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EXTREMAL PROBLEMS FOR A POLYNOMIAL AND ITS POLAR DERIVATIVE

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Abstract. This paper considers the well known Erdös-Lax and Turán-type inequalities that relate the uniform norm of a univariate complex coefficient polynomial to that of its derivative on the unit circle in the plane. Here, we establish some new inequalities that relate the uniform norm of a polynomial and its polar derivative while taking into account the placement of the zeros and the extremal coefficients of the polynomial. The obtained results strengthen some recently proved Erdös-Lax and Turán-type inequalities for constrained polynomials and also produce various inequalities that are sharper than the previous ones known in the literature on this subject.

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1. Introduction

The inequalities for polynomials and their derivatives generalizing the classical inequalities for various norms and with various constraints on using different methods of the geometric function theory is a fertile area in analysis. Various inequalities in both directions relating the norm of the derivative and the polynomial itself play a key role in the literature for proving the inverse theorems in approximation theory and, of course have their own intrinsic interest. These inequalities for constrained polynomials have been the subject of many research papers which is witnessed by many recent articles (for example, see [8], [12], [14], [15], [19]-[23]). The unit disk in the complex plane serves as the prototype of a bounded domain for studying extremal properties of polynomials and their derivatives. If one is interested in how "big" a polynomial or its derivative can be in the unit disk, then, because of the maximum modulus principle, it suffices to study the values on the boundary. A well-known classical result is the Bernstein-inequality [4] for the uniform norm on the unit circle in the plane: namely, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin. It might easily be observed that the restriction on the zeros of $P(z)$ imply an improvement in (1.1). It turns out that to have any hope of a lower bound or an improved upper bound, one must have some control over the location of the zeros of polynomial $P(z)$. It was conjectured by P. Erdös and later proved by Lax [16] that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

On the other hand, in 1939 (see [26]), Turán obtained a lower bound for the maximum of $|P'(z)|$ on $|z| = 1$, by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Thus in (1.2) and (1.3) equality holds for those polynomials of degree n having all their zeros on $|z| = 1$. As a generalization of (1.3), Govil [10] proved that if $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{1.4}$$

whereas, for the class of polynomials not vanishing in $|z| < k,\ k \leq 1$, the precise estimate of maximum of $|P'(z)|$ on $|z| = 1$ is not easily obtainable. In 1980, it was again Govil [9], who generalized (1.2) by proving that if $P(z)$ does not vanish in $|z| < k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{1.5}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. As is easy to see that (1.4) and (1.5) become equalities if $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomial having all zeros on $|z| = k$, then it may be possible to improve the bounds in (1.4) and (1.5). In this connection, it was shown by Govil [11] that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}, \tag{1.6}$$

whereas, the corresponding improvement of (1.5) was obtained by Aziz and Ahmad [3] in 1997. In fact, they proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\}, \tag{1.7}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$.
Over the last four decades many different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on the multiplicity of zero at $z=0$, the modulus of largest root of $P(z)$, restriction on coefficients etc. The one such generalization is moving from the domain of ordinary derivative to their polar derivative. Let us remind that the polar derivative of a polynomial $P(z)$ of degree n with respect to point $\alpha \in \mathbb{C}$ (see [17]) is defined as

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\} = P'(z),$$

uniformly with respect to z for $|z| \le R,\ R > 0$.
For more information on the polar derivative of polynomials, one can consult the comprehensive books of Marden [17], Milovanonić et al. [18] or Rahman and Schmeisser [25]. In 1998, Aziz and Rather [2] established the polar derivative generalization of (1.4) by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for every complex number α with $|\alpha| \ge k$,

$$\max_{|z|=1}|D_\alpha P(z)| \ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|, \tag{1.8}$$

whereas, the corresponding polar derivative analogue of (1.6) and a refinement of (1.8) was given by Dewan et al. [7]. They proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge k$,

$$\max_{|z|=1}|D_\alpha P(z)| \ge \frac{n}{1+k^n}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)| + \left(|\alpha| + \frac{1}{k^{n-1}}\right)\min_{|z|=k}|P(z)|\right\}. \tag{1.9}$$

In 2010, Dewan et al. [5] established an interesting generalization of (1.9) and proved that if $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s}z^{n-s}),\ 0 \le s \le n$, is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge k$,

$$\max_{|z|=1}|D_\alpha P(z)| \ge \frac{n}{1+k^{n-s}}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)| + \left(\frac{|\alpha|}{k^s} + \frac{1}{k^{n-1}}\right)\min_{|z|=k}|P(z)|\right\}. \tag{1.10}$$

Very recently, Kumar and Dhankhar [15] used a new version of Schwarz lemma and obtained some inequalities for the derivative of constrained polynomials giving extensions and refinements of (1.4) and (1.5) in the form of the following results.

