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WEIGHTED NORM INEQUALITIES FOR CALDERÓN-ZYGMUND OPERATORS OF ϕ-TYPE AND THEIR COMMUTATORS

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Abstract. In this paper, we introduce new weighted Morrey spaces $L^{p,\kappa}_{\theta,\omega}(\phi)$ associated with a nondecreasing function ϕ of upper type β with $\beta > 0$, where $\omega \in A^{\theta}_{p}(\phi)$ and $\phi(\alpha t) \leqslant C\alpha^{\beta}\phi(t)$, then we obtain the weighted strong type and weak endpoint estimates for Calderón-Zygmund operators of ϕ-type and their commutators $[b, T]$ on new weighted Morrey spaces $L^{p,\kappa}_{\theta,\omega}(\phi)$, where $b \in \mathrm{BMO}^{\theta}(\phi)$.

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1. Introduction

The groundbreaking work of Calderón and Zygmund in the 1950s [1] is basis for what is today named after them Calderón-Zygmund theory, it has an important role in harmonic analysis. And they proved that Calderón-Zygmund singular integral operator is bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$.

Since the pioneering work of Calderón [2] in 1965, many researchers have been interested in commutators. In 1976, Coifman, Rochberg and Weiss [3] introduced the commutators which are defined by

$$[b, T]f(x) := b(x)T(f)(x) - T(bf)(x),$$

where b is a locally integrable function in $\mathbb{R}^n$, usually called the symbol, and T is a Calderón-Zygmund singular integral operator. They also proved that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, T]$ is a bounded operator on $L^p(\mathbb{R}^n), 1 < p < \infty$.

It is well known that Morrey [4] first introduced the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ to investigate the local behavior of solutions to second-order elliptic partial differential equations in 1938. Subsequently, there has been an explosion of interest in studying the boundedness of operators on Morrey-type spaces.

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In 1969, Peetre [5] proved that the Calderón-Zygmund singular integral operator is bounded on the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$. In 1991, Fazio and Ragusa [6] obtained the boundedness of commutators of Calderón-Zygmund operators on the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$.

On the other hand, in 1991, Mizuhara [7] introduced the generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ and established the boundedness of Calderón-Zygmund operators, where φ is a positive increasing function in $(0,\infty)$ and satisfies the doubling condition. In 1994, Nakai [8] used the nonnegative function ψ to replace doubling condition of φ and obtained the boundedness of Calderón-Zygmund operators on the generalized Morrey spaces $L^{p,\psi}(\mathbb{R}^n)$. In 2009, Komori and Shirai [9] introduced the weighted Morrey spaces $L^{p,\kappa}(\omega)$ ($0 \le \kappa < 1, \omega$ is a nonnegative and locally integrable function) and studied the boundedness of some classical operators in harmonic analysis on their Morrey spaces.

In 2018, Wu and Wang [10] introduced new classes of weights, new BMO functions and obtained the weighted norm inequalities for Calderón-Zygmund operators of ϕ-type and their commutators. We will give the definition of the new class of weights $A_p^{\theta}(\phi)$ and new BMO spaces $\mathrm{BMO}^{\theta}(\phi)$ and their related properties in the second section.

In 2021, Zhao and Zhou [11] studied the new Morrey-type spaces $M^{p,q}_{\alpha,\lambda}(u,\omega)$ ($\lambda \in [0,1)$, $\alpha \in (-\infty,\infty)$, u,ω be two weights) and obtained the some weighted norm inequalities for certain classes of multilinear operators and their commutators. The purpose of this paper is to study the Calderón-Zygmund singular integral operator of ϕ-type on a new class of weighted Morrey spaces $L^{p,\kappa}_{\theta,\omega}(\phi)$.

We recall following necessary definition. For a nonnegative and nondecreasing function ϕ mapping from $[0,\infty)$ to $[1,\infty)$, we shall mean that it is of upper type β with $\beta > 0$, if there exists a positive constant C such that

$$\phi(\alpha t) \leqslant C\alpha^{\beta}\phi(t), \tag{1.1}$$

for all $\alpha \geqslant 1$ and $t \geqslant 0$. We always assume that $\phi(1) > 1$.

Definition 1.1. Let $1 < p < \infty, 0 \leqslant \kappa < 1$ and ω be a weight, function ϕ is of upper type β with $\beta > 0$. For given $0 \leqslant \theta < \infty$, the weighted Morrey space $L^{p,\kappa}_{\theta,\omega}(\phi)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ satisfing $\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)} < \infty$, where

$$\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)} := \sup_{Q} \phi(|Q|)^{-\theta}\left(\frac{1}{\omega(Q)^{\kappa}}\int_Q |f(x)|^p\omega(x)dx\right)^{1/p} < \infty,$$

where the supremum is taken over all cubes Q. Define $L^{p,\kappa}_{\infty,\omega}(\phi) := \bigcup_{\theta>0} L^{p,\kappa}_{\theta,\omega}(\phi)$.

Let $\omega = 1$, this new space is the space $L^{p,\psi}(\mathbb{R}^n)$ defined in [8]. If we take $\theta = 0$, then $L^{p,\kappa}_{0,\omega}(\phi) = L^{p,\kappa}(\omega)$, which was first defined by Komori and Shirai in [9].

Definition 1.2. Let $p = 1, 0 \leqslant \kappa < 1$ and ω be a weight, function ϕ is of upper type β with $\beta > 0$. For given $0 \le \theta < \infty$, the weighted weak Morrey space $WL^{1,\kappa}_{\theta,\omega}(\phi)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ satisfing $\|f\|_{WL^{1,\kappa}_{\theta,\omega}(\phi)} < \infty$, where

$$\|f\|_{WL^{1,\kappa}_{\theta,\omega}(\phi)} := \sup_Q \phi(|Q|)^{-\theta} \frac{1}{\omega(Q)^\kappa} \sup_{t>0} t\omega(\{x \in Q : |f(x)| > t\}) < \infty.$$

where the supremum is taken over all cubes Q. Define $WL^{1,\kappa}_{\infty,\omega}(\phi) := \bigcup_{\theta>0} WL^{1,\kappa}_{\theta,\omega}(\phi)$.

If we take $\theta = 0$, this space is the weighted weak Morrey space $WL^{1,\kappa}(\omega)$ in [12].

According to the above definitions, we have $L^{p,\kappa}(\omega) \subset L^{p,\kappa}_{\theta_1,\omega}(\phi) \subset L^{p,\kappa}_{\theta_2,\omega}(\phi)$ and $WL^{1,\kappa}(\omega) \subset WL^{1,\kappa}_{\theta_1,\omega}(\phi) \subset WL^{1,\kappa}_{\theta_2,\omega}(\phi)$ for $0 \le \theta_1 < \theta_2 < \infty$. Hence $L^{p,\kappa}(\omega) \subset L^{p,\kappa}_{\infty,\omega}(\phi)$ for $(p,\kappa) \in [1,\infty) \times [0,1)$ and $WL^{1,\kappa}(\omega) \subset WL^{1,\kappa}_{\infty,\omega}(\phi)$ for $0 \leqslant \kappa < 1$.

