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A SCHRÖDINGER - POISSON SYSTEM WITH THE CRITICAL GROWTH ON THE FIRST HEISENBERG GROUP

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Abstract. In this paper, we study the Schrödinger-Poisson system with the critical growth on the first Heisenberg group. With the aid of the Green's representation formula, the concentratio-compactness and the critical point theory, the existence of ground state solution.

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1. Introduction and main results

In recent years, Heisenberg group has attracted the attention of many scholars, and in quantum mechanics, partial differential equations, harmonic analysis, number theory and other branches plays an important role. The first mathematicians who study of subelliptic analysis on the Heisenberg group were Folland and Stein in [15], who consistently created a generalisation of the analysis for more general stratified groups [12]. And it can also be noted that Rothschild and Stein generalised these results for general vector fields satisfying the Hormander's conditions, see [26]. These results were published in the famous book by Folland and Stein [14] which laid the anisotropic analysis. And it is worth noting that homogeneous Lie group is nilpotent.

In the present paper, we are concerned the following Schrödinger–Poisson system in the Heisenberg:

$$\begin{cases} -\Delta_H u - \phi|u|u = \mu|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta_H \phi = |u|^3, & \text{in } \Omega, \\ \phi = u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_H is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$ and $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $2 < q < 4$ and $\mu > 0$ some real parameters. $Q^* := 2Q/(Q-2) = 4$ is Sobolev critical exponent for $\Omega \subset \mathbb{H}^1$, $Q = 4$ is the homogeneous dimension of $\mathbb{H}^1$.

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In the last few decades, Schrödinger–Poisson systems have been studied extensively due to its strong physical background. For more detailed physical aspects of Schrödinger–Poisson systems and for further mathematical and physical interpretation, we refer to [2, 6, 7] and the references therein.

The investigation of (1.1) was motivated by some works appeared in recent years. In [3], An and Liu studied the following Schrödinger-Poisson type system on the Heisenberg group:

$$\begin{cases} -\Delta_H u + \lambda\phi u = \mu|u|^{q-2}u + |u|^2 u, & \text{in } \Omega, \\ -\Delta_H \phi = u^2, & \text{in } \Omega, \\ \phi = u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

by the Green's representation formula and the critical point theory, they obtained at least two positive solutions and a positive ground state solution. In [22], A. Loiudice proved that problem (1.2) with $q = 2$ and $\lambda = 0$ admits at least one positive solution. And then, this result was extended to a critical semilinear boundary problem with singular nonlinearities, see [20]. On some recent results recovering the Heisenberg group, we refer to [10, 19, 22, 23, 24] and the references therein.

On the other hand, in the Euclidean case, in [5], Azzollini et al. have been studied the following Schrödinger–Poisson system with critical growing

$$\begin{cases} -\Delta u = \lambda u + q\phi|u|^3 u, & \text{in } B_R, \\ -\Delta\phi = q|u|^5, & \text{in } B_R, \\ \phi = u = 0, & \text{on } \partial B_R, \end{cases} \tag{1.3}$$

where $\lambda \in \mathbb{R}$ and B_R is the ball in $\mathbb{R}^3$ centered in 0 with radius R.

Inspired by the works in the above references, our main purpose in this paper is to study the existence of ground state solution for problem (1.1). In addition, since the first equation of (1.1) contains a nonlocal term $\phi|u|u$, proving the existence of two solutions to (1.1) be much more complicated and more difficult than proving the case of a single equation with a critical nonlinearity. However, we can prove the existence of ground state solution for problem (1.1) by using the classical techniques of Brézis–Nirenberg [11], the Green's representation formula of [8] and some more accurate estimates for related expressions.

We are now in position to state the existence result of ground state solution as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{H}^1$ be a smooth bounded domain, $2 < q < 4$, then problem (1.1) has ground state solution.*

The plan of the paper is as follows. In Section 2, we present some necessary preliminary knowledge on the Heisenberg group functional setting. In Section 3, we prove the some basic lemmas. In Section 4, we complete the proof of Theorem 1.1.

2. Variational setting and preliminaries

In this section we briefly recall some basic facts on the first Heisenberg group and the functional space $S_0^1(\Omega)$. For a complete treatment, we refer to [13, 16, 18, 21].