Theorem A. If $P(z) = z^s(a_0 + a_1z + a_2z^2 + ... + a_{n-s}z^{n-s}),\ 0 \le s \le n$, is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then

(1.11)
$$\max_{|z|=1} |P'(z)| \ge \left(\frac{n}{1+k^{n-s}} + \frac{n(k^n|a_{n-s}| - k^s|a_0|)(k-1)}{2(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0|)} \right) \max_{|z|=1} |P(z)|.$$

Equality in (1.11) holds for $P(z) = z^n + k^n$.

Theorem B. Let $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$, be a polynomial of degree n having no zeros in $|z| < k,\ k \le 1$, and $Q(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then

$$\max_{|z|=1} |P'(z)| \le \left(\frac{n}{1+k^n} - \frac{nk^{n-1}(|a_0| - k^n|a_n|)(1-k)}{2(1+k^n)(|a_0| + k^{n-1}|a_n|)} \right) \max_{|z|=1} |P(z)|. \tag{1.12}$$

Equality in (1.12) holds for $P(z) = z^n + k^n$.

In the same paper, Kumar and Dhankhar also obtained the polar derivative generalizations of (1.11) and (1.12) in the form of the following results.

Theorem C. Let $P(z) = z^s(a_0 + a_1z + a_2z^2 + ... + a_{n-s}z^{n-s}),\ 0 \le s \le n$, be a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge k$,

$$\begin{aligned} &\max_{|z|=1} |D_\alpha P(z)| \\ &\ge \left(\frac{n(|\alpha| - k)}{1+k^{n-s}} + \frac{n(|\alpha| - k)(k^n|a_{n-s}| - k^s|a_0|)(k-1)}{2(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0|)} \right) \max_{|z|=1} |P(z)|. \end{aligned} \tag{1.13}$$

Theorem D. Let $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$, be a polynomial of degree n having no zeros in $|z| < k,\ k \le 1$, and $Q(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for any complex number α with $|\alpha| \ge 1$,

$$\begin{aligned} &\max_{|z|=1} |D_\alpha P(z)| \\ &\le \left(\frac{n(|\alpha| + k^n)}{1+k^n} - \frac{n(|\alpha| - 1)k^{n-1}(|a_0| - k^n|a_n|)(1-k)}{2(1+k^n)(|a_0| + k^{n-1}|a_n|)} \right) \max_{|z|=1} |P(z)|. \end{aligned} \tag{1.14}$$

Note: Dividing both sides of (1.13) and (1.14) by $|\alpha|$ and let $|\alpha| \to \infty$, we get respectively (1.11) and (1.12).

The purpose of this paper is to further strengthen the inequalities (1.8)–(1.14). Besides, the obtained results produce refinements of inequalities (1.6), (1.7) and related Erdös-Lax and Turán-type inequalities as well. Moreover, some concrete numerical examples are presented, showing that in some situations, the bounds obtained by our results can be considerably sharper than the ones previously known.

2. Main results

In this section, we state our main results and their proofs are given in the next section. We begin by presenting the following strengthening of (1.10) and (1.13).

Theorem 1. Let $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s}z^{n-s}),\ 0 \le s \le n$, be a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge k$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| & \\ &\ge \frac{n}{1+k^{n-s}}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)| + \left(\frac{|\alpha|}{k^s} + \frac{1}{k^{n-1}}\right)m\right\} \\ &\quad + \frac{n(|\alpha|-k)(k^n|a_{n-s}| - k^s|a_0| - m)(k-1)}{2(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - m)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\},\end{aligned} \tag{2.1}$$

where here and throughout this paper $m = \min_{|z|=k} |P(z)|$.

Taking $s = 0$ in Theorem 1, we get the following refinement of (1.9).

