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ON UNIQUENESS OF MEROMORPHIC SOLUTIONS TO DELAY DIFFERENTIAL EQUATION

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Abstract. In this paper, we investigate uniqueness of finite-order transcendental meromorphic solutions of the following two equations:

$$f(z+1)-f(z-1)+a(z)\frac{f'(z)}{f(z)}=R(z,f)=\frac{\sum_{m=0}^{3}a_m f^m(z)}{\sum_{n=0}^{2}b_n f^n(z)},$$

and

$$f(z+1)f(z-1)+a(z)\frac{f'(z)}{f(z)}=R(z,f)=\frac{\sum_{m=0}^{4}a_m f^m(z)}{\sum_{n=0}^{3}b_n f^n(z)},$$

where $R(z,f)$ is an irreducible rational function in $f(z)$, $a(z)$, a_m and b_n are small functions of $f(z)$. Such solutions $f(z)$ are uniquely determined by their poles and the zeros of $f(z)-e_j$ (counting multiplicities) for two complex numbers $e_1 \neq e_2$.

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1. Introduction and main results

We assume the reader is familiar with the elementary Nevanlinna theory, see, e.g. [3, 9, 12]. As usual, the abbreviation CM stands for "counting multiplicities", while IM means "ignoring multiplicities".

In 1929, Nevanlinna [10] raised the classic results in the uniqueness theory of meromorphic functions. He obtained:

Theorem A (five-point theorem)**.** *If two meromorphic functions f, g share five distinct values in the extended complex plane IM, then $f \equiv g$.*

Theorem B (four-point theorem)**.** *If two meromorphic functions f, g share four distinct values in the extended complex plane CM, then $f \equiv T \circ g$, where T is a Möbius transformation.*

Nevanlinna value distribution theory is a useful tool to research the uniqueness of meromorphic functions. Many scholars got many important results, see, e.g. [13]. In the last decade, Nevanlinna value distribution theory is widely used in complex

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difference and difference equations [4, 5]. Recently, Qi et al. [11] studyed some shared value properties for finite-order meromorphic solutions of the difference Painlevé IV equation. They showed that:

Theorem C ([11]). *Suppose that $f(z)$ is a finite-order transcendental meromorphic solution of*

$$(f(z+1)+f(z))(f(z)+f(z-1)) = R(z,f) = \frac{\sum_{m=0}^{4} a_m f^m(z)}{\sum_{n=0}^{2} b_n f^n(z)},$$

where $R(z,f)$ is an irreducible rational function in $f(z)$, a_m, b_n are small functions of $f(z)$ with $a_4 b_2 \neq 0$. Let e_1, e_2 be two distinct finite numbers such that $\Phi(z,e_1) \not\equiv 0$, $\Phi(z,e_2) \not\equiv 0$. Here

$$\Phi(z,f) = (f(z+1)+f(z))(f(z)+f(z-1))\sum_{n=0}^{2} b_n f^n(z) - \sum_{m=0}^{4} a_m f^m(z).$$

If $f(z)$ and another meromorphic function $g(z)$ share the values e_1, e_2 and ∞ CM, then $f(z) \equiv g(z)$.

Recently, Halburd and Korhonen researched some properties of the following delay differential equation

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z,w(z)) = \frac{P(z,w(z))}{Q(z,w(z))}. \tag{1.1}$$

One of the main conclusions is as follows:

Theorem D ([7]). *Suppose that $w(z)$ is a non-rational meromorphic solutions of (1.1), where $a(z)$ is rational, $P(z,w)$ is a polynomial in w with rational coefficients in z, and $Q(z,f)$ is a polynomial in $w(z)$ with roots that are nonzero rational functions of z and not roots of $P(z,f)$. If the hyper-order of $w(z)$ is less than one, then*

$$\deg_w(P) = \deg_w(Q) + 1 \leq 3 \quad or \quad \deg_w(R) \leq 1.$$

In this paper, we consider the sharing value of meromorphic solutions of (1.1). In particular, we assume $\deg_w P(z,w(z)) = 3$, $\deg_w Q(z,w(z)) = 2$, and obtained the following result:

Theorem 1.1. *Suppose that $f(z)$ is a finite-order transcendental meromorphic solution of*

$$f(z+1) - f(z-1) + a(z)\frac{f'(z)}{f(z)} = \frac{P(z,f(z))}{Q(z,f(z))} = \frac{\sum_{m=0}^{3} a_m f^m(z)}{\sum_{n=0}^{2} b_n f^n(z)}, \tag{1.2}$$

$P(z,f)$, $Q(z,f)$ are coprime rational functions in $f(z)$, $a(z)$, a_m and b_n are small functions of $f(z)$ satisfying $b_2 = 1$, $a_3 \neq 0$. Let e_1, e_2 be two distinct finite numbers

such that $P(z,e_1) \not\equiv 0$, $P(z,e_2) \not\equiv 0$. If $f(z)$ and another meromorphic function $g(z)$ share the values e_1, e_2 and ∞ CM, then $f(z) \equiv g(z)$.

Lately, we researched the following equation

$$f(z+1)f(z-1) + a(z)\frac{f'(z)}{f(z)} = \frac{P(z,f(z))}{Q(z,f(z))},$$

where $a(z)$ is rational, $P(z,f)$ and $Q(z,f)$ are coprime rational functions of $f(z)$. The roots of $Q(z,f)$ are all rational functions of $f(z)$. And obtained that $\deg_f(P) \leq 4$ and $\deg_f(Q) \leq 3$ [2]. As for the uniqueness of meromorphic solutions of the above equation, we get

Theorem 1.2. *Suppose that $f(z)$ is a finite-order transcendental meromorphic solution of*

$$f(z+1)f(z-1) + a(z)\frac{f'(z)}{f(z)} = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{m=0}^{4} a_m f^m(z)}{\sum_{n=0}^{3} b_n f^n(z)}, \tag{1.3}$$

where $P(z,f)$ and $Q(z,f)$ are coprime rational functions of $f(z)$. $a(z)$, a_m, b_n are small functions of $f(z)$ and $b_3 = 1$, $a_4 \neq 0$. Let e_1, e_2 be two distinct finite numbers such that $\Psi(z,e_1) \not\equiv 0$, $\Psi(z,e_2) \not\equiv 0$, where $\Psi(z,f) = [f(z)f(z+1)f(z-1)+a(z)f'(z)]\sum_{n=0}^{3} b_n f^n(z) - \sum_{m=0}^{4} a_m f^{m+1}(z)$. If $f(z)$ and another meromorphic function $g(z)$ share the values e_1, e_2 and ∞ CM, then $f(z) \equiv g(z)$.

2. Lemmas

In this section, we present some lemmas which play an important role in the following proofs. The first lemma is an analogue of the logarithmic derivative lemma on difference.

Lemma 2.1 ([1], [5]). *Let $f(z)$ be a meromorphic function of finite order $\sigma(f)$. Then we have*

$$m(r, \frac{f(z+c)}{f(z)}) + m(r, \frac{f(z)}{f(z+c)}) = S(r,f),$$

where $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as r tends to infinity outside of an exceptional set E of finite logarithmic measure, i.e.,

$$\lim_{r\to\infty} \int_{E\cap[1,r)} dt/t < \infty.$$

Lemma 2.2 is an analogue of Clunie lemma on delay differential equation.

Lemma 2.2 ([8]). *Let f be a transcendental meromorphic solution of hyper-order $\sigma_2(f) < 1$ of*

$$R(f)Q(f) = P(f),$$

with $\deg P(f) \leq \deg Q(f)$*, where* $P(f)$ *and* $R(f)$ *are differential difference polynomials of* f*,* $Q(f)$ *is a difference polynomial of* f*. Assume that there is only unique monomial of degree* $\deg Q(f)$ *in* $Q(f)$*. Then*

$$m(r, R(f)) = S(r, f)$$

holds possibly outside an exceptional of finite logarithmic measure.

