients) in $\mathbb{R}^n$ and $\mathcal{P}_d(\mathbb{R}^n)$ the subspace consisting of all polynomials of degree at most d. It is well-known that $N := \dim \mathcal{P}_d(\mathbb{R}^n) = \binom{n+d}{d}$. The vector space $\mathcal{P}(\mathbb{R}^n)$ is endowed with the norm

$$\|P\|_\infty = \max_{|\alpha|\leq d} |c_\alpha| \quad \text{with} \quad P(\mathbf{x}) = \sum_{|\alpha|\leq d} c_\alpha \mathbf{x}^\alpha.$$

A subset $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ of $\mathbb{R}^n$ that consists of N distinct points is said to be unisolvent for $\mathcal{P}_d(\mathbb{R}^n)$ (or degree d) if, for every function f defined on X, there exists a unique $P \in \mathcal{P}_d(\mathbb{R}^n)$ such that $f(\mathbf{x}) = P(\mathbf{x})$ for all $\mathbf{x} \in X$. This polynomial is called the Lagrange interpolation polynomial of f at X and is denoted by $\mathbf{L}[X; f]$. Note that it is difficult to check whether a certain set of N distinct points is unisolvent of degree d as soon as $n \geq 2$. Some geometric configurations in $\mathbb{R}^n$ give unisolvent sets, e.g., the natural lattices, principal lattices and Bos configurations on algebraic hypersurfaces [4]. Now we choose a basis $\mathcal{B} = \{p_1, \dots, p_N\}$ for $\mathcal{P}_d(\mathbb{R}^n)$. Then

$$\mathrm{VDM}(\mathcal{B}; X) = \det[p_i(\mathbf{x}_j)]_{1\leq i,j\leq N}$$

is called the Vandermonde determinant. Here j is the row index of the matrix. It is well-known that X is unisolvent if and only if $\mathrm{VDM}(\mathcal{B}; X) \neq 0$. We have

$$\mathbf{L}[X; f](\mathbf{x}) = \sum_{p\in\mathcal{B}} \frac{\mathrm{VDM}(\mathcal{B}[p \leftarrow f]; X)}{\mathrm{VDM}(\mathcal{B}; X)} p, \tag{1.1}$$

where $\mathcal{B}[p \leftarrow f]$ means that we substitute f for p in $\mathcal{B}$. We are concerned with the following problem which was stated in [1].

Problem. *Suppose that the points of the unisolvent set X^m for $\mathcal{P}_d(\mathbb{R}^n)$ tend to the origin when $m \to \infty$. Determine conditions on X^m such that, for a sufficiently smooth function f, $\mathbf{L}[X^m; f]$ converges to $\mathbf{T}_0^d(f)$ (the Taylor expansion of f at 0 to the order d).*

In the one dimensional case $n = 1$, the convergence result holds without any condition on X^m (see [2, Theorem 1.4]). This fact comes from the Newton representation of the univariate Lagrange interpolation and the continuity property of divided difference with respect to the interpolation points. Unfortunately, the analogous property is not true in the multivariate case (see [1, Example 1.2]). As far as we have known, there are a few results focusing on the problem. Coatmelec showed in [5] that when the X^m are images of a fixed unisolvent set X under scalings by ratio r_m with $r_m \to 0$ composed with a rotation R_m of $\mathbb{R}^n$, $\mathbf{L}[X^m; f] \to \mathbf{T}_0^d(f)$ for any function f of class C^d. In [1], Bloom and Calvi gave a sufficient condition. The condition is that $\mathbf{L}[X^m; f]$ converges to 0 for any homogeneous polynomial f of degree $d + 1$. Using the Bloom-Calvi condition, Phung in [8, Proposition 4.6] showed that the X^m's can be chosen suitably on concentric circles centered at the origin. In [6], the authors treated the case when X^m is a natural lattice. Using a beautiful error formula of de Boor [3], they proved that when X^m satisfies a natural geometric condition the corresponding Lagrange interpolation polynomial (of fixed degree) of a sufficient smooth function converges to a Taylor polynomial. In this paper, we are interested in solving the problem when X^m is a principal lattice.