Corollary 1. Let $P(z) = \sum\limits_{v=0}^{n} a_v z^v$, is a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for any complex number α with $|\alpha| \ge k$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| & \\ &\ge \frac{n}{1+k^{n}}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)| + \left(|\alpha| + \frac{1}{k^{n-1}}\right)m\right\} \\ &\quad + \frac{n(|\alpha|-k)(k^n|a_{n}| - |a_0| - m)(k-1)}{2(1+k^{n})(k^n|a_{n}| + k|a_0| - m)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\}.\end{aligned} \tag{2.2}$$

If we divide both sides of (2.1) and (2.2) by $|\alpha|$ and let $|\alpha| \to \infty$, we easily get the following refinements of (1.11) and (1.6) respectively.

Corollary 2. Let $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s}z^{n-s}),\ 0 \le s \le n$, be a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$,

$$\begin{aligned}\max_{|z|=1} |P'(z)| & \\ &\ge \frac{n}{1+k^{n-s}}\left\{\max_{|z|=1}|P(z)| + \frac{m}{k^s}\right\} \\ &\quad + \frac{n(k^n|a_{n-s}| - k^s|a_0| - m)(k-1)}{2(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - m)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\}.\end{aligned} \tag{2.3}$$

Corollary 3. If $P(z) = \sum\limits_{v=0}^{n} a_v z^v$, is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\begin{aligned} \max_{|z|=1} |P'(z)| & \\ &\geq \frac{n}{1+k^n}\left\{ \max_{|z|=1} |P(z)| + m \right\} \\ &+ \frac{n(k^n|a_n| - |a_0| - m)(k-1)}{2(1+k^n)(k^n|a_n| + k|a_0| - m)}\left\{ \max_{|z|=1} |P(z)| - \frac{m}{k^n} \right\}. \end{aligned} \tag{2.4}$$

Equality in (2.4) holds for $P(z) = z^n + k^n$.

Remark 1. It may be remarked that, in general, for any polynomial of degree n of the form $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s}),\ 0 \leq s \leq n$, having all its zeros in $|z| \leq k,\ k \geq 1$, the inequalities (2.1) and (2.3) would give improvements over the bounds obtained from the inequalities (1.13) and (1.11) respectively, excepting the case when some or all the zeros of $P(z)$ lie on $|z| = k$. For the class of polynomials having a zero on $|z| = k$ and $k \neq 1$, the inequalities (2.2) and (2.4) will give bounds that are sharper than obtainable from the inequalities (1.9) and (1.6) respectively. Also, (2.1) implies a considerable improvement of (1.10) for $k \neq 1$. One can also observe that for the class of polynomials having all their zeros in $|z| < k$, the inequalities (2.2) and (2.4) respectively improve the inequalities (1.9) and (1.6) considerably when $k^n|a_n| - |a_0| - m \neq 0$ and $k > 1$. We shall illustrate this by means of the following example.

Example 1. Let $P(z) = z^2(z^4 - 2z^3 + 4z - 4)$. Then $P(z)$ is a polynomial of degree 6 having a zero of order 2 at $z = 0$ and the remaining zeros $\{-\sqrt{2},\ \sqrt{2},\ 1-i,\ 1+i\}$ on the circle $|z| = \sqrt{2}$. For this polynomial, we find that

$$\max_{|z|=1} |P(z)| = 9.614 \text{ (approximately)}$$

and

$$m = \min_{|z|=k} |P(z)| = k^2(k^2 - 2)[(k-1)^2 + 1].$$

If we take $k = 2$, so that $P(z)$ has all its zeros in $|z| \leq k = 2$. Taking $\alpha = \frac{7+i\sqrt{11}}{2}$, so that $|\alpha| = 3.873$ (approximately). By Theorem C, we obtain

$$\max_{|z|=1} |D_\alpha P(z)| \geq 7.944,$$

while as Theorem 1 yields

$$\max_{|z|=1} |D_\alpha P(z)| \geq 10.414.$$

Also, by Theorem A, we obtain

$$\max_{|z|=1}|P'(z)| \geq 4.241,$$

while as Corollary 2 gives

$$\max_{|z|=1}|P'(z)| \geq 5.560.$$

This shows that (2.1) and (2.3) give considerable improvements over the bounds obtained from (1.13) and (1.11) respectively.

Note. For the same polynomial $P(z)$ as in Example 1, if we take $k=\sqrt{2}$, then the bound obtained in Theorem 1 will be same as the bound obtained in Theorem C and the bound obtained in Corollary 2 will be same as the bound obtained in Theorem A.

Our next result is a polar derivative generalization of (1.7) which also provides a refinement of Theorem D. The obtained inequality gives a refinement of Theorem B as well.