Next, we introduce the Calderón-Zygmund operators of ϕ-type in [10]. Let T be an operator initially defined on Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and take values into the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. We study the Calderón-Zygmund operators of ϕ-type T which satisfies the following conditions:

(1) If there exists a function $K(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ such that

$$T(f)(x) := \int_{\mathbb{R}^n} K(x,y) f(y) dy,$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$;

(2) For any $N \geqslant 0$, there exists a positive constant C such that

$$|K(x,y)| \leqslant \frac{C}{|x-y|^n \phi(|x-y|^n)^N};$$

(3) For some $\varepsilon > 0$ and any $N \geqslant 0$, there exists a positive constant C such that

$$|K(x,y) - K(x',y)| \leqslant \frac{C|x-x'|^\varepsilon}{(|x-y| + |x'-y|)^{n+\varepsilon} \phi\big((|x-y| + |x'-y|)^n\big)^N},$$

whenever $|x - x'| \leqslant \frac{1}{2}\max\{|x-y|, |x'-y|\}$ and

$$|K(x,y) - K(x,y')| \leqslant \frac{C|y-y'|^\varepsilon}{(|x-y| + |x-y'|)^{n+\varepsilon} \phi\big((|x-y| + |x-y'|)^n\big)^N},$$

whenever $|y - y'| \leqslant \frac{1}{2}\max\{|x-y|, |x-y'|\}$;

(4) T is a bounded linear operator on $L^2(\mathbb{R}^n)$.

It is clear that if T satisfies (2)-(4), then T falls within the scope of the Calderón-Zygmund theory. Since T has an extension that maps $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, and by interpolation and duality, T also maps $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$.

If the operator T satisfies (2)-(4) and $\phi(t) = 1+t$, we know that pseudodifferential operators with smooth symbols are only the special case of T, see [13, 14, 15].

The aim of this paper is to obtain the weighted norm inequalities of Calderón-Zygmund operators of ϕ-type and their commutators on new weighted Morrey spaces $L^{p,\kappa}_{\theta,\omega}(\phi)$.

Next, we state our main results as follows.

Theorem 1.1. *Assume that T satisfies $(2)-(4)$. Let $0 \le \kappa < 1$, and function ϕ is of upper type β with $\beta > 0$.*

(1) *If $1 < p < \infty$, and $\omega \in A^{\infty}_p(\phi)$, then*

$$\|Tf\|_{L^{p,\kappa}_{\infty,\omega}(\phi)} \le C\|f\|_{L^{p,\kappa}_{\infty,\omega}(\phi)}.$$

(2) *If $p = 1$, and $\omega \in A^{\infty}_1(\phi)$, then for all $\lambda > 0$ and any cube Q,*

$$\frac{1}{\omega(Q)^\kappa}\lambda\omega(\{x \in Q : |Tf(x)| > \lambda\}) \le C\phi(|Q|)^\upsilon \|f\|_{L^{1,\kappa}_{\infty,\omega}(\phi)}.$$

Theorem 1.2. *Assume that T satisfies $(2)-(4)$. Let $1 < p < \infty$, $0 \le \kappa < 1$ and function ϕ is of upper type β with $\beta > 0$. If $b \in BMO^{\infty}(\phi)$ and $\omega \in A^{\infty}_p(\phi)$, then $[b, T]$ is a bounded operator from $L^{p,\kappa}_{\infty,\omega}(\phi)$ to $L^{p,\kappa}_{\infty,\omega}(\phi)$.*

Theorem 1.3. *Assume that T satisfies $(2)-(4)$. Let $0 \le \kappa < 1$, $0 \leqslant \theta < \infty$ and function ϕ is of upper type β with $\beta > 0$. If $b \in BMO^{\infty}(\phi)$ and $\omega \in A^{\infty}_1(\phi)$, then for any $\lambda > 0$ and any cube Q, there exist positive constants C and ν such that*

$$\frac{1}{\omega(Q)^\kappa}\omega(\{x \in Q : |[b, T]f(x)| > \lambda\}) \leqslant C\phi(|Q|)^\nu \left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}$$

holds for those functions f such that $\Phi(|f|) \in L^{1,\kappa}_{\theta,\omega}(\phi)$, where $\Phi(t) = t(1 + \log^+ t)$.

2. Some Preliminaries and Notations

In this section, we first recall some notations. For a measurable set E, we define $|E|$ as the Lebesgue measure of E and χ_E as the characteristic function of E. $Q(x, r)$ denotes the cube centered at x with the sidelength r and $aQ(x, r) = Q(x, ar)$. For a locally integrable function f, f_Q denotes the average $f_Q := \frac{1}{|Q|}\int_Q f(y)dy$. A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight ω and a measurable set E, we define $\omega(E) := \int_E \omega(y)dy$. The letter C will denote a positive constant not necessarily the same at each occurrence.

2.1. $A_p^\theta(\phi)$ **and** $A_p^\infty(\phi)$ **Weights.** In this section, we recall the definition of the new class of weights introduced by [10].

A weight will always mean a positive function which is locally integrable. We say that a weight ω belongs to the class $A_p^\theta(\phi)$ for $0 \leqslant \theta < \infty$ and $1 < p < \infty$, if there is a positive constant C such that for all cubes Q

$$\Big(\frac{1}{\phi(|Q|)^\theta |Q|}\int_Q \omega(y)dy\Big)\Big(\frac{1}{\phi(|Q|)^\theta |Q|}\int_Q \omega(y)^{-\frac{1}{p-1}}dy\Big)^{p-1} \leqslant C.$$

In particular case, when $p = 1$, $A_1^\theta(\phi)$ is understood

$$\frac{1}{\phi(|Q|)^\theta |Q|}\int_Q \omega(y)dy \leqslant C \inf_{x\in Q} \omega(x).$$

We also write $A_\infty^\theta(\phi) := \bigcup_{p\geqslant 0} A_p^\theta(\phi)$, $A_p^\infty(\phi) := \bigcup_{\theta\geqslant 0} A_p^\theta(\phi)$ and $A_\infty^\infty(\phi) := \bigcup_{p\geqslant 1} A_p^\infty(\phi)$. If $\theta = 0$, remark that $A_p^0(\phi)$ coincides with the Muckenhoupt's class of weightes A_p in [19], for all $1 \leqslant p < \infty$. When ϕ is a constant function, $A_p^\theta(\phi)$ also coincides with A_p for any $\theta \in [0,\infty)$. However, in general, the class $A_p^\infty(\phi)$ is strictly larger than the class A_p for all $1 \leqslant p < \infty$. Let $\theta \geq 0$ and $0 \leqslant \gamma \leqslant n\theta$, it is easy to check that $\omega(x) = (1+|x|)^{-(n+\gamma)} \notin A_\infty$ and $\omega(x)dx$ is not a doubling measure, but $\omega(x) = (1+|x|)^{-(n+\gamma)} \in A_1^\theta(\phi)$(see [13]).

The following Lemma hold for the new classes $A_p^\theta(\phi)$, see Proposition 15 in [10].