For convenience, we recall some facts about principal lattices. For $d \geq 1$, we set

$$\mathcal{S}_d = \big\{\beta : \beta = (\beta_0, \ldots, \beta_n) \in \mathbb{N}^{n+1} : |\beta| := \beta_0 + \cdots + \beta_n = d\big\},$$

where $\mathbb{N}$ is the set of all non-negative integers. Let $A = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ such that the $\mathbf{a}_i$'s form a simplex in $\mathbb{R}^n$. For abbreviation, we say that A is a simplex in $\mathbb{R}^n$. Let us denote by $\mathrm{PL}_d(A)$ the set of points

$$\mathrm{PL}_d(A) := \Big\{\mathbf{x} = \sum_{i=0}^{n} \frac{\beta_i}{d}\mathbf{a}_i : \beta \in \mathcal{S}_d\Big\}.$$

We call $\mathrm{PL}_d(A)$ the principal lattice of degree d generated by A. We have known that $\mathrm{PL}_d(A)$ is a unisolvent set of degree d in $\mathbb{R}^n$. It is the intersections of certain hyperplanes in $\mathbb{R}^n$. Moreover, the fundamental Lagrange interpolation polynomials are the products of affine polynomials. For a deeper discussion of the principal lattice and its generalization, we refer the reader to [7] and the references given there.

Observe that the Lagrange interpolation operator has bad behavior when the interpolation points tend to a hyperplane. Hence we must give a condition on the simplexes A^m such that the principal lattices $\mathrm{PL}_d(A^m)$ do not become more and more flat. The precise condition is given below.

Let $A = \{\mathbf{a}_0, \dots, \mathbf{a}_n\}$ be a simplex in $\mathbb{R}^n$. Let $\mathcal{H}_{A\setminus\{\mathbf{a}_i\}}$ be the unique affine hyperplane passing through all points in $A \setminus \{\mathbf{a}_i\}$. We define the quantities

$$D_A = \max_{0\le i<j\le n} \|\mathbf{a}_i - \mathbf{a}_j\| \quad \text{and} \quad H_A = \min_{0\le i\le n} \operatorname{dist}(\mathbf{a}_i, \mathcal{H}_{A\setminus\{\mathbf{a}_i\}}),$$

where we denote by $\|\mathbf{a}\|$ the Euclidean norm of $\mathbf{a} \in \mathbb{R}^n$. Our first main result focuses on a special kind of simplex in $\mathbb{R}^n$.

Theorem 1.1. *Let $\delta > 1$ and $A = \{\mathbf{a}_0, \dots, \mathbf{a}_n\}$ be a simplex in $\mathbb{R}^n$ such that*

$$\max_{1\le i\le n} \|\mathbf{a}_i - \mathbf{a}_0\| \le \delta \min_{1\le i\le n} \operatorname{dist}(\mathbf{a}_i, \mathcal{H}_{A\setminus\{\mathbf{a}_i\}}). \tag{1.2}$$

Then there exists a constant $\epsilon > 1$ depending only on n and δ such that

$$D_A \le \epsilon H_A, \tag{1.3}$$

Conversely, if (1.3) holds, then there exists $\delta > 1$ such that (1.2) and similar relations corresponding to $\mathbf{a}_i$, $i = 1, \dots, n$, also hold.

The above theorem gives the definition of admissible simplexes.

Definition 1.1. *The sequence $A^m = \{\mathbf{a}_0^m, \dots, \mathbf{a}_n^m\}$ of simplexes in $\mathbb{R}^n$ is said to be admissible if there exists $\epsilon > 1$ such that*

$$D_{A^m} \le \epsilon H_{A^m}, \quad \forall m \ge 1. \tag{1.4}$$

The following theorem is another main result of our paper.

Theorem 1.2. *For $d \in \mathbb{N}^*$, let $A^m = \{\mathbf{a}_0^m, \dots, \mathbf{a}_n^m\}$ be a sequence of admissible simplexes in $\mathbb{R}^n$ such that, for $i = 0, \dots, n$, $\mathbf{a}_i^m \to 0$ as $m \to \infty$. Then for every function f of class C^{d+1} in a neighborhood of 0 we have*

$$\lim_{m\to\infty} \mathbf{L}[\mathrm{PL}_d(A^m); f] = \mathbf{T}_0^d(f),$$

where $\mathrm{PL}_d(A^m)$ is the principal lattice of degree d generated by A^m.