Theorem 2. Let $P(z)=a_0+a_1z+a_2z^2+...+a_nz^n$, be a polynomial of degree n having no zeros in $|z|<k,\ k\leq 1$, and $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for any complex number α with $|\alpha|\geq 1$, we have

$$\begin{aligned}\max_{|z|=1}&|D_\alpha P(z)|\\ &\leq \frac{n}{1+k^n}\left\{(|\alpha|+k^n)\max_{|z|=1}|P(z)|-(|\alpha|-1)m\right\}\\ &\quad-\frac{n(|\alpha|-1)k^{n-1}(|a_0|-k^n|a_n|-m)(1-k)}{2(1+k^n)(|a_0|+k^{n-1}|a_n|-m)}\left\{\max_{|z|=1}|P(z)|-m\right\}.\end{aligned} \tag{2.5}$$

Dividing both sides of (2.5) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the following refinement of Theorem B.

Corollary 4. Let $P(z)=a_0+a_1z+a_2z^2+...+a_nz^n$, be a polynomial of degree n having no zeros in $|z|<k,\ k\leq 1$, and $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then

$$\begin{aligned}\max_{|z|=1}&|P'(z)|\\ &\leq \frac{n}{1+k^n}\left\{1-\frac{k^{n-1}(|a_0|-k^n|a_n|-m)(1-k)}{2(|a_0|+k^{n-1}|a_n|-m)}\right\}\left\{\max_{|z|=1}|P(z)|-m\right\}.\end{aligned} \tag{2.6}$$

Equality in (2.6) holds for $P(z)=z^n+k^n$.

Remark 2. The condition that $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$ in Theorem 2 and Corollary 4 is needed only for $0 < k < 1$. For $k = 1$, both these results hold with out this condition, for example see Dewan et al. ([7], Theorem 1 for $k = t = 1$) and Aziz and Dawood [1].

Remark 3. In fact, excepting the case when some or all the zeros of $P(z)$ lie on $|z| = k$, the bounds obtained in (2.5) and (2.6) are always sharper than the bounds obtained from (1.14) and (1.12) respectively. As an illustration we consider the following example to compare the bounds.

Example 2. Let $P(z) = z^3 - z^2 + z - 1$. Clearly $P(z)$ has all its zeros $\{1, i, -i\}$ which all lie on $|z| = 1$. Also $Q(z) = \overline{P\left(\frac{1}{\bar{z}}\right)} = -P(z)$, so that $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. For this polynomial, we find that

$$\max_{|z|=1} |P(z)| = 4 \text{ and } m = \min_{|z|=k} |P(z)| = (1-k)(1+k^2),$$

with $0 \le k \le 1$. If we take $k = \frac{1}{2}$, we have $P(z) \neq 0$ in $|z| < k = \frac{1}{2}$.
Taking $\alpha = 3 + 4i$, so that $|\alpha| = 5$. By Theorem D, we obtain

$$\max_{|z|=1} |D_\alpha P(z)| \le 52.80,$$

while as Theorem 2 gives

$$\max_{|z|=1} |D_\alpha P(z)| \le 47.10.$$

Also, by Theorem B, we obtain

$$\max_{|z|=1} |P'(z)| \le 10.20,$$

while as Corollary 4 yields

$$\max_{|z|=1} |P'(z)| \le 7.425.$$

3. Auxiliary results

We need the following lemmas to prove our theorems. The following lemma is due to Mir et al. [24].

Lemma 1. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $R \ge 1$ and $0 \le t \le 1$, we have

$$\begin{aligned} \max_{|z|=R} |P(z)| \le & \left(\frac{(1+R^n)(|a_0| + R|a_n| - tm_1)}{(1+R)(|a_0| + |a_n| - tm_1)} \right) \max_{|z|=1} |P(z)| \\ & - \left(\frac{(1+R^n)(|a_0| + R|a_n| - tm_1)}{(1+R)(|a_0| + |a_n| - tm_1)} - 1 \right) tm_1, \end{aligned} \tag{3.1}$$

where $m_1 = \min_{|z|=1} |P(z)|$. Equality in (3.1) holds for $P(z) = \frac{\alpha + \beta^n}{2}$, $|\alpha| = |\beta| = 1$.