Lemma 2.1. [10] *Let $\theta \geq 0$, the following statements hold:*
(i) If $1 \leqslant p_1 < p_2 < \infty$, then $A_{p_1}^\theta(\phi) \subset A_{p_2}^\theta(\phi)$.
(ii) $\omega \in A_p^\theta(\phi)$ if and only if $\omega^{1-p'} \in A_{p'}^\theta(\phi)$, where $1/p + 1/p' = 1$.
(iii) If $\omega_1, \omega_2 \in A_p^\theta(\phi), p \geqslant 1$, then $\omega_1^\alpha \omega_2^{1-\alpha} \in A_p^\theta(\phi)$ for any $0 < \alpha < 1$.
(iv) If $\omega \in A_p^\theta(\phi)$ for $1 \leqslant p < \infty$, then

$$\frac{1}{\phi(|Q|)^\theta |Q|}\int_Q |f(y)|dy \leqslant C\bigg(\frac{1}{\omega(5Q)}\int_{5Q} |f(y)|^p\omega(y)dy\bigg)^{1/p}.$$

(v) If $\omega \in A_p^\theta(\phi)$ with $p \geqslant 1$, then there exist positive numbers δ, η, and C such that for all cubes Q

$$\bigg(\frac{1}{|Q|}\int_Q \omega(x)^{1+\delta}dx\bigg)^{1/(1+\delta)} \leqslant C\bigg(\frac{1}{|Q|}\int_Q \omega(x)dx\bigg)\phi(|Q|)^\eta.$$

(vi) If $\omega \in A_p^\infty(\phi)$ with $p > 1$ then there exists $\varepsilon > 0$ such that $\omega \in A_{p-\varepsilon}^\infty(\phi)$.

Applying Lemma 2.1(v) and the Hölder inequality, we can get Lemma 2.2.

Lemma 2.2. *Let $0 \le \theta < \infty, 1 \leqslant p < \infty$. If $\omega \in A_p^\theta(\phi)$, then there exist positive constants $0 < \delta < 1, \eta$ and C such that*

$$\frac{\omega(E)}{\omega(Q)} \leqslant C\phi(|Q|)^\eta \Big(\frac{|E|}{|Q|}\Big)^\delta,$$

for any measurable subset E of a ball Q.

Lemma 2.3. *Let $0 \le \theta < \infty, 1 \leqslant p < \infty$. If $\omega \in A_p^\theta(\phi)$, then there exist two positive constants $\rho > 1$ and C such that*

$$\omega(\rho Q) \leqslant C\phi(|\rho Q|)^{p\theta}\omega(Q).$$

Proof. For $1 < p < \infty$, by Hölder's inequality and the definition of $A_p^\theta(\phi)$, we obtain

$$\begin{aligned}\frac{1}{|\rho Q|}\int_{\rho Q}|f(x)|dx &\leqslant \frac{1}{|\rho Q|}\Big(\int_{\rho Q}|f(x)|^p\omega(x)dx\Big)^{1/p}\Big(\int_{\rho Q}\omega(x)^{-p'/p}dx\Big)^{1/p'}\\ &\leqslant \frac{C}{\omega(\rho Q)^{1/p}}\Big(\int_{\rho Q}|f(x)|^p\omega(x)dx\Big)^{1/p}\phi(|\rho Q|)^\theta.\end{aligned}$$

If we take $f(x) := \chi_Q(x)$, then

$$\omega(\rho Q) \leqslant C\phi(|\rho Q|)^{p\theta}\omega(Q).$$

For $p = 1$, from the definition of $A_1^\theta(\phi)$, it follows that

$$\begin{aligned}\frac{1}{|\rho Q|}\int_{\rho Q}|f(x)|dx &\leqslant \frac{C}{\omega(\rho Q)}\cdot\inf_{x\in\rho Q}\omega(x)\Big(\int_{\rho Q}|f(x)|dx\Big)\phi(|\rho Q|)^\theta\\ &\leqslant \frac{C}{\omega(\rho Q)}\Big(\int_{\rho Q}|f(x)|\omega(x)dx\Big)\phi(|\rho Q|)^\theta.\end{aligned}$$

Taking $f(x) := \chi_Q(x)$, yields $\omega(\rho Q) \leqslant C\phi(|\rho Q|)^\theta\omega(Q)$. □

2.2. **$\mathrm{BMO}^\theta(\phi)$ and $\mathrm{BMO}^\infty(\phi)$ spaces.**

In this section, we will recall the definition and some basic properties of the new BMO function spaces. According to [10], we say a locally integrable function b is in $\mathrm{BMO}_p^\theta(\phi)$ with $p \geqslant 1$ and $\theta \geqslant 0$, if there exists a positive constant C such that for any cube Q

$$\Big(\frac{1}{|Q|}\int_Q|b(y)-b_Q|^p dy\Big)^{1/p} \le C\phi(|Q|)^\theta, \tag{2.1}$$

where $b_Q := \frac{1}{|Q|}\int_Q b(y)dy$. A norm for $b \in \mathrm{BMO}_p^\theta(\phi)$, denoted by $\|b\|_{\mathrm{BMO}_p^\theta(\phi)}$, is given by the infimum of the constants satisfying (2.1).

When $\theta = 0$ or ϕ is a constant function, $\mathrm{BMO}^\theta(\phi) = \mathrm{BMO}(\mathbb{R}^n)$; and $\mathrm{BMO}^{\theta_1}(\phi) \subset \mathrm{BMO}^{\theta_2}(\phi)$ for $0 \le \theta_1 \leqslant \theta_2$. We define $\mathrm{BMO}^\infty(\phi) := \bigcup_{\theta\geqslant 0}\mathrm{BMO}^\theta(\phi)$. In [16], Morvidone proved that these spaces are independent of the scale p, so we denote $\mathrm{BMO}^\theta(\phi)$ simply.

The following result can be considered to be a variant of John-Nirenberg inequality for the spaces $\mathrm{BMO}^\theta(\phi)$.

Lemma 2.4. [10] *Let* $q \geqslant 1$. *If* $b \in BMO^\theta(\phi)$, *then for all cubes* Q

(i) $\left(\frac{1}{|Q|}\int_Q |b(y)-b_Q|^q dy\right)^{\frac{1}{q}} \leqslant C\phi(|Q|)^\theta$;

(ii) $\left(\frac{1}{|2^kQ|}\int_{2^kQ} |b(y)-b_Q|^q dy\right)^{\frac{1}{q}} \leqslant Ck\phi(|2^kQ|)^\theta$, *for all* $k \in \mathbb{N}$.

Lemma 2.5. [18] *If* $f \in BMO^\theta(\phi)$, *then there exist positive constants* C_1 *and* C_2 *such that, for given any cube* Q *in* $\mathbb{R}^n$ *and any* $\gamma > 0$,

$$|\{x \in Q : |f(x)-f_Q| > \gamma\}| \leqslant C_1|Q| \exp\left\{-\frac{C_2\gamma}{\|f\|_{BMO^\theta(\phi)}\phi(|Q|)^\theta}\right\}.$$

The proof of this lemma is similar to the Property 4.2 of [18], so we omit it.