Let $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$ be the first Heisenberg group. If $\xi = (x, y, t) \in \mathbb{H}^1$ and $\xi' = (x', y', t') \in \mathbb{H}^1$, then the group law is defined by

$$\tau : \mathbb{H}^1 \longrightarrow \mathbb{H}^1, \quad \tau_\xi(\xi^{'}) = \xi \circ \xi^{'},$$

where

$$\xi \circ \xi' = (x + x',\ y + y',\ t + t' + 2(x'y - y'x)).$$

A natural group of dilations on $\mathbb{H}^1$ is given by $\delta_s(\xi) = (sx, sy, s^2t)$ for any $s > 0$. Hence, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. The homogeneous dimension of $\mathbb{H}^1$ is $Q = 4$. The gauge norm $|\cdot|_H$ in $\mathbb{H}^1$ is defined as

$$|\xi|_H = [(x^2 + y^2)^2 + t^2]^{\frac{1}{4}}$$

for any $\xi \in \mathbb{H}^1$. It is also the Korányi norm. Although the Korányi distance dose not refect the sub-Riemannian structure of the Heisenberg group, the calculation is relatively simple. The Kohn – Laplacian Δ_H on $\mathbb{H}^1$ is defined as

$$\Delta_H u = \mathrm{div}_H(\nabla_H u),$$

where

$$\nabla_H u = (X, Y), \quad X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t},$$

and ∇_H is the horizontal gradient, X and Y is a basis for Lie algebra of left-invariant vector fields on $\mathbb{H}^1$. The left-invariant distance d_H on $\mathbb{H}^1$ is accordingly defined by

$$d_H(\xi_0 \circ \xi) = |\xi^{-1} \circ \xi|_H,$$

where $\xi^{-1} = -\xi$. It is well known that Δ_H is a very degenerate elliptic operator and Bony's maximum principle is satisfed (see [9]).

Also, the Heisenberg ball of radius r centered at ξ_0 is the set

$$B_H(\xi_0, r) = \left\{ \xi \in \mathbb{H}^1 : d_H(\xi_0 \circ \xi) < r \right\}.$$

The natural volume in $\mathbb{H}^1$ is the Haar measure, which coincides with the Lebesgue measure L^3 in $\mathbb{R}^3$ (see [25]); then $B_H(\xi_0, r) = \alpha_Q r^Q$, where $\alpha_Q = |B_H(0, 1)|$. It implies $\tau_\xi(B_r(0)) = B_r(\xi)$ and $\delta_r(B_1(0)) = B_r(0)$. As a consequence, for every $0 \le a < b$ and for every measurable function $f : [a,b] \to R$, we have

$$\int_{B_d(0,b)\setminus B_d(0,a)} f(d_0(\xi))\mathrm{d}\xi = Q|B_d(0,1)| \int_a^b f(r)r^{Q-1}\mathrm{d}r.$$

The Folland–Stein space $S_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{S_0^1(\Omega)} = \int_\Omega |\nabla_H u|^2 \mathrm{d}\xi.$$

For brevity, we use the notation $\|u\| = \|u\|_{S_0^1(\Omega)}$ and $|\cdot|_p$ denotes the usual L^p-norm, that is,

$$|u|_p^p = \int_\Omega |u|^p \mathrm{d}\xi, \qquad u \in L^p(\Omega).$$

We denote by B_ρ the closed ball of radius ρ centered at zero in the Folland-Stein space $S_0^1(\Omega)$, and by S_ρ its relative boundary, that is,

$$B_\rho = \Big\{ u \in S_0^1(\Omega) : \|u\| \le \rho \Big\}, \qquad S_\rho = \Big\{ u \in S_0^1(\Omega) : \|u\| = \rho \Big\}.$$

By [15], the Folland–Stein space $(S_0^1(\Omega), \|\cdot\|)$ is a Hilbert space and the embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact when $1 \le p < Q^* = 4$, while it is only continuous if $p = Q^* = 4$. In particular, Jerison and Lee [17] proved that the best Sobolev constant

$$S = \inf_{u \in S_0^1(\mathbb{H}^1)} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2 \mathrm{d}\xi}{\left(\int_{\mathbb{H}^1} |u|^{Q^*} \mathrm{d}\xi\right)^{\frac{2}{Q^*}}} \tag{2.1}$$

is achieved by the C^∞ function

$$U(x,y,t) = \frac{c_0}{\sqrt{(1+x^2+y^2)^2 + t^2}},$$

where c_0 is a suitable positive constant. In other words, the function U is a positive solution of the following equation:

$$-\Delta_H u = u^3, \quad u \in S_0^1(\Omega) \tag{2.2}$$

and satisfies

$$\int_{\mathbb{H}^1} |\nabla_H U|^2 \mathrm{d}\xi = \int_{\mathbb{H}^1} |U|^4 \mathrm{d}\xi = S^2.$$

Let

$$u_\varepsilon(\xi) = \eta(\xi) U_\varepsilon(\xi) = \frac{c_0 \varepsilon \eta(\xi)}{\sqrt{(\varepsilon + x^2 + y^2)^2 + t^2}} \tag{2.3}$$

where $\eta \in C_0^\infty(B_H(0,r_0))$, $0 \le \eta \le 1$ and $\eta = 1$ in $B_H(0,\frac{r_0}{2})$. Then, one has $\|u_\varepsilon\|^2 = S^2 + O(\varepsilon^2)$ and

$$|u_\varepsilon|_p^p = \begin{cases} O(\varepsilon^p), & \text{if } 0 < p < 2, \\ O(\varepsilon^{1+\alpha}), & \text{if } p = 2, \\ O(\varepsilon^{4-p}), & \text{if } 2 < p < 4, \\ S^2 + O(\varepsilon^4), & \text{if } p = 4, \end{cases} \tag{2.4}$$

as $\varepsilon \to 0$, where $0 < \alpha < 1$ (see [3]).