Note that an error formula for the Lagrange interpolation polynomial at the principal lattice is available (see for instance [7]). But it quite difficult to use it to prove Theorem 1.2. Now we outline the method of the proof. Observe that a principal lattice in $\mathbb{R}^n$ can be regarded as the image of the standard principal lattice under a linear or an affine transformation of $\mathbb{R}^n$. Here the standard principal lattice is the lattice spanned by the standard simplex in $\mathbb{R}^n$. Hence, we can use [1, Corollary 2.2] to reduce the convergence property in Theorem 1.2 to a condition on certain linear transforms. We show that the condition on linear transforms holds when the principal lattices are generated by a sequence of admissible simplexes, and the theorem follows.

2. Proofs of the main results

In this section, we give the proofs of the two theorems stated in the previous section.

Proof of Theorem 1.1. The reverse conclusion is obviously true. Assume that (1.2) holds. For convenience, we set

$$D^0 = \max_{1\le i\le n} \|\mathbf{a}_i - \mathbf{a}_0\| \quad \text{and} \quad H^0 = \min_{1\le i\le n} \operatorname{dist}(\mathbf{a}_i, \mathcal{H}_{A\setminus\{\mathbf{a}_i\}}).$$

By definition, we have $D^0 \le \delta H^0$. It is easily seen that

$$D^0 \le D_A \le 2D^0 \quad \text{and} \quad H_A = \min\{H^0, \operatorname{dist}(\mathbf{a}_0, \mathcal{H}_{A\setminus\{\mathbf{a}_0\}})\}.$$

Therefore, it suffices to show that there exists a constant $\epsilon_n > 0$ depending only on n and δ such that

$$D^0 \le \epsilon_n \operatorname{dist}(\mathbf{a}_0, \mathcal{H}_{A\setminus\{\mathbf{a}_0\}}). \tag{2.1}$$

To prove above claim, we will verify that there is a positive constant c_n depending only on n and δ such that

$$\operatorname{vol}(A) \ge c_n (D^0)^n, \tag{2.2}$$

where $\operatorname{vol}(A)$ is the volume of the polyhedron in $\mathbb{R}^n$ generated by A. The proof is by induction on n. If $n = 2$, then it is obvious that we can take $c_2 = \frac{1}{2\delta}$, because when $D^0 = \|\mathbf{a}_2 - \mathbf{a}_0\|$ we can write

$$\operatorname{vol}(A) = \frac{1}{2}\|\mathbf{a}_2 - \mathbf{a}_0\| \cdot \operatorname{dist}(\mathbf{a}_1, \mathcal{H}_{\{\mathbf{a}_2,\mathbf{a}_0\}}) \ge \frac{1}{2} D^0 \frac{D^0}{\delta} = \frac{(D^0)^2}{2\delta}.$$

Assume the estimate holds for $n-1$; we will prove it for n. Observe that

$$\|\mathbf{a}_i - \mathbf{a}_0\| \ge \operatorname{dist}(\mathbf{a}_i, \mathcal{H}_{A\setminus\{\mathbf{a}_i\}}) \ge \frac{1}{\delta} D^0, \quad i = 1, \dots, n.$$

It follows that

$$D^0 \ge \widetilde{D}^0 \ge \frac{1}{\delta} D^0, \tag{2.3}$$

where

$$\widetilde{D}^0 := \max_{1\le i\le n-1} \|\mathbf{a}_i - \mathbf{a}_0\|. \tag{2.4}$$

For $B \subset A$, we will denote by $\mathcal{H}_B$ the $(\operatorname{card}(B) - 1)$-dimensional plane passing through all points in B. Evidently, if $B \subset B' \subset A$, then $\operatorname{dist}(\mathbf{a}, \mathcal{H}_B) \ge \operatorname{dist}(\mathbf{a}, \mathcal{H}_{B'})$ for every $\mathbf{a} \in \mathbb{R}^n$. Therefore,

$$\min_{1\le i\le n-1} \operatorname{dist}(\mathbf{a}_i, \mathcal{H}_{A\setminus\{\mathbf{a}_i,\mathbf{a}_n\}}) \ge H^0 \ge \frac{1}{\delta} D^0. \tag{2.5}$$