Lemma 2.6. *If* $f \in BMO^\theta(\phi)$ *and* $\omega \in A_\infty^\infty(\phi)$, *then there exist positive constants* C *and* s *such that, for every cube* Q,

$$\left(\frac{1}{\omega(Q)}\int_Q |f(x)-f_Q|^p\omega(x)dx\right)^{1/p} \leqslant C\phi(|Q|)^{s/p}\|f\|_{BMO^\theta(\phi)}.$$

Proof. Applying Lemma 2.2 and Lemma 2.5, we find that

$$\omega(\{x \in Q : |f(x)-f_Q| > \gamma\}) \leqslant CC_1^\delta\phi(|Q|)^\eta \exp\left\{-\frac{C_2\gamma}{\|f\|_{\mathrm{BMO}^\theta(\phi)}\phi(|Q|)^\theta}\right\}^\delta \omega(Q).$$

Let $s = \eta + p\theta$, then for any cube Q,

$$\frac{1}{\omega(Q)}\int_Q |f(x)-f_Q|^p\omega(x)dx = \frac{p}{\omega(Q)}\int_0^\infty \gamma^{p-1}\omega(\{x \in Q : |f(x)-f_Q| > \gamma\})d\gamma$$

$$\leqslant CC_1^\delta p\phi(|Q|)^\eta \int_0^\infty \gamma^{p-1}\exp\left\{-\frac{C_2\gamma}{\|f\|_{\mathrm{BMO}^\theta(\phi)}\phi(|Q|)^\theta}\right\}^\delta d\gamma \leqslant C\phi(|Q|)^s\|f\|^p_{\mathrm{BMO}^\theta(\phi)}.\square$$

Lemma 2.7. [10] *Supposing that* $f \in BMO^\theta(\phi)$, *there exist positive constants* c_1 *and* c_2 *such that*

$$\sup_Q \frac{1}{|Q|}\int_Q \exp\left\{\frac{c_1|f(x)-f_Q|}{\|f\|_{BMO^\theta(\phi)}\phi(|Q|)^\theta}\right\}dx \leqslant c_2.$$

2.3. **Orlicz Norms.**

For $\Phi(t) = t(1+\log^+ t)$ and a cube Q in $\mathbb{R}^n$, we will consider the average $\|f\|_{\Phi,Q}$ of a function f given by the Luxemburg norm

$$\|f\|_{\Phi,Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \leqslant 1\right\}. \tag{2.2}$$

We also have the equivalent definition of (2.2) (see [17]).

$$\|f\|_{\Phi,Q} \approx \inf_{\gamma>0}\left\{\gamma + \frac{\gamma}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx\right\}. \tag{2.3}$$

As we know, $\Psi(t) = e^t - 1$ is also a young function, the corresponding average is denoted by $\|f\|_{\Psi,Q} = \|f\|_{\exp L,Q}$. Then there is a generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|dx \leqslant 2\|f\|_{\exp L,Q}\|g\|_{L\log L,Q}. \tag{2.4}$$

By Lemma 2.7 and (2.4), it shows that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q||g(x)|dx \leqslant 2\phi(|Q|)^\theta \|f\|_{\mathrm{BMO}^\theta(\phi)}\|g\|_{L\log L,Q}. \tag{2.5}$$

To get to Theorems 1.3 − 1.5, we need the following lemmas.

Lemma 2.8. [10] *Assume that T satisfies (2)−(4). Let $\omega \in A_p^\infty(\phi)$ with $1 \leqslant p < \infty$, then T is bounded from $L^p(\omega)$ to $L^p(\omega)$ for $1 < p < \infty$ and $L^1(\omega)$ to $L^{1,\infty}(\omega)$.*

Lemma 2.9. [10] *Assume that T satisfies (2) − (4). Let $b \in BMO^\theta(\phi)$ for $\theta \geqslant 0$ and $\omega \in A_p^\infty(\phi)$ with $1 < p < \infty$, then there exists a positive constant C such that*

$$\|[b,T](f)\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}.$$

Lemma 2.10. [10] *Assume that T satisfies (2) − (4). Let $b \in BMO^\theta(\phi)$ for $\theta \geqslant 0$ and $\omega \in A_1^\infty(\phi)$, then there exists a positive constant C such that*

$$\omega(\{x \in \mathbb{R}^n : |[b,T](f)(x)| > \lambda\}) \leqslant C\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\omega(x)dx.$$

3. Proof of the main results

3.1. **Proof of Theorem 1.1.** (i) Let $1 < p < \infty$, $0 \leq \kappa < 1$ and $\omega \in A_p^\infty(\phi)$, we only need to show that there exist positive constants C and ν such that for any given cube $Q = Q(x,r)$,

$$\Big(\frac{1}{\omega(Q)^\kappa}\int_Q |T(f)(x)|^p\omega(x)dx\Big)^{1/p} \leqslant C\phi(|Q|)^\nu \tag{3.1}$$

holds for any function $f \in L^{p,\kappa}_{\infty,\omega}(\phi)$.

Suppose that $f \in L^{p,\kappa}_{\theta,\omega}(\phi)$ for some $\theta \geq 0$ and $\omega \in A_p^{\theta'}(\phi)$ for some $\theta' \geq 0$. We split $f = f_1 + f_2$, where $f_1 = f\chi_{4Q}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus 4Q}$. Then, the linearity of T gives us that

$$\Big(\frac{1}{\omega(Q)^\kappa}\int_Q |T(f)(x)|^p\omega(x)dx\Big)^{1/p}$$
$$\leqslant \Big(\frac{1}{\omega(Q)^\kappa}\int_Q |T(f_1)(x)|^p\omega(x)dx\Big)^{1/p} + \Big(\frac{1}{\omega(Q)^\kappa}\int_Q |T(f_2)(x)|^p\omega(x)dx\Big)^{1/p} := \mathrm{I} + \mathrm{II}.$$

For the term I. Since $\omega \in A_p^{\theta'}(\phi)$ with $1 < p < \infty$ and $\theta' \geq 0$, then by Lemma 2.3, we obtain

$$
\begin{aligned}
\mathrm{I} &\leqslant C\frac{1}{\omega(Q)^{\kappa/p}}\left(\int_{\mathbb{R}^n}|f_1(x)|^p\omega(x)dx\right)^{1/p} = C\frac{1}{\omega(Q)^{\kappa/p}}\left(\int_{4Q}|f(x)|^p\omega(x)dx\right)^{1/p}\\
&\leqslant C\frac{\omega(4Q)^{\kappa/p}}{\omega(Q)^{\kappa/p}}\phi(|4Q|)^{\theta}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)} \leqslant C\phi(|Q|)^{\nu'}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)},
\end{aligned}
$$

where $\nu' = \kappa\theta' + \theta$. Notice that the first inequality we have used Lemma 2.8.