In addition, we say that $(u,\phi) \in S_0^1(\Omega) \times S_0^1(\Omega)$ is a solution of problem (1.1) if and only if

$$\int_\Omega \nabla_H u \nabla_H v \mathrm{d}\xi - \int_\Omega \phi|u|uv\mathrm{d}\xi - \mu\int_\Omega |u|^{q-2}uv\mathrm{d}\xi = 0$$

and

$$\int_\Omega \nabla_H \phi \nabla_H w \mathrm{d}\xi - \int_\Omega w|u|^3\mathrm{d}\xi = 0$$

for any $v, w \in S_0^1(\Omega)$. Further, if u and ϕ are both positive, then we say that (u,ϕ) is positive solution of problem (1.1). We define the functional $J(u,\phi) : S_0^1(\Omega) \times S_0^1(\Omega) \to \mathbb{R}$, $\forall\ (u,\phi) \in S_0^1(\Omega) \times S_0^1(\Omega)$,

$$J(u,\phi) = \frac{1}{2}\int_\Omega |\nabla_H u|^2\mathrm{d}\xi + \frac{1}{6}\int_\Omega |\nabla_H \phi|^2\mathrm{d}\xi - \frac{1}{3}\int_\Omega \phi|u|^3\mathrm{d}\xi - \frac{\mu}{q}\int_\Omega |u|^q\mathrm{d}\xi.$$

Then J is C^1 on $S_0^1(\Omega) \times S_0^1(\Omega)$ and its critical points are the solutions of (1.1). Indeed, let $J'_u(u,\phi)$, $J'_\phi(u,\phi)$ denote the partial derivatives of J at (u,ϕ), that is, for any $(v,w) \in S_0^1(\Omega) \times S_0^1(\Omega)$,

$$J'_u(u,\phi)[v] = \int_\Omega \nabla_H u \nabla_H v\mathrm{d}\xi - \int_\Omega \phi|u|uv\mathrm{d}\xi - \mu\int_\Omega |u|^{q-2}uv\mathrm{d}\xi,$$

$$J'_\phi(u,\phi)[w] = \frac{1}{3}\int_\Omega \nabla_H \phi \nabla_H w\mathrm{d}\xi - \frac{1}{3}\int_\Omega w|u|^3\mathrm{d}\xi.$$

By Sobolev inequalities and that $S_0^1(\Omega)$ is continuously embedded into $L^4(\Omega)$, then standard computations show that J'_u (respectively J'_ϕ) maps continuously $S_0^1(\Omega) \times S_0^1(\Omega)$ in $S^{-1}(\Omega)$, where $S^{-1}(\Omega)$ denotes the dual space of $S_0^1(\Omega)$. So we conclude that J is C^1 on $S_0^1(\Omega) \times S_0^1(\Omega)$ and

$$J'_u(u,\phi) = J'_\phi(u,\phi) = 0$$

if and only if (u,ϕ) is a solution of problem (1.1).

The properties of the function ϕ are given in the following lemma. It is similar to the properties of the function ϕ in Euclidean case, see [4, Lemma 2.1].

Lemma 2.1. *If $u \in S_0^1(\Omega)$, then there exists a unique nonnegative function $\phi_u \in S_0^1(\Omega)$ satisfying*

$$\begin{cases} -\Delta_H \phi = |u|^3, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Moreover, $\phi_u > 0$ *if* $u \neq 0$ *and:*

(1) For any positive constant s*, then* $\phi_{su} = s^3\phi_u$ *and*

$$\int_\Omega \phi_u |u|^3 d\xi = \int_\Omega |\nabla_H \phi|^2 d\xi \leq S^{-1}|u|_4^6. \tag{2.6}$$

(2) For every $u, v \in S_0^1(\Omega)$*,*

$$\int_\Omega \phi_u |v|^3 d\xi = \int_\Omega \phi_v |u|^3 d\xi.$$

(3) For every $u, u_1 \cdots, u_k \in S_0^1(\Omega)$*,*

$$\left|\phi_u - \sum_{i=1}^k \phi_{u_i}\right|_4 \leq \frac{1}{S}\left||u|^3 - \sum_{i=1}^k |u_i|^3\right|_{\frac{4}{3}}.$$