We now consider the $(n-1)$-dimensional space $\mathcal{H} := \mathcal{H}_{A\setminus\{\mathbf{a}_n\}}$. Looking at (2.3), (2.5) and using the induction hypothesis for $\mathcal{H}$, we find that

$$\mathrm{vol}\big(A \setminus \{\mathbf{a}_n\}\big) \geq c_{n-1}\big(\widetilde{D}^0\big)^{n-1} \geq \frac{c_{n-1}}{\delta^{n-1}}\big(D^0\big)^{n-1}.$$

It follows that

$$\begin{aligned}\mathrm{vol}(A) &= \frac{1}{n}\mathrm{vol}\big(A \setminus \{\mathbf{a}_n\}\big)\mathrm{dist}(\mathbf{a}_n, \mathcal{H}_{A\setminus\{\mathbf{a}_n\}}) \\ &\geq \frac{1}{n}\frac{c_{n-1}}{\delta^{n-1}}\big(D^0\big)^{n-1}\frac{D^0}{\delta} = c_n\big(D^0\big)^n, \quad c_n = \frac{c_{n-1}}{n\delta^n},\end{aligned}$$

which completes the proof of the estimate. Now since $\|\mathbf{a}_i - \mathbf{a}_j\| \leq 2D^0$ for every $i \neq j$, we have

$$\mathrm{vol}\big(A \setminus \{\mathbf{a}_0\}\big) \leq \big(2D^0\big)^{n-1}.$$

Combining the last relation with (2.2) we obtain

$$\mathrm{dist}(\mathbf{a}_0, \mathcal{H}_{A\setminus\{\mathbf{a}_0\}}) = \frac{n\mathrm{vol}\big(A\big)}{\mathrm{vol}\big(A \setminus \mathbf{a}_0\big)} \geq \frac{nc_n}{2^{n-1}}D^0,$$

which gives (2.1), and the proof is complete. □

The tool to prove Theorem 1.2 comes from a result of Bloom and Calvi [1]. Note that the rate of convergence is obtained from the proof of the result.

Theorem 2.1. *(Bloom-Calvi) Let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ be a unisolvent set of degree d in $\mathbb{R}^n$. Let $\{\Phi_m\}$ be a sequence of linear automorphism of $\mathbb{R}^n$. Assume that $\|\Phi_m\|^{d+1}\|\Phi_m^{-1}\|^d \to 0$ as $m \to \infty$, where $\|\cdot\|$ is any matrix norm. Then for every function f of class C^{d+1} in a neighborhood of 0 we have*

$$\lim_{m\to\infty} \mathbf{L}[\Phi_m(X); f] = \mathbf{T}_0^d(f).$$

Furthermore,

$$\|\mathbf{L}[\Phi_m(X); f] - \mathbf{T}_0^d(f)\|_\infty = O\big(\|\Phi_m\|^{d+1}\|\Phi_m^{-1}\|^d\big),$$

where the constant in O depends on n, d, X and f.

To use Theorem 2.1, it is necessary to study the norm of the inverse matrix. Let $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be n distinct points in $\mathbb{R}^n$ with $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$ such that $A_0 := \{0, \mathbf{a}_1, \dots, \mathbf{a}_n\}$ is a simplex in $\mathbb{R}^n$. This condition holds if and only if $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is linearly independent. We consider the square matrix $M = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$, where $\mathbf{a}_j$ is the j-th column of M. Let us define the following two norms of M

$$\|M\|_C = \max\{\|\mathbf{a}_1\|, \dots, \|\mathbf{a}_n\|\} \quad \text{and} \quad \|M\|_R = \|M^T\|_C,$$

where M^T is the transpose of M. We always denote by $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ the standard basis for $\mathbb{R}^n$.

Lemma 2.1. *We have*

$$\|M^{-1}\|_R = \frac{1}{\min_{1\le i\le n} \operatorname{dist}(\mathbf{a}_i, \mathcal{H}_{A_0\setminus\{\mathbf{a}_i\}})}.$$

Proof. Let C be the cofactor matrix of M and $\operatorname{adj}(M) = C^T$, the adjugate of M. We have

$$M^{-1} = \frac{1}{\det M}\operatorname{adj}(M).$$

It is easily seen that the first row of $\operatorname{adj}(M)$ is the vector

$$\mathbf{u}_1 = \det(\mathbf{e}, \mathbf{a}_2, \ldots, \mathbf{a}_n) := \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{2.6}$$