For the term II. From the size condition (2) of K, it follows that

$$
\begin{aligned}
|T(f_2)(x)| &\leqslant \int_{\mathbb{R}^n\backslash 4Q}|K(x,y)||f(y)|dy \leqslant C\int_{\mathbb{R}^n\backslash 4Q}\frac{|f(y)|}{|x-y|^n\phi(|x-y|^n)^N}dy\\
&\leqslant C\sum_{k=1}^{\infty}\int_{4^{k+1}Q\backslash 4^kQ}\frac{|f(y)|}{|x-y|^n\phi(|x-y|^n)^N}dy\\
&\leqslant C\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|f(y)|dy.
\end{aligned}
$$

From the definition of $A_p^{\theta'}(\phi)$ and Hölder's inequality, then conclude that

$$
\begin{aligned}
|T(f_2)(x)| &\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}}{\phi(|4^{k+1}Q|)^N\omega(4^{k+1}Q)^{1/p}}\left(\int_{4^{k+1}Q}|f(y)|^p\omega(y)dy\right)^{1/p}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}\omega(4^{k+1}Q)^{\kappa/p}}{\phi(|4^{k+1}Q|)^N\omega(4^{k+1}Q)^{1/p}}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}. \qquad (3.2)
\end{aligned}
$$

Since $\omega \in A_p^{\theta'}(\phi)$. Then from Lemma 2.2 and let $\theta' + \theta + \frac{\eta(1-\kappa)}{p} - \frac{\delta(1-\kappa)}{p\beta} < N < \theta' + \theta + \frac{\eta(1-\kappa)}{p}$, it follows that

$$
\begin{aligned}
&\left(\frac{1}{\omega(Q)^{\kappa}}\int_Q|T(f_2)(x)|^p\omega(x)dx\right)^{1/p}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}\omega(4^{k+1}Q)^{\kappa/p}\omega(Q)^{1/p}}{\phi(|4^{k+1}Q|)^N\omega(4^{k+1}Q)^{1/p}\omega(Q)^{\kappa/p}}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}\\
&= C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}\omega(Q)^{(1-\kappa)/p}}{\phi(|4^{k+1}Q|)^N\omega(4^{k+1}Q)^{(1-\kappa)/p}}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}}{\phi(|4^{k+1}Q|)^N}\phi(|4^{k+1}Q|)^{\eta(1-\kappa)/p}\left(\frac{|Q|}{|4^{k+1}Q|}\right)^{\delta(1-\kappa)/p}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}.
\end{aligned}
$$

We see that $\phi(|4^{k+1}Q|) \leqslant C4^{(k+1)n\beta}\phi(|Q|)$, therefore,

$$
\mathrm{II} = \left(\frac{1}{\omega(Q)^{\kappa}}\int_Q|T(f_2)(x)|^p\omega(x)dx\right)^{1/p} \leqslant C\phi(|Q|)^{\upsilon}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)},
$$

where $\upsilon = \theta' + \theta + \frac{\eta(1-\kappa)}{p} - N$.

Combining this inequality with the estimate of I and making $\nu := \max\{\nu', \upsilon\}$, we get the desired result (3.1).

(ii) As for the case $p = 1$. For $\lambda > 0$, by Chebyshev inequality we have

$$\begin{aligned}
&\frac{1}{\omega(Q)^\kappa}\lambda\omega(\{x \in Q : |Tf(x)| > \lambda\}) \\
&\leqslant \frac{1}{\omega(Q)^\kappa}\lambda\omega(\{x \in Q : |Tf_1(x)| > \lambda/2\}) \quad + \frac{1}{\omega(Q)^\kappa}\lambda\omega(\{x \in Q : |Tf_2(x)| > \lambda/2\}) \\
&\leqslant C\frac{1}{\omega(Q)^\kappa}\int_Q |Tf_1(x)|\omega(x)dx + C\frac{1}{\omega(Q)^\kappa}\int_Q |Tf_2(x)|\omega(x)dx.
\end{aligned}$$

The rest of the proof is similar to the case $p > 1$, so we omit it. This finishes the proof of Theorem 1.3.

3.2. **Proof of Theorem 1.4.** Let $1 < p < \infty$, $0 \leq \kappa < 1$, $\omega \in A_p^\infty(\phi)$ and $b \in \mathrm{BMO}^\infty(\phi)$, we only need to show that there exist positive constants C and ν such that for any given cube $Q = Q(x, r)$,

$$\left(\frac{1}{\omega(Q)^\kappa}\int_Q |[b,T](f)(x)|^p\omega(x)dx\right)^{1/p} \leqslant C\phi(|Q|)^\nu \tag{3.3}$$

holds for any function $f \in L^{p;\kappa}_{\infty,\omega}(\phi)$.

Suppose that $f \in L^{p,\kappa}_{\theta,\omega}(\phi)$ for some $\theta \geq 0$, $\omega \in A_p^{\theta'}(\phi)$ for some $\theta' \geq 0$ and $b \in \mathrm{BMO}^{\theta''}(\phi)$ for some $\theta'' \geq 0$. We split $f = f_1 + f_2$, where $f_1 = f\chi_{4Q}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus 4Q}$, then

$$\begin{aligned}
&\left(\frac{1}{\omega(Q)^\kappa}\int_Q |[b,T]f(x)|^p\omega(x)dx\right)^{1/p} \\
&\leqslant \left(\frac{1}{\omega(Q)^\kappa}\int_Q |[b,T]f_1(x)|^p\omega(x)dx\right)^{1/p} \quad + \left(\frac{1}{\omega(Q)^\kappa}\int_Q |[b,T]f_2(x)|^p\omega(x)dx\right)^{1/p} \\
&:= \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

For the term III. By Lemma 2.9 and Lemma 2.3, we conclude that

$$\begin{aligned}
\mathrm{III} &\leqslant C\frac{1}{\omega(Q)^{\kappa/p}}\left(\int_{\mathbb{R}^n} |f_1(y)|^p\omega(y)dy\right)^{1/p} \\
&\leqslant C\frac{\omega(4Q)^{\kappa/p}}{\omega(Q)^{\kappa/p}}\phi(|4Q|)^\theta\|f\|_{L^{p;\kappa}_{\theta,\omega}(\phi)} \leqslant C\phi(|Q|)^{\nu'}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)},
\end{aligned} \tag{3.4}$$

where $\nu' = \kappa\theta' + \theta$.