Lemma 2. Let $P(z) = z^s(a_0 + a_1 z + a_2 z^2 + ... + a_{n-s} z^{n-s})$, $0 \le s \le n$, be a polynomial of degree n having all its zeros in $|z| \le k,\ k \ge 1$, then for $0 \le t \le 1$, we have

$$\max_{|z|=k}|P(z)| \ge \left[\frac{2k^n}{1+k^{n-s}} + \frac{k^n(k^n|a_{n-s}|-k^s|a_0|-tm)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm)}\right]\max_{|z|=1}|P(z)|$$
$$+\left[\frac{k^{n-s}-1}{k^{n-s}+1} - \frac{(k^n|a_{n-s}|-k^s|a_0|-tm)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm)}\right]tm. \tag{3.2}$$

Equality in (3.2) holds for $P(z) = z^n + k^n$.

Proof of Lemma 2. Let $T(z) = P(kz)$. Since $P(z)$ has all its zeros in $|z| \le k,\ k \ge 1$, the polynomial $T(z)$ has all its zeros in $|z| \le 1$. Let $H(z) = z^nT\left(\frac{1}{z}\right)$ be the reciprocal polynomial of $T(z)$, then $H(z)$ is a polynomial of degree $n-s$ having no zeros in $|z| < 1$. Hence applying (3.1) of Lemma 1 to the polynomial $H(z)$, we get for $k \ge 1$ and $0 \le t \le 1$,

$$\max_{|z|=k}|H(z)| \le \frac{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm^*)}{(1+k)(k^n|a_{n-s}|+k^s|a_0|-tm^*)}\max_{|z|=1}|H(z)|$$
$$-\left(\frac{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm^*)}{(1+k)(k^n|a_{n-s}|+k^s|a_0|-tm^*)}-1\right)tm^*, \tag{3.3}$$

where $m^* = \min_{|z|=1}|H(z)|$.

Since $|H(z)| = |T(z)|$ on $|z| = 1$, therefore,

$$m^* = \min_{|z|=1}|H(z)| = \min_{|z|=1}\left|z^nP\left(\frac{k}{z}\right)\right| = \min_{|z|=k}|P(z)| = m,$$
$$\max_{|z|=1}|H(z)| = \max_{|z|=1}|T(z)| = \max_{|z|=k}|P(z)|,$$

and

$$\max_{|z|=k}|H(z)| = \max_{|z|=k}\left|z^nP\left(\frac{k}{z}\right)\right| = k^n\max_{|z|=1}|P(z)|.$$

The above when substituted in (3.3) gives

$$\max_{|z|=k}|P(z)| \ge \left(\frac{(1+k)(k^n|a_{n-s}|+k^s|a_0|-tm)}{(1+k^{n-s})(k^n|a_{n-s}|+k^s|a_0|-tm)}\right)k^n\max_{|z|=1}|P(z)|$$
$$+\left(1-\frac{(1+k)(k^n|a_{n-s}|+k^s|a_0|-tm)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm)}\right)tm. \tag{3.4}$$

Using the fact that

$$\frac{(1+k)(k^n|a_{n-s}|+k^s|a_0|-tm)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm)}$$
$$= \frac{2}{1+k^{n-s}} + \frac{(k^n|a_{n-s}|-k^s|a_0|-tm)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm)},$$

in (3.4), we get

$$\max_{|z|=k}|P(z)| \ge \left(\frac{2k^n}{1+k^{n-s}} + \frac{k^n(k^n|a_{n-s}|-k^s|a_0|-tm)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm)}\right)\max_{|z|=1}|P(z)|$$
$$+\left(\frac{k^{n-s}-1}{k^{n-s}+1} - \frac{(k^n|a_{n-s}|-k^s|a_0|-tm)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}|+k^{s+1}|a_0|-tm)}\right)tm,$$

which is (3.2) and this completes the proof of Lemma 2.

Lemma 3. If $P(z)$ is a polynomial of degree n and, $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then on $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|.$$

The above lemma is due to Govil and Rahman [13].

Lemma 4. If $P(z)$ is a polynomial of degree n, then for $R \geq 1$,

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|.$$

The above lemma is a simple consequence of the Maximum Modulus Principle (e.g, see [18]).

Lemma 5. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for any complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{2}\left\{(|\alpha| - 1) \max_{|z|=1} |P(z)| + (|\alpha| + 1) \min_{|z|=1} |P(z)|\right\}.$$

The above lemma is due to Aziz and Rather [2].