Lemma 2.3 ([11])**.** *If* $f(z)$ *is a meromorphic function of finite order, then*

$$N(r, f(z+c)) \leq N(r+|c|, f) = N(r, f) + S(r, f),$$

and

$$N(r, \frac{1}{f(z+c)}) \leq N(r+|c|, \frac{1}{f}) = N(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.4 ([12])**.** *Suppose that* $f_j(z)(j = 1, \cdots, n)(n \geq 2)$ *are meromorphic functions and* $g_j(z)(j = 1, \cdots, n)$ *are entire functions satisfying the following conditions:*

(1) $\displaystyle\sum_{j=1}^{n} f_j(z) e^{g_j(z)} = 0.$

(2) $1 \leq j \leq k \leq n,\ g_j(z) - g_k(z)$ *are not constants for* $1 \leq j \leq k \leq n$*.*

(3) *For* $1 \leq j \leq n,\ 1 \leq h \leq k \leq n,$

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, r \to \infty, r \notin E,$$

where $E \subset (1, \infty)$ *is of linear measure. Then* $f_j = 0$ *for* $j = 1, \cdots, n$*.*

The following lemma is an analogue of Mohon'ko theorem on delay differential equation (see [6, Remark 5.3]).

Lemma 2.5. *Let* $w(z)$ *be a non-rational meromorphic solution of* $P(z, w) = 0$*, where* $P(z, w)$ *is a differential difference polynomial in* $w(z)$ *with rational coefficients, and let* $a(z)$ *be a rational function satisfying* $P(z, a(z)) \not\equiv 0$*. If* $\rho_2(w) < 1$*, then* $m(r, \frac{1}{w-a}) = S(r, w)$*.*

Lemma 2.6. *Let* $f(z)$ *be a finite order transcendental meromorphic solution of* (1.2)*, then*

$$N(r, f) \geq \frac{1}{2} T(r, f) + S(r, f),$$

namely,

$$m(r, f) \leq \frac{1}{2} T(r, f) + S(r, f).$$

Proof. Taking the Nevanlinna characteristic function of both sides of (1.2), by Lemmas 2.1, 2.3 and standard Valiron-Mohon'ko identity, we have

$$\begin{aligned} 3T(r,f) &= T(r,\frac{\sum_{m=0}^{m=3} a_m f^m(z)}{\sum_{n=0}^{n=2} b_n f^n(z)}) + S(r,f) = T(r, f(z+1) - f(z-1) + a(z)\frac{f'(z)}{f(z)}) \\ &\leq m(r, f(z)(\frac{f(z+1)}{f(z)} - \frac{f(z-1)}{f(z)})) + m(r, a(z)\frac{f'(z)}{f(z)}) \\ &\quad + N(r, f(z+1) - f(z-1) + a(z)\frac{f'(z)}{f(z)}) + S(r,f) \\ &\leq m(r,f) + 2N(r,f) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + S(r,f). \end{aligned}$$

Then

$$2T(r,f) \leq 2N(r,f) + \overline{N}(r,\frac{1}{f}) + S(r,f), \quad N(r,f) \geq \frac{1}{2}T(r,f) + S(r,f).$$

3. Proofs of the Theorems

In this section, the proofs of our results are given. Some ideas in the proofs come from [11], but we do not have $m(r,f) = S(r,f)$ in Theorem 1.1.

Proof of Theorem 1.1. From the assumptions that $P(z,e_1) \not\equiv 0, P(z,e_2) \not\equiv 0$ and Lemma 2.5, we have

$$m(r,\frac{1}{f-e_1}) = S(r,f), \quad m(r,\frac{1}{f-e_2}) = S(r,f). \tag{3.1}$$

$f(z)$ and $g(z)$ sharing e_1, e_2 CM gives that

$$\frac{f-e_1}{g-e_1} = e^{A(z)}, \quad \frac{f-e_2}{g-e_2} = e^{B(z)}, \tag{3.2}$$

where $A(z)$ and $B(z)$ are two entire