Here the determinant in (2.6) is taken pointwisely according to the first row. Since $\{\mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly independent, $\mathbf{u}_1$ is nonzero. Moreover, it is a normal vector of the hyperplane $\mathcal{H}_{A_0\setminus\{\mathbf{a}_1\}}$, because $\langle \mathbf{a}, \mathbf{u}_1\rangle = \det(\mathbf{a}, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ for $\mathbf{a} \in \mathbb{R}^n$, and hence $\langle \mathbf{a}_j, \mathbf{u}_1\rangle = 0$ for $j = 2, \ldots, n$. We thus get $\mathcal{H}_{A_0\setminus\{\mathbf{a}_1\}} = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}_1, \mathbf{x}\rangle = 0\}$ and

$$\operatorname{dist}(\mathbf{a}_1, \mathcal{H}_{A_0\setminus\{\mathbf{a}_1\}}) = \frac{|\langle \mathbf{a}_1, \mathbf{u}_1\rangle|}{\|\mathbf{u}_1\|} = \frac{|\det M|}{\|\mathbf{u}_1\|}.$$

The same relation holds for the k-th row $\mathbf{u}_k$ of $\operatorname{adj}(M)$. It follows that

$$\begin{aligned} \|M^{-1}\|_R &= \frac{1}{|\det M|}\|\operatorname{adj}(M)\|_R \\ &= \frac{1}{|\det M|}\max\big\{\|\mathbf{u}_1\|, \ldots, \|\mathbf{u}_n\|\big\} \\ &= \max_{1\le i\le n} \frac{1}{\operatorname{dist}(\mathbf{a}_i, \mathcal{H}_{A_0\setminus\{\mathbf{a}_i\}})}. \end{aligned}$$

The proof is complete. □

Proof of Theorem 1.2. The proof will be divided into two steps.
Step 1. We first assume that $\mathbf{a}_0^m = 0$ for every $m \ge 1$. Let Φ_m be the unique invertible linear automorphism of $\mathbb{R}^n$ such that $\Phi_m(\mathbf{e}_i) = \mathbf{a}_i^m$ for $i = 1, \ldots, n$. Then the matrix of Φ_m is the square matrix $M_m = [\mathbf{a}_1^m\ \mathbf{a}_2^m \cdots \mathbf{a}_n^m]$. By definition, we have $\|M_m\|_C = \max_{1\le i\le n} \|\mathbf{a}_i^m\|$. Using Lemma 2.1, we get

$$\|M_m^{-1}\|_R = \frac{1}{\min_{1\le i\le n} \operatorname{dist}(\mathbf{a}_i^m, \mathcal{H}_{A^m\setminus\{\mathbf{a}_i^m\}})}.$$

Let E_d be the principal lattice of degree d generated by $\{0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Then E_d is a unisolvent set of degree d in $\mathbb{R}^n$. Evidently, $\Phi_m(E_d) = \mathrm{PL}_d(A^m)$. Theorem 2.1 and the hypothesis that the sequence of simplexes is admissible now yield

$$\|\mathbf{L}[\mathrm{PL}_d(A^m); f] - \mathbf{T}_0^d(f)\|_\infty = O\Big((\|M_m\|_C)^{d+1}(\|M_m^{-1}\|_R)^d\Big) = O(\max_{1\le i\le n}\|\mathbf{a}_i^m\|), \tag{2.7}$$

where the constant in O depends on n, d, ϵ and f. In particular, $\mathbf{L}[\mathrm{PL}_d(A^m); f] \to \mathbf{T}_0^d(f)$ as $m \to \infty$.

Step 2. We prove the theorem in the general case. Using arguments in the first step, we conclude from (2.7) that

$$\|\mathbf{L}[\mathrm{PL}_d(A^m); f] - \mathbf{T}^d_{\mathbf{a}_0^m}(f)\|_\infty = O(\max_{1\le i\le n} \|\mathbf{a}_i^m - \mathbf{a}_0^m\|). \tag{2.8}$$

On the other hand, since f is of class C^{d+1} in a neighborhood of 0, we easily seen that

$$\|\mathbf{T}^d_{\mathbf{a}_0^m}(f) - \mathbf{T}^d_0(f)\|_\infty = \max_{|\alpha|\le d} \frac{1}{\alpha!}|D^\alpha f(\mathbf{a}_0^m) - D^\alpha f(0)| = O(\|\mathbf{a}_0^m\|).$$