For the term IV. Notice that

$$[b,T]f(x) = (b(x) - b_Q)Tf(x) - T((b - b_Q)f)(x).$$

Hence,

$$|[b,T]f_2(x)| = \left| \int_{\mathbb{R}^n} (b(x)-b_Q)Tf_2(x) - T((b-b_Q)f_2)(x)dx \right|$$
$$\leqslant |b(x)-b_Q| \int_{\mathbb{R}^n} |K(x,y)f_2(y)|dy + \int_{\mathbb{R}^n} |b(y)-b_Q||K(x,y)f_2(y)|dy$$
$$:= \mathrm{IV}_1 + \mathrm{IV}_2.$$

Next, for IV_1. From the estimate (3.2) and Lemma 2.6, we see that

$$\left(\frac{1}{\omega(Q)^\kappa}\int_Q |IV_1|^p \omega(x)dx\right)^{1/p}$$
$$\leqslant C\frac{1}{\omega(Q)^{\kappa/p}}\left(\int_Q |b(x)-b_Q|^p\omega(x)dx\right)^{1/p}\sum_{k=1}^\infty \frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|f(y)|dy$$
$$\leqslant C\frac{\omega(Q)^{1/p}}{\omega(Q)^{\kappa/p}}\phi(|Q|)^{s/p}\sum_{k=1}^\infty \frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}\omega(4^{k+1}Q)^{\kappa/p}}{\phi(|4^{k+1}Q|)^N\omega(4^{k+1}Q)^{1/p}}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}$$
$$= C\phi(|Q|)^{s/p}\sum_{k=1}^\infty \frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}\omega(Q)^{(1-\kappa)/p}}{\phi(|4^{k+1}Q|)^N\omega(4^{k+1}Q)^{(1-\kappa)/p}}\|f\|_{L^{p,\kappa}_{\theta,\phi}(\phi)}.$$

By Lemma 2.2 and let $\theta'+\theta+\frac{\eta(1-\kappa)}{p}-\frac{\delta(1-\kappa)}{p\beta} < N < \theta'+\theta+\frac{\eta(1-\kappa)}{p}+\frac{s}{p}$. Then

$$\left(\frac{1}{\omega(Q)^\kappa}\int_Q |IV_1|^p\omega(x)dx\right)^{1/p} \leqslant C\phi(|Q|)^{\upsilon'}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}, \tag{3.5}$$

where $\upsilon' = \theta'+\theta+\frac{\eta(1-\kappa)}{p}+\frac{s}{p}-N$.

For IV_2. From the size condition (2) of K, it follows that

$$IV_2 = \int_{\mathbb{R}^n\setminus 4Q}|b(y)-b_Q||K(x,y)f(y)|dy \leqslant C\int_{\mathbb{R}^n\setminus 4Q}\frac{|b(y)-b_Q||f(y)|}{|x-y|^n\phi(|x-y|^n)^N}dy$$
$$\leqslant C\sum_{k=1}^\infty\int_{4^{k+1}Q\setminus 4^kQ}\frac{|b(y)-b_Q||f(y)|}{|x-y|^n\phi(|x-y|^n)^N}dy$$
$$\leqslant C\sum_{k=1}^\infty\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|b(y)-b_Q||f(y)|dy$$
$$\leqslant C\sum_{k=1}^\infty\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|b(y)-b_{4^{k+1}Q}||f(y)|dy$$
$$+ C\sum_{k=1}^\infty\frac{|b_{4^{k+1}Q}-b_Q|}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|f(y)|dy.$$

The Hölder inequality implies that

$$\int_{4^{k+1}Q}|b(y)-b_{4^{k+1}Q}||f(y)|dy$$
$$\leqslant \left(\int_{4^{k+1}Q}|b(y)-b_{4^{k+1}Q}|^{p'}\omega(y)^{-p'/p}dy\right)^{1/p'}\left(\int_{4^{k+1}Q}|f(y)|^p\omega(y)dy\right)^{1/p}. \tag{3.6}$$

Since $\omega \in A_p^{\theta'}(\phi)$, the Lemma 2.1 (ii) gives us that $\omega^{-p'/p} \in A_{p'}^{\theta'}(\phi)$. From Lemma 2.6 and (3.6), it follows that

$$\sum_{k=1}^{\infty} \frac{1}{\phi(|4^{k+1}Q|)^N |4^{k+1}Q|} \int_{4^{k+1}Q} |b(y) - b_{4^{k+1}Q}||f(y)|dy \leqslant \sum_{k=1}^{\infty} \frac{1}{\phi(|4^{k+1}Q|)^N |4^{k+1}Q|}$$
$$\cdot \left(\int_{4^{k+1}Q} |b(y) - b_{4^{k+1}Q}|^{p'} \omega(y)^{-p'/p} dy \right)^{1/p'} \left(\int_{4^{k+1}Q} |f(y)|^p \omega(y) dy \right)^{1/p}$$
$$\leqslant C \sum_{k=1}^{\infty} \frac{\phi(|4^{k+1}Q|)^{s/p} \phi(|4^{k+1}Q|)^{\theta'}}{\phi(|4^{k+1}Q|)^N |4^{k+1}Q| \omega(4^{k+1}Q)^{1/p}} \left(\int_{4^{k+1}Q} |f(y)|^p \omega(y) dy \right)^{1/p}$$
$$\leqslant C \sum_{k=1}^{\infty} \frac{\phi(|4^{k+1}Q|)^{s/p} \phi(|4^{k+1}Q|)^{\theta'} \phi(|4^{k+1}Q|)^{\theta} \omega(4^{k+1}Q)^{\kappa/p}}{\phi(|4^{k+1}Q|)^N |4^{k+1}Q| \omega(4^{k+1}Q)^{1/p}} \|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}.$$

Further, by Lemma 2.2 and let

$$\theta' + \theta + \frac{\eta(1-\kappa)}{p} + \frac{s}{p} - \frac{\delta(1-\kappa)}{p\beta} < N < \theta' + \theta + \frac{\eta(1-\kappa)}{p} + \frac{s}{p}$$

we get

(3.7)
$$\left(\frac{1}{\omega(Q)^\kappa} \int_Q \left| \sum_{k=1}^{\infty} \frac{1}{\phi(|4^{k+1}Q|)^N |4^{k+1}Q|} \int_{4^{k+1}Q} |b(y) - b_{4^{k+1}Q}||f(y)|dy \right|^p \omega(x) dx \right)^{1/p}$$
$$\leqslant C \frac{\omega(Q)^{1/p}}{\omega(Q)^{\kappa/p}} \sum_{k=1}^{\infty} \frac{\phi(|4^{k+1}Q|)^{s/p} \phi(|4^{k+1}Q|)^{\theta'} \phi(|4^{k+1}Q|)^{\theta} \omega(4^{k+1}Q)^{\kappa/p}}{\phi(|4^{k+1}Q|)^N |4^{k+1}Q| \omega(4^{k+1}Q)^{1/p}} \|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}$$
$$\doteq C \sum_{k=1}^{\infty} \frac{\phi(|4^{k+1}Q|)^{s/p} \phi(|4^{k+1}Q|)^{\theta'} \phi(|4^{k+1}Q|)^{\theta} \omega(Q)^{(1-\kappa)/p}}{\phi(|4^{k+1}Q|)^N |4^{k+1}Q| \omega(4^{k+1}Q)^{(1-\kappa)/p}} \|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}$$
$$\leqslant C \phi(|Q|)^{\upsilon'} \|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)},$$

where $\upsilon' = \theta' + \theta + \frac{\eta(1-\kappa)}{p} + \frac{s}{p} - N$.