(4) If $\{u_n\} \subset S_0^1(\Omega)$ *and* $u \in S_0^1(\Omega)$ *are such that* $u_n \rightharpoonup u$ *in* $S_0^1(\Omega)$*, then, up to subsequences,* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $S_0^1(\Omega)$ *and strongly in* $L^p(\Omega)$ *for all* $p \in [1, 4)$*. Moreover*

$$\int_\Omega \phi_{u_n}|u_n|^3 d\xi - \int_\Omega \phi_{u_n - u}|u_n - u|^3 d\xi = \int_\Omega \phi_u |u|^3 d\xi + o_n(1). \tag{2.7}$$

(5) $\|\phi_u\| \leq S^{-2}\|u\|^3$*, where* S *is the best Sobolev constant.*

Proof. For each $u \in S_0^1(\Omega)$, define $T_u : S_0^1(\Omega) \to \mathbb{R}$,

$$T_u(w) = \int_\Omega |u|^3 w \mathrm{d}\xi, \qquad \forall\ w \in S_0^1(\Omega).$$

Set $w_n \to w \in S_0^1(\Omega)$ as $n \to \infty$. By the Hölder's inequality, we have

$$\begin{aligned} |T_u(w_n) - T_u(w)| &\leq \left(\int_\Omega |w_n - w|^4\right)^{\frac{1}{4}} \left(\int_\Omega |u|^4\right)^{\frac{3}{4}} \\ &\leq S^{-\frac{1}{2}}|u|_4^3 \|w_n - w\| \to 0, \end{aligned}$$

as $n \to \infty$. This means that T_u is a continuous linear functional. It follows from the Lax–Milgram theorem that there exists a unique $\phi_u \in S_0^1(\Omega)$ such that

$$\int_\Omega \nabla_H \phi_u \nabla_H w \mathrm{d}\xi = \int_\Omega w|u|^3 \mathrm{d}\xi, \quad \forall\ w \in S_0^1(\Omega),$$

this is, ϕ_u is the unique solution of (2.5). Furthermore, by the principle of maximum, we get $\phi_u > 0$ and $\phi_u \geq 0$ if $u \neq 0$.

Next, we prove (1). In fact, for any positive constant s, we have

$$-\Delta_H \phi_{su} = s^3|u|^3 = s^3(-\Delta_H \phi_u) = -\Delta_H(s^3\phi_u).$$

It follows from the uniqueness that $\phi_{su} = s^3\phi_u$. At the same time, since $\phi_u \in S_0^1(\Omega)$, ϕ_u can be taken as a test function in (2.5). Then, from the Hölder inequality and

(2.1), one has

$$\int_\Omega |\nabla_H \phi_u|^2 \mathrm{d}\xi \le \Big(\int_\Omega |\phi_u|^4\Big)^{\frac14}\Big(\int_\Omega |u|^4\Big)^{\frac34} \le S^{-\frac12}\Big(\int_\Omega |\nabla_H \phi_u|^2 \mathrm{d}\xi\Big)^{\frac12} |u|_4^3,$$

which implies (2.6). To obtain (2) we observe that

$$\int_\Omega \phi_u |v|^3 \mathrm{d}\xi = \int_\Omega \nabla\phi_u \nabla\phi_v \mathrm{d}\xi = \int_\Omega \phi_v |u|^3 \mathrm{d}\xi.$$

By (2), we derive that

$$\begin{aligned}
\Big|\phi_u - \sum_{i=1}^k \phi_{u_i}\Big|_4^2 &\le \frac1S \Big\|\phi_u - \sum_{i=1}^k \phi_{u_i}\Big\|^2 \\
&= \frac1S \int_\Omega \Big|\nabla\phi_u - \sum_{i=1}^k \nabla\phi_{u_i}\Big|^2 \mathrm{d}\xi \\
&= \frac1S \int_\Omega \Big(|\nabla\phi_u|^2 - 2\phi_u \sum_{i=1}^k \nabla\phi_{u_i} + \sum_{i=1}^k\sum_{j=1}^k \nabla\phi_{u_i}\nabla\phi_{u_j}\Big)\mathrm{d}\xi \\
&= \frac1S \int_\Omega \Big(\phi_u - \sum_{i=1}^k \phi_{u_i}\Big)\Big(|u|^3 - \sum_{i=1}^k |u_i|^3\Big)\mathrm{d}\xi \\
&\le \frac1S \Big|\phi_u - \sum_{i=1}^k \phi_{u_i}\Big|_4 \Big||u|^3 - \sum_{i=1}^k |u_i|^3\Big|_{\frac43}
\end{aligned}$$

and (3) follows.