From what has already been proved, we have

$$\begin{aligned}\|\mathbf{L}[\mathrm{PL}_d(A^m); f] - \mathbf{T}^d_0(f)\|_\infty &\le \|\mathbf{L}[\mathrm{PL}_d(A^m); f] - \mathbf{T}^d_{\mathbf{a}_0^m}(f)\|_\infty + \|\mathbf{T}^d_{\mathbf{a}_0^m}(f) - \mathbf{T}^d_0(f)\|_\infty \\ &= O(\max_{1\le i\le n}\|\mathbf{a}_i^m - \mathbf{a}_0^m\|) + O(\|\mathbf{a}_0^m\|) \\ &= O(\max_{0\le i\le n}\|\mathbf{a}_i^m\|).\end{aligned}$$

It follows that

$$\lim_{m\to\infty} \mathbf{L}[\mathrm{PL}_d(A^m); f] = \mathbf{T}^d_0(f),$$

and the proof is complete. □

The following result is a direct consequence of the proof of Theorem 1.2.

Corollary 2.1. *For $d \in \mathbb{N}^*$, let $A_0^m := \{0, \mathbf{a}_1^m, \ldots, \mathbf{a}_n^m\}$ be a sequence of simplexes in $\mathbb{R}^n$ such that*

$$\lim_{m\to\infty} \frac{\big(\max_{1\le i\le n}\|\mathbf{a}_i^m\|\big)^{d+1}}{\big(\min_{1\le i\le d}\mathrm{dist}(\mathbf{a}_i^m, \mathcal{H}_{A_0^m\setminus\{\mathbf{a}_i^m\}})\big)^d} = 0.$$

Then for every function f of class C^{d+1} in a neighborhood of 0 *we have*

$$\lim_{d\to\infty} \mathbf{L}[\mathrm{PL}_d(A_0^m); f] = \mathbf{T}^d_0(f).$$

Example 1. This example generalizes [1, Example 1.2]. It shows that the condition to be admissible of $\{A^m\}$ in Theorem 1.2 can not be removed.

Let $\mathcal{B} = \{1, x_1, \ldots, x_n\}$ be a basis for $\mathcal{P}_1(\mathbb{R}^n)$. Let us take $\mathbf{a}_0^m = 0$, $\mathbf{a}_i^m = \frac{1}{m}\mathbf{e}_i$ for $i = 1, \ldots, n-1$ and $\mathbf{a}_n^m = (0, \ldots, 0, \frac{1}{m^\alpha}, \frac{1}{m^\beta})$ with $\beta > 2\alpha > 2$. Then $A^m := \{\mathbf{a}_0^m, \ldots, \mathbf{a}_n^m\}$ is a simplex in $\mathbb{R}^n$ for $m \ge 1$. We have

$$H_{A^m} \le \mathrm{dist}(\mathbf{a}_n^m, \mathcal{H}_{A^m\setminus\{\mathbf{a}_n^m\}}) = \frac{1}{m^\beta} \quad \text{and} \quad D_{A^m} = \frac{\sqrt{2}}{m}.$$

Hence, the sequence $\{A^m\}$ is not admissible. Easily computations give

$$\mathrm{VDM}(\mathcal{B}; A^m) = \frac{1}{m^{n+\beta-1}}.$$

If we choose $f(\mathbf{x}) = x_{n-1}^2$, then

$$\mathrm{VDM}(\mathcal{B}[x_n \leftarrow f]; A^m) = \frac{1}{m^{n-2}}(\frac{1}{m^{2\alpha+1}} - \frac{1}{m^{\alpha+2}}).$$

Consequently, in view of (1.1), we see that the coefficient of x_n in $\mathbf{L}[X^m; f]$ is equal to

$$\frac{\mathrm{VDM}(\mathcal{B}[x_n \leftarrow f]; A^m)}{\mathrm{VDM}(\mathcal{B}; A^m)} = \frac{\frac{1}{m^{2\alpha+1}} - \frac{1}{m^{\alpha+2}}}{\frac{1}{m^{\beta+1}}},$$

which tends to $-\infty$ when $m \to \infty$. On the other hand, since A^m is a unisolvent set of degree 1 in $\mathbb{R}^n$, we have $\mathbf{T}_0^1(f) = 0$. Hence $\mathbf{L}[X^m; f]$ does not convege to $\mathbf{T}_0^1(f)$ as $m \to \infty$.

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