4. Proofs of the main results

Proof of Theorem 1. Recall that $P(z)$ has all its zeros in $|z| \leq k,\ k \geq 1$, therefore, all the zeros of the polynomial $G(z) = P(kz)$ lie in $|z| \leq 1$. Applying Lemma 5 to the polynomial $G(z)$ and noting that $\frac{|\alpha|}{k} \geq 1$, we get

$$\max_{|z|=1} \left|D_{\frac{\alpha}{k}} G(z)\right| \geq \frac{n}{2}\left\{\left(\frac{|\alpha|}{k} - 1\right) \max_{|z|=1} |G(z)| + \left(\frac{|\alpha|}{k} + 1\right) m\right\}. \tag{4.1}$$

where $m = \min_{|z|=1} |G(z)| = \min_{|z|=1} |P(kz)| = \min_{|z|=k} |P(z)|$.

The above inequality (4.1) is equivalent to

$$\begin{aligned} &\max_{|z|=1} \left|nP(kz) + \left(\frac{\alpha}{k} - z\right) kP'(kz)\right| \\ &\qquad \geq \frac{n}{2k}(|\alpha| - k) \max_{|z|=k} |P(z)| + \frac{n}{2k}(|\alpha| + k)m. \end{aligned} \tag{4.2}$$

Using the fact that

$$\max_{|z|=1} \left|nP(kz) + \left(\frac{\alpha}{k} - z\right) kP'(kz)\right| = \max_{|z|=k} |D_\alpha P(z)|,$$

and on applying Lemma 2 (for $t = 1$), the above expression (4.2) gives

$$\begin{aligned}
&\max_{|z|=k} |D_\alpha P(z)| \\
&\qquad \geq \frac{n}{2k}(|\alpha| - k)\Bigg\{ \left[\frac{2k^n}{1+k^{n-s}} + \frac{k^n(k^n|a_{n-s}| - k^s|a_0| - m)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - m)}\right] \\
&\qquad \times \max_{|z|=1}|P(z)| + \left[\frac{k^{n-s}-1}{k^{n-s}+1} - \frac{(k^n|a_{n-s}| - k^s|a_0| - m)(k-1)}{(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - m)}\right] m\Bigg\} \\
&\qquad + \frac{n}{2k}(|\alpha| + k)m.
\end{aligned} \tag{4.3}$$

Since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, and $k \geq 1$, applying Lemma 4 to the polynomial $D_\alpha P(z)$, we get

$$\max_{|z|=k} |D_\alpha P(z)| \leq k^{n-1} \max_{|z|=1} |D_\alpha P(z)|,$$

which on using in (4.3) gives

$$\begin{aligned}
k^{n-1} \max_{|z|=1} |D_\alpha P(z)| \geq{}& \frac{n}{2k(1+k^{n-s})}\Bigg\{2k^n(|\alpha| - k)\max_{|z|=1}|P(z)| \\
&+ (k^{n-s}-1)(|\alpha| - k)m + (k^{n-s}+1)(|\alpha| + k)m\Bigg\} \\
&+ \frac{nk^n(|\alpha| - k)(k^n|a_{n-s}| - k^s|a_0| - m)(k-1)}{2k(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - m)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\}.
\end{aligned}$$

The above inequality is equivalent to

$$\begin{aligned}
\max_{|z|=1} |D_\alpha P(z)| \geq{}& \frac{n}{1+k^{n-s}}\left\{(|\alpha| - k)\max_{|z|=1}|P(z)| + \left(\frac{|\alpha|}{k^s} + \frac{1}{k^{n-1}}\right)m\right\} \\
&+ \frac{n(|\alpha| - k)(k^n|a_{n-s}| - k^s|a_0| - m)(k-1)}{2(1+k^{n-s})(k^n|a_{n-s}| + k^{s+1}|a_0| - m)}\left\{\max_{|z|=1}|P(z)| - \frac{m}{k^n}\right\},
\end{aligned}$$

which is exactly (2.1). This completes the proof of Theorem 1.