Furthermore, by applying (2), we get

$$\begin{aligned}
&\int_\Omega \phi_{u_n}|u_n|^3 \mathrm{d}\xi - \int_\Omega \phi_{u_n-u}|u_n-u|^3 \mathrm{d}\xi \\
&= \int_\Omega (\phi_{u_n} - \phi_{u_n-u})(|u_n|^3 - |u_n-u|^3)\mathrm{d}\xi + \int_\Omega \phi_{u_n-u}|u_n|^3 \mathrm{d}\xi \\
&\quad + \int_\Omega \phi_{u_n}|u_n-u|^3 - 2\phi_{u_n-u}|u_n-u|^3 \mathrm{d}\xi \\
&= \int_\Omega (\phi_{u_n} - \phi_{u_n-u})(|u_n|^3 - |u_n-u|^3)\mathrm{d}\xi \\
&\quad + 2\int_\Omega (\phi_{u_n} - \phi_{u_n-u})|u_n-u|^3 \mathrm{d}\xi.
\end{aligned}$$

An easy variant of the classical Brezis–Lieb Lemma yields that

$$|u_n|^3 - |u_n-u|^3 \to |u|^3 \qquad \text{in } L^{\frac43}(\Omega) \text{ as } n\to\infty$$

and applying (3), we get that

$$\phi_{u_n} - \phi_{u_n-u} \to \phi_u \qquad \text{in } L^4(\Omega) \text{ as } n\to\infty. \tag{2.8}$$

So

$$\int_\Omega (\phi_{u_n} - \phi_{u_n-u})(|u_n|^3 - |u_n-u|^3)\mathrm{d}\xi \to \int_\Omega \phi_u|u|^3 \mathrm{d}\xi \quad \text{as } n\to\infty.$$

Moreover, we get $|u_n - u|^3 \rightharpoonup 0$ in $L^{\frac{4}{3}}(\Omega)$. Hence, since $\phi_u \in L^4(\Omega)$ and (2.8), we have

$$\begin{aligned}&\int_\Omega (\phi_{u_n} - \phi_{u_n - u})|u_n - u|^3 \mathrm{d}\xi \\ &= \int_\Omega (\phi_{u_n} - \phi_{u_n - u} - \phi_u)|u_n - u|^3 \mathrm{d}\xi + \int_\Omega \phi_u |u_n - u|^3 \mathrm{d}\xi \longrightarrow 0\end{aligned}$$

as $n \to \infty$. This completes the proof of (4).

Finally, multiplying the second equation of (1.1) by ϕ_u and integrating we have

$$\|\phi_u\|^2 = \int_\Omega \phi_u |u|^3 \mathrm{d}\xi \le |\phi_u|_4 |u|_4^3 \le S^{-2} \|u\|^3 \|\phi_u\|$$

and then (5). □

Lemma 2.2. *(*[3, Lemma3.2]*) Let $\Psi(u) = \phi_u$ for any $u \in S_0^1(\Omega)$, where ϕ_u is as in Lemma 2.1. Let*

$$X = \left\{ (u, \phi) \in S_0^1(\Omega) \times S_0^1(\Omega) : J'_\phi(u, \phi) = 0 \right\}.$$

Then Ψ is C^1 and X is the graph of Ψ.

We define the functional I_μ as follows

$$\begin{aligned}I_\mu(u) &= J(u, \phi_u) \\ &= \frac{1}{2} \int_\Omega |\nabla_H u|^2 \mathrm{d}\xi - \frac{1}{6} \int_\Omega \phi_u |u|^3 \mathrm{d}\xi - \frac{\mu}{q} \int_\Omega |u|^q \mathrm{d}\xi\end{aligned}$$

for $u \in S_0^1(\Omega)$.

Lemma 2.3. *(*[3, Lemma 3.3]*) Let $(u, \phi) \in S_0^1(\Omega) \times S_0^1(\Omega)$. Then (u, ϕ) is a critical point of J if and only if u is a critical point of I_μ and $\phi = \Psi(u)$, where Ψ is as in Lemma 2.2.*

Hence, we know that a critical point u of the functional I_μ with $\phi = \Psi(u)$ corresponds to a solution (u, ϕ_u) of problem (1.1) and

$$I'_\mu(u)[v] = \int_\Omega \nabla_H u \nabla_H v \mathrm{d}\xi - \int_\Omega \phi_u |u| u v \mathrm{d}\xi - \mu \int_\Omega |u|^{q-2} u v \mathrm{d}\xi.$$

Based on the above arguments, we will strive to prove the existence of critical points of the functional I_μ by critical point theory and some analytical techniques. In addition, in this paper, where we say that (u, ϕ_u) with $u \in S_0^1(\Omega)$ is a ground state solution of problem (1.1), we mean that (u, ϕ_u) is a solution of problem (1.1) which has the least energy among all solutions of problem (1.1), that is, $I'_\mu(u) = 0$ and

$$I_\mu(u) = \inf \left\{ I_\mu(v) : v \in S_0^1(\Omega) \backslash \{0\},\ \langle I'_\mu(u), v \rangle = 0 \right\}.$$

3. Some basic lemmas

In this section, we prove that the functional I_μ satisfy the Palais-Smale condition in the cases $2 < q < 4$. First, we recall that a C^1 functional I_μ on Banach space $S_0^1(\Omega)$ is said to satisfy the Palais-Smale condition at level c ($(PS)_c$ in short) if every sequence $\{u_n\}_n \subset S_0^1(\Omega)$ satisfying $I_\mu(u_n) \to c$ and $I'_\mu(u_n) \to 0$ $(n \to \infty)$ has a convergent subsequence.