Furthermore, by the definition of $\mathrm{BMO}^{\theta''}(\phi)$, Lemma 2.4 (ii) and Hölder's inequality, we obtain

$$|b_{4^{k+1}Q} - b_Q| = \frac{1}{|4^{k+1}Q|} \int_{4^{k+1}Q} |b(x) - b_Q| dx$$
$$\leqslant \left(\frac{1}{|4^{k+1}Q|} \int_{4^{k+1}Q} |b(x) - b_Q|^p dx \right)^{1/p} \leqslant C k \phi(|4^{k+1}Q|)^{\theta''}. \tag{3.8}$$

Combining (3.2) and (3.8), by Lemma 2.2 and let $\theta''+\theta'+\theta+\frac{\eta(1-\kappa)}{p}+\frac{s}{p}-\frac{\delta(1-\kappa)}{p\beta} < N < \theta''+\theta'+\theta+\frac{\eta(1-\kappa)}{p}+\frac{s}{p}$. Then

(3.9)

$$\begin{aligned}&\left(\frac{1}{\omega(Q)^\kappa}\int_Q\left|\sum_{k=1}^{\infty}\frac{|b_{4^{k+1}Q}-b_Q|}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|f(y)|dy\right|^p\omega(x)dx\right)^{1/p}\\&\leqslant C\frac{\omega(Q)^{1/p}}{\omega(Q)^{\kappa/p}}\sum_{k=1}^{\infty}\frac{k\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{s/p}\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}\omega(4^{k+1}Q)^{\kappa/p}}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|\omega(4^{k+1}Q)^{1/p}}\\&\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}\leqslant C\sum_{k=1}^{\infty}\frac{k\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{s/p}\phi(|4^{k+1}Q|)^{\theta'}\phi(|4^{k+1}Q|)^{\theta}\omega(Q)^{(1-\kappa)/p}}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|\omega(4^{k+1}Q)^{(1-\kappa)/p}}\\&\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}\leqslant C\phi(|Q|)^{\upsilon''}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)},\end{aligned}$$

where $\upsilon''=\theta''+\theta'+\theta+\frac{\eta(1-\kappa)}{p}+\frac{s}{p}-N$.

Therefore, by (3.7) and (3.9), setting $\widetilde{\upsilon}:=\max\{\upsilon',\upsilon''\}$, then

$$\left(\frac{1}{\omega(Q)^\kappa}\int_Q|IV_2|^p\omega(x)dx\right)^{1/p}\leqslant C\phi(|Q|)^{\widetilde{\upsilon}}\|f\|_{L^{p,\kappa}_{\theta,\omega}(\phi)}. \tag{3.10}$$

Finally, summing up (3.4), (3.5) and (3.10) and letting $\nu:=\max\{\nu',\widetilde{\upsilon}\}$, we obtain the desired estimate (3.3). We complete the proof of Theorem 1.4.

3.3. **Proof of Theorem 1.5.** Let $0\le\kappa<1$, $\theta\ge 0$, $\omega\in A_1^\infty(\phi)$ and $b\in \mathrm{BMO}^\infty(\phi)$, we only need to show that there exist positive constants C and ν such that for any given cube $Q=Q(x,r)$,

$$\frac{1}{\omega(Q)^\kappa}\omega(\{x\in Q:|[b,T](f)(x)|>\lambda\})\leqslant C\phi(|Q|)^\nu\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)} \tag{3.11}$$

holds for any function f such that $\Phi(|f|)\in L^{1,\kappa}_{\theta,\omega}(\phi)$.

Suppose that $\omega\in A_1^{\theta'}(\phi)$ for some $\theta'\ge 0$ and $b\in \mathrm{BMO}^{\theta''}(\phi)$ for some $\theta''\ge 0$. We split $f=f_1+f_2$, where $f_1=f\chi_{4Q}$ and $f_2=f\chi_{\mathbb{R}^n\setminus 4Q}$. Then for any $\lambda>0$, we can write

$$\begin{aligned}\frac{1}{\omega(Q)^\kappa}\omega(\{x\in Q:|[b,T]f(x)|>\lambda\})&\leqslant\frac{1}{\omega(Q)^\kappa}\omega(\{x\in Q:|[b,T]f_1(x)|>\lambda/2\})\quad+\\&+\frac{1}{\omega(Q)^\kappa}\omega(\{x\in Q:|[b,T]f_2(x)|>\lambda/2\}):=\mathrm{V}+\mathrm{VI}.\end{aligned}$$

For the term V. By Lemma 2.10 and Lemma 2.3, we obtain

$$\begin{aligned}\mathrm{V}&=\frac{1}{\omega(Q)^\kappa}\omega(\{x\in Q:|[b,T]f_1(x)|>\lambda/2\})\leqslant C\frac{1}{\omega(Q)^\kappa}\int_{\mathbb{R}^n}\Phi\Big(\frac{|f_1(x)|}{\lambda}\Big)\omega(x)dx\\&\leqslant C\frac{\omega(4Q)^\kappa}{\omega(Q)^\kappa}\phi(|4Q|)^\theta\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}\leqslant C\phi(|Q|)^{\nu'}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)},\end{aligned}$$

where $\nu'=\kappa\theta'+\theta$.

For the term VI. Notice that

$$
\begin{aligned}
|[b,T]f_2(x)| &\leqslant |b(x)-b_Q|\int_{\mathbb{R}^n}|K(x,y)f_2(y)|dy+\int_{\mathbb{R}^n}|b(y)-b_Q||K(x,y)f_2(y)|dy\\
&:= \mathrm{IV}_1+\mathrm{IV}_2.
\end{aligned}
$$

Then we have

$$
\mathrm{VI}\leqslant \frac{1}{\omega(Q)^{\kappa}}\omega(\{x\in Q: IV_1>\lambda/4\}) \quad +\frac{1}{\omega(Q)^{\kappa}}\omega(\{x\in Q: IV_2>\lambda/4\}) := \mathrm{VI}_1+\mathrm{VI}_2.
$$

For VI_1. From the pointwise inequality (3.2) and Lemma 2.6, it follows that

$$
\begin{aligned}
\mathrm{VI}_1 &\leqslant \frac{1}{\omega(Q)^{\kappa}}\frac{4}{\lambda}\int_Q IV_1\omega(x)dx\\
&\leqslant \frac{C}{\omega(Q)^{\kappa}}\int_Q|b(x)-b_Q|\omega(x)dx\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}\frac{|f(y)|}{\lambda}dy\\
&\leqslant C\frac{\omega(Q)}{\omega(Q)^{\kappa}}\phi(|Q|)^s\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}\frac{|f(y)|}{\lambda}dy.
\end{aligned}
$$

Since $\omega\in A_1^{\theta'}(\phi)$ for some $\theta'\geq 0$, by Lemma 2.1 (iv), yields

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}\frac{|f(y)|}{\lambda}dy\\
&\leqslant C\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}\Phi\left(\frac{|f(y)|}{\lambda}\right)dy\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}}{\phi(|4^{k+1}Q|)^N\omega(5\cdot 4^{k+1}Q)}\int_{5\cdot 4^{k+1}Q}\Phi\left(\frac{|f(y)|}{\lambda}\right)\omega(y)dy\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}}{\phi(|4^{k+1}Q|)^N\omega(5\cdot 4^{k+1}Q)^{1-\kappa}}\left\|\Phi\left(\frac{|f(x)|}{\lambda}\right)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}. \qquad (3.12)
\end{aligned}
$$