Proof of Theorem 2. Let $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. Since $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, the polynomial $Q(z)$ of degree n has all its zeros in $|z| \leq \frac{1}{k}$, $\frac{1}{k} \geq 1$. On applying Corollary 3 to $Q(z)$, we get

$$\begin{aligned}
\max_{|z|=1} |Q'(z)| \geq{}& \frac{n}{1+\frac{1}{k^n}}\left\{\max_{|z|=1}|P(z)| + \frac{m}{k^n}\right\} \\
&+ \frac{n\left(\frac{|a_0|}{k^n} - |a_n| - \frac{m}{k^n}\right)\left(\frac{1}{k} - 1\right)}{2\left(1+\frac{1}{k^n}\right)\left(\frac{|a_0|}{k^n} + \frac{|a_n|}{k} - \frac{m}{k^n}\right)}\left\{\max_{|z|=1}|P(z)| - m\right\},
\end{aligned} \tag{4.4}$$

because

$$\min_{|z|=\frac{1}{k}} |Q(z)| = \min_{|z|=\frac{1}{k}} \left|z^n\overline{P\left(\frac{1}{\bar{z}}\right)}\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m}{k^n},$$

and

$$\max_{|z|=1}|Q(z)| = \max_{|z|=1}|P(z)|.$$

The above inequality (4.4) is equivalent to

$$\begin{aligned}\max_{|z|=1}|Q'(z)| \geq \frac{nk^n}{1+k^n}&\left\{\max_{|z|=1}|P(z)| + \frac{m}{k^n}\right\}\\ &+ \frac{nk^{n-1}(|a_0| - k^n|a_n| - m)(1-k)}{2(1+k^n)(|a_0| + k^{n-1}|a_n| - m)}\left\{\max_{|z|=1}|P(z)| - m\right\}.\end{aligned} \tag{4.5}$$

Since $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we have

$$\max_{|z|=1}(|P'(z)| + |Q'(z)|) = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{4.6}$$

On combining (4.5), (4.6) and Lemma 3, we get

$$\begin{aligned}n\max_{|z|=1}|P(z)| \geq \max_{|z|=1}|P'(z)| + \frac{nk^n}{1+k^n}&\left\{\max_{|z|=1}|P(z)| + \frac{m}{k^n}\right\}\\ &+ \frac{nk^{n-1}(|a_0| - k^n|a_n| - m)(1-k)}{2(1+k^n)(|a_0| + k^{n-1}|a_n| - m)}\left\{\max_{|z|=1}|P(z)| - m\right\},\end{aligned}$$

which gives

$$\begin{aligned}\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^n}&\left\{\max_{|z|=1}|P(z)| - m\right\}\\ &- \frac{nk^{n-1}(|a_0| - k^n|a_n| - m)(1-k)}{2(1+k^n)(|a_0| + k^{n-1}|a_n| - m)}\left\{\max_{|z|=1}|P(z)| - m\right\}.\end{aligned} \tag{4.7}$$

Also, it is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{4.8}$$

Note that for any complex number α, the polar derivative of $P(z)$ with respect to α is

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z),$$

which implies by (4.8) with $|\alpha| \geq 1$ and $|z| = 1$, that

$$\begin{aligned}|D_\alpha P(z)| &\leq |nP(z) - zP'(z)| + |\alpha||P'(z)|\\ &= |Q'(z)| + |\alpha||P'(z)|\\ &= |Q'(z)| + |P'(z)| - |P'(z)| + |\alpha||P'(z)|\\ &\leq n\max_{|z|=1}|P(z)| + (|\alpha| - 1)|P'(z)|. \quad \text{(by Lemma 3)}\end{aligned}$$

This gives by using (4.7), that

$$\max_{|z|=1}|D_\alpha P(z)| \leq \frac{n}{1+k^n}\left\{(|\alpha|+k^n)\max_{|z|=1}|P(z)|-(|\alpha|-1)m\right\}$$
$$-\frac{n(|\alpha|-1)k^{n-1}(|a_0|-k^n|a_n|-m)(1-k)}{2(1+k^n)(|a_0|+k^{n-1}|a_n|-m)}\left\{\max_{|z|=1}|P(z)|-m\right\},$$

which is (2.5) and this completes the proof of Theorem 2.

Concluding remark: In the past few years, a series of papers related to some Erdös-Lax and Turán-type inequalities has been published and significant advances have been achieved. This type of inequalities are of interest both in mathematics and in the application areas such as physical systems. In this paper, we continue the study of this type of inequalities for a certain class of polynomials, following up on a study started by various authors in the recent past. More precisely, the author establishes for a certain class of polynomials some new lower and upper bounds for the derivative and polar derivative of a polynomial on the unit disk while taking into account the placement of the zeros and extremal coefficients of the underlying polynomial. Moreover, some concrete numerical examples are presented, showing that in some situations, the bounds obtained by our results can be considerably sharper than the ones previously known.

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