We first begin giving the following general mountain pass theorem (see[1]).

Theorem 3.1. *Let X is a real Banach space and $I_\mu \in C^1(X, \mathbb{R})$, with $I_\mu(0) = 0$. Assume that*
(1) there exist $r, \alpha > 0$ such that $I_\mu(u) \geq \alpha$ for all $u \in X$, with $\|u\| = r$;
(2) there exist $\|e\| > r$ satisfying $\|u\|_X > r$ such that $I_\mu(e) < 0$.
Define $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0$ and $I(\gamma(1)) < 0\}$.

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t)) \geq \alpha, \tag{3.1}$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n \in X$.

Now, we begin proving that I_μ satisfies the assumptions of the mountain pass theorem.

Lemma 3.1. *Suppose that $2 < q < 4$ is satisfied. Then the functional I_μ satisfies the mountain pass geometry, that is,*
(1) there exist $r, \alpha > 0$ such that $I_\mu(u) \geq \alpha$ for any $u \in S_0^1(\Omega)$ such that $\|u\| = r$;
(2) there exists $e \in S_0^1(\Omega)$ with $\|u\| > r$ such that $I_\mu(e) < 0$.

Proof. By the Hölder inequality, (2.1) and (2.6), we have

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int_\Omega |\nabla_H u|^2 \mathrm{d}\xi - \frac{1}{6}\int_\Omega \phi_u |u|^3 \mathrm{d}\xi - \frac{\mu}{q}\int_\Omega |u|^q \mathrm{d}\xi \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{6}S^{-4}\|u\|^6 - \frac{\mu}{q}\int_\Omega |u|^q \mathrm{d}\xi \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{6}S^{-4}\|u\|^6 - \frac{\mu}{q}S^{-\frac{q}{2}}|\Omega|^{\frac{4-q}{4}}\|u\|^q. \end{aligned} \tag{3.2}$$

Since $2 < q < 4$, then we can choose $r, \alpha > 0$ such that $I_\mu(u) \geq \alpha$ for $\|u\| = r$.

On the other hand, let $u \in S_0^1(\Omega) \backslash \{0\}$, and $2 < q < 4$, we have

$$I_\mu(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^6}{6}\int_\Omega \phi_u |u|^3 \mathrm{d}\xi - \frac{\mu t^q}{q}\int_\Omega |u|^q \mathrm{d}\xi.$$

It is obvious that $I_\mu(tu) \to -\infty$ as $t \to +\infty$. Thus, there exists $e \in S_0^1(\Omega) \backslash \{0\}$ such that $I_\mu(e) < 0$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Assume that $2 < q < 4$ is satisfied. Then for each $\mu > 0$, there exists a positive constant M which is independent of μ such that*

$$\limsup_{n\to\infty} \|u_n\| \le M.$$

Proof. We assume that $\{u_n\} \subset S_0^1(\Omega)$ satisfies

$$\begin{aligned} c + o_n(1) &= I_\mu(u_n) \\ &= \frac{1}{2}\int_\Omega |\nabla_H u_n|^2 \mathrm{d}\xi - \frac{1}{6}\int_\Omega \phi_{u_n}|u_n|^3 \mathrm{d}\xi - \frac{\mu}{q}\int_\Omega |u_n|^q \mathrm{d}\xi \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} o_n(1)\|u_n\| = \langle I'_\mu(u_n), v\rangle &= \int_\Omega \nabla_H u_n \nabla_H v \mathrm{d}\xi - \int_\Omega \phi_{u_n}|u_n|u_n v \mathrm{d}\xi \\ &- \mu\int_\Omega |u_n|^{q-2}u_n v \mathrm{d}\xi. \end{aligned} \tag{3.4}$$

So, by (2.6), (3.3) and (3.4), we have

$$\begin{aligned} c + o_n(1)\|u_n\| &= I_\mu(u_n) - \frac{1}{q}\langle I'_\mu(u_n), u_n\rangle \\ &= \frac{q-2}{2q}\|u_n\|^2 - \frac{q-6}{6q}\int_\Omega \phi_{u_n}|u_n|^3 \mathrm{d}\xi \\ &= \frac{q-2}{2q}\|u_n\|^2 + \frac{6-q}{6q}\|\phi_{u_n}\|^2 \ge \frac{q-2}{2q}\|u_n\|^2. \end{aligned} \tag{3.5}$$

This means that $\{u_n\}$ is also bounded in $S_0^1(\Omega)$ since $2 < q < 4$. Thus for each $\mu > 0$, there exists a positive constant M which is independent of μ such that

$$\limsup_{n\to\infty} \|u_n\| \le M.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that $2 < q < 4$ is satisfied. Then for each $\mu > 0$, the functional I_μ satisfies the $(PS)_c$ condition with $c < \dfrac{1}{3}S^2$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for I_μ at $c < \frac{1}{3}S^2$, by Lemma 3.2, $\{u_n\}$ is bounded in $S_0^1(\Omega)$. Since $S_0^1(\Omega)$ is reflexible. Therefore, we may still assume that $u_n \rightharpoonup u_0$ weakly in $S_0^1(\Omega)$ and $u_n \to u_0$ strongly in $L^p(\Omega)$ with $1 \le p < 4$.