Notice that the first inequality above here because of

$$
t\leqslant t(1+\log^+ t)=\Phi(t),\quad \text{for any } t>0. \qquad (3.13)
$$

Therefore, by Lemma 2.2 and let $\theta'+\theta+\eta(1-\kappa)-\frac{\delta(1-\kappa)}{\beta} < N < \theta'+\theta+\eta(1-\kappa)+s$, we obtain

$$\begin{aligned}
\mathrm{VI}_1 &\leqslant C\frac{\omega(Q)}{\omega(Q)^\kappa}\phi(|Q|)^s\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}\frac{|f(y)|}{\lambda}dy\\
&\leqslant C\frac{\omega(Q)}{\omega(Q)^\kappa}\phi(|Q|)^s\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}}{\phi(|4^{k+1}Q|)^N\omega(5\cdot 4^{k+1}Q)^{1-\kappa}}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|Q|)^s\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}\omega(Q)^{1-\kappa}}{\phi(|4^{k+1}Q|)^N\omega(5\cdot 4^{k+1}Q)^{1-\kappa}}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|Q|)^s\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}\phi(|5\cdot 4^{k+1}Q|)^{\eta(1-\kappa)}}{\phi(|4^{k+1}Q|)^N}\\
&\qquad\times\left(\frac{|Q|}{|5\cdot 4^{k+1}Q|}\right)^{\delta(1-\kappa)}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}\\
&\leqslant C\phi(|Q|)^{\rho}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}, \qquad (3.14)
\end{aligned}$$

where $\rho = \theta'+\theta+\eta(1-\kappa)+s-N$.

For VI_2. From the estimate of IV_2 in Theorem 1.4, it follows that

$$\begin{aligned}
\mathrm{VI}_2 &\leqslant \frac{1}{\omega(Q)^\kappa}\frac{4}{\lambda}\int_Q IV_2\ \omega(x)dx\\
&\leqslant C\frac{\omega(Q)}{\omega(Q)^\kappa}\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|b(y)-b_{4^{k+1}Q}|\cdot\frac{|f(y)|}{\lambda}dy\\
&\quad+C\frac{\omega(Q)}{\omega(Q)^\kappa}\sum_{k=1}^{\infty}\frac{|b_{4^{k+1}Q}-b_Q|}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}\frac{|f(y)|}{\lambda}dy := A+B.
\end{aligned}$$

For A. Since $\omega\in A_1^{\theta'}(\phi)$ for some $\theta'\geq 0$, from (2.5), (2.3) and Lemma 2.1(iv), we deduce that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{\phi(|4^{k+1}Q|)^N|4^{k+1}Q|}\int_{4^{k+1}Q}|b(y)-b_{4^{k+1}Q}|\cdot\frac{|f(y)|}{\lambda}dy\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta''}}{\phi(|4^{k+1}Q|)^N}\|b\|_{BMO^{\theta''}(\phi)}\left\|\frac{|f|}{\lambda}\right\|_{L\log L,4^{k+1}Q}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta''}}{\phi(|4^{k+1}Q|)^N}\inf_{\gamma>0}\left\{\gamma+\frac{\gamma}{|4^{k+1}Q|}\int_{4^{k+1}Q}\frac{|f(y)|}{\gamma}\log\left(1+\frac{|f(y)|}{\gamma}\right)dy\right\}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{\theta'}}{\phi(|4^{k+1}Q|)^N\omega(5\cdot 4^{k+1}Q)}\int_{5\cdot 4^{k+1}Q}\frac{|f(y)|}{\lambda}\log\left(1+\frac{|f(y)|}{\lambda}\right)\omega(y)dy\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}}{\phi(|4^{k+1}Q|)^N\omega(5\cdot 4^{k+1}Q)^{1-\kappa}}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}.
\end{aligned}$$

Let $\theta''+\theta'+\theta+\eta(1-\kappa)-\frac{\delta(1-\kappa)}{\beta}<N<\theta''+\theta'+\theta+\eta(1-\kappa)$. Applying Lemma 2.2 and (1.1) yield that

(3.15)

$$\begin{aligned} A &\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}\omega(Q)^{1-\kappa}}{\phi(|4^{k+1}Q|)^{N}\omega(5\cdot 4^{k+1}Q)^{1-\kappa}}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)} \\ &\leqslant C\sum_{k=1}^{\infty}\frac{\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}\phi(|5\cdot 4^{k+1}Q|)^{\eta(1-\kappa)}}{\phi(|4^{k+1}Q|)^{N}} \\ &\quad\times\left(\frac{|Q|}{|5\cdot 4^{k+1}Q|}\right)^{\delta(1-\kappa)}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}\leqslant C\phi(|Q|)^{\rho'}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}, \end{aligned}$$

where $\rho'=\theta''+\theta'+\theta+\eta(1-\kappa)-N$.

For the term of B, similar to (3.12). Let $\theta''+\theta'+\theta+\eta(1-\kappa)-\frac{\delta(1-\kappa)}{\beta}<N<\theta''+\theta'+\theta+\eta(1-\kappa)$, by Lemma 2.2 and (3.8), we have

$$\begin{aligned} B &\leqslant C\sum_{k=1}^{\infty}\frac{k\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}\omega(Q)^{1-\kappa}}{\phi(|4^{k+1}Q|)^{N}\omega(5\cdot 4^{k+1}Q)^{1-\kappa}}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)} \\ &\leqslant C\sum_{k=1}^{\infty}\frac{k\phi(|4^{k+1}Q|)^{\theta''}\phi(|4^{k+1}Q|)^{\theta'}\phi(|5\cdot 4^{k+1}Q|)^{\theta}\phi(|5\cdot 4^{k+1}Q|)^{\eta(1-\kappa)}}{\phi(|4^{k+1}Q|)^{N}} \\ &\quad\times\left(\frac{|Q|}{|5\cdot 4^{k+1}Q|}\right)^{\delta(1-\kappa)}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}\leqslant C\phi(|Q|)^{\rho'}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}, \end{aligned}$$

where $\rho'=\theta''+\theta'+\theta+\eta(1-\kappa)-N$.

Summing up the above estimates for VI_1 and VI_2. Let $\widetilde{\rho}:=\max\{\rho,\rho'\}$, we obtain

(3.16)

$$\mathrm{VI}=\frac{1}{\omega(Q)^{\kappa}}\omega(\{x\in Q:|[b,T]f_2(x)|>\lambda/2\})\leqslant C\phi(|Q|)^{\widetilde{\rho}}\left\|\Phi\Big(\frac{|f(x)|}{\lambda}\Big)\right\|_{L^{1,\kappa}_{\theta,\omega}(\phi)}.$$

Therefore, combining (3.16) with the estimate of V and letting $\nu:=\{\nu',\widetilde{\rho}\}$, we get the desired inequality (3.11). The proof of Theorem 1.5 is finished.

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