Next, inspired by [4], we set $f(s) := |s|s$. Since $\{u_n\}$ is bounded in $L^4(\Omega)$, then $\{f(u_n)\}$ is bounded in $L^2(\Omega)$ and so, in a standard way, it follows that $f(u_n) \rightharpoonup f(u)$ in $L^2(\Omega)$. Then, for all $\varphi \in C_0^\infty(\Omega)$, using (4) of Lemma 2.1, Hölder and Sobolev inequalities, and since $\phi_u\varphi \in L^2(\Omega)$,

$$\begin{aligned} &\left|\int_\Omega f(u_n)\phi_{u_n}\varphi \mathrm{d}\xi - \int_\Omega f(u)\phi_u\varphi \mathrm{d}\xi\right| \\ &\le \left|\int_\Omega (\phi_{u_n} - \phi_u) f(u_n)\varphi \mathrm{d}\xi\right| + \left|\int_\Omega (f(u_n) - f(u))\phi_u\varphi \mathrm{d}\xi\right| \\ &\le C|\varphi|_\infty \|u_n\|^2 |\phi_{u_n} - \phi_u|_2 + o_n(1) \to 0 \end{aligned}$$

as $n \to \infty$.

Since $u_n \rightharpoonup u_0$ in $S_0^1(\Omega)$, we get

$$\|u_n\|^2 = \|u_n - u_0\|^2 + \|u_0\|^2 + o_n(1).$$

Then, by using the strong convergence $u_n \to u_0$ in $L^2(\Omega)$ and (2.7) we get

$$I_\mu(u_n) = I_\mu(u_0) + I(u_n - u_0) + o_n(1) \tag{3.6}$$

with

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{6}\|\phi_u\|^2.$$

Furthermore,

$$\begin{aligned} o_n(1) &= I'_\mu(u_n)[u_n - u_0] = (I'_\mu(u_n) - I'_\mu(u_0))[u_n - u_0] \\ &= \|u_n - u_0\|^2 - \int_\Omega \phi_{u_n} f(u_n)(u_n - u_0)\mathrm{d}\xi \\ &\quad + \int_\Omega \phi_{u_0} f(u_0)(u_n - u_0)\mathrm{d}\xi - \mu|u_n - u_0|^q. \end{aligned} \tag{3.7}$$

Since $u_n \rightharpoonup u_0$ in $L^4(\Omega)$ and $\phi_{u_0} f(u_0) \in L^{\frac{4}{3}}(\Omega)$,

$$\int_\Omega \phi_{u_0} f(u_0)(u_n - u_0)\mathrm{d}\xi = o_n(1). \tag{3.8}$$

Moreover

$$\begin{aligned} \int_\Omega \phi_{u_n} f(u_n)(u_n - u_0)\mathrm{d}\xi &= \int_\Omega \phi_{u_n}|u_n|^3\mathrm{d}\xi - \int_\Omega \phi_{u_0}|u_0|^3\mathrm{d}\xi \\ &- \int_\Omega (\phi_{u_n} - \phi_{u_0}) f(u_n) u_0 \mathrm{d}\xi - \int_\Omega \phi_{u_0} u_0 (f(u_n) - f(u_0))\mathrm{d}\xi. \end{aligned} \tag{3.9}$$

Since the sequence $((\phi_{u_n} - \phi_{u_0}) f(u_n))$ is bounded in $L^{\frac{4}{3}}(\Omega)$, $\phi_{u_n} \to \phi_{u_0}$ and $f(u_n) \to f(u_0)$ a.e. in Ω, by [27, Proposition 5.4.7] we have

$$\int_\Omega (\phi_{u_n} - \phi_{u_0}) f(u_n) u_0 \mathrm{d}\xi = o_n(1). \tag{3.10}$$

Analogously, we prove that

$$\int_\Omega \phi_{u_0} u_0 (f(u_n) - f(u_0))\mathrm{d}\xi = o_n(1). \tag{3.11}$$

Then, using (2.7), (3.10) and (3.11) in (3.9), we obtain

$$\int_\Omega \phi_{u_n} f(u_n)(u_n - u_0)\mathrm{d}\xi = \int_\Omega \phi_{u_n - u_0}|u_n - u_0|^3\mathrm{d}\xi + o_n(1). \tag{3.12}$$

Moreover, by (3.7), (3.8) and (3.12) we get

$$\|u_n - u_0\|^2 - \int_\Omega \phi_{u_n - u_0}|u_n - u_0|^3\mathrm{d}\xi = o_n(1) \tag{3.13}$$

and so

$$\begin{aligned} I(u_n - u_0) &= \frac{1}{2}\|u_n - u_0\|^2 - \frac{1}{6}\|u_n - u_0\|^2 + o_n(1) \\ &= \frac{1}{3}\|u_n - u_0\|^2 + o_n(1). \end{aligned} \tag{3.14}$$

On the other hand,

$$I'_\mu(u_n)[\varphi] \to I'_\mu(u)[\varphi]$$

and, by density, we get

$$0 = I'_\mu(u)[u] = \|u\|^2 - \|\phi_u\|^2 - \mu|u|_q^q,$$

from which

$$I_\mu(u) = \frac{q-2}{2q}\|u\|^2 + \frac{6-q}{6q}\|\phi_u\|^2 \geq 0. \tag{3.15}$$

From (3.6) and (3.15), we get

$$I(u_n - u_0) = I_\mu(u_n) - I_\mu(u_0) + o_n(1) \leq c + o_n(1) < \frac{1}{3}S^2.$$

Then it follows that

$$\limsup_{n\to\infty} \|u_n - u_0\|^2 < S^2,$$

by (3.13) and (5) of Lemma 2.1,

(3.16)

$$\begin{aligned} o_n(1) &= \|u_n - u_0\|^2 - \int_\Omega \phi_{u_n-u_0}|u_n - u_0|^3 \mathrm{d}\xi \geq \|u_n - u_0\|^2 - S^{-4}\|u_n - u_0\|^6 \\ &= \|u_n - u_0\|^2\left[1 - \frac{\|u_n - u_0\|^4}{S^4}\right] \geq C\|u_n - u_0\|^2. \end{aligned}$$

Hence $u_n \to u_0$ in $S_0^1(\Omega)$. This ends the proof. □

4. Proof of Theorem 1.1

In this section, we prove that problem (1.1) has a positive ground state solution, where $2 < q < 4$. To this end, we define

$$\psi = \inf_{u\in\mathcal{N}} I_\mu(u), \tag{4.1}$$

where $\mathcal{N} = \{u \in S_0^1(\Omega)\backslash\{0\} : \langle I'_\mu(u), u\rangle = 0\}$.

Proof of Theorem 1.1. Obviously, if $u \in \mathcal{N}$, we has $I_\mu(|u|) = I_\mu(u)$, so we consider a nonnegative minimizing sequence $\{u_n\} \subset \mathcal{N}$ and such that

$$I_\mu(u_n) \to \psi, \quad \text{as } n \to \infty. \tag{4.2}$$

By $I_\mu(u_\lambda) < 0$ and Lemma 3.2, we can see that $\psi < 0$ and $\{u_n\}$ is bounded in $S_0^1(\Omega)$. We may assume that $u_n \rightharpoonup u_1$ weakly in $S_0^1(\Omega)$ and $u_n \to u_1$ strongly in $L^p(\Omega)$ with $1 \leq p < 4$, then $u_1 \neq 0$. If $u_1 \equiv 0$ then $\lim_{n\to\infty} \|u_n\|^2 = 0$, furthermore $\lim_{n\to\infty} I_\mu(u_n) = 0$, this is a contradiction from (4.2). Therefore, we have $u_1 \neq 0$ in $S_0^1(\Omega)$.

It follows from Lemma 3.3 that $u_n \to u_1$ in $S_0^1(\Omega)$. It means that u_1 is a positive solution of problem (1.1) and $I_\mu(u_1) \geq \psi$.

On the other hand, we prove $I_\mu(u_1) \le \psi$. By Fatou's Lemma, we have

$$\begin{aligned}\psi &= \lim_{n\to\infty}\left(I_\mu(u_n) - \frac{1}{6}\langle I'_\mu(u_n), u_n\rangle\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{3}\|u_n\|^2 - \frac{\mu(6-q)}{6q}\int_\Omega |u_n|^q \mathrm{d}\xi\right)\\ &\ge \liminf_{n\to\infty}\left(\frac{1}{3}\|u_n\|^2 - \frac{\mu(6-q)}{6q}\int_\Omega |u_n|^q \mathrm{d}\xi\right)\\ &\ge \frac{1}{3}\|u_1\|^2 - \frac{\mu(6-q)}{6q}\int_\Omega |u_1|^q \mathrm{d}\xi\\ &= I_\mu(u_1) - \frac{1}{6}\langle I'_\mu(u_1), u_1\rangle = I_\mu(u_1).\end{aligned}$$

This means that $I_\mu(u_1) \le \psi$ and thus $I_\mu(u_1) = \psi$. Obviously, this proves that u_1 is a positive ground state solution of problem (1.1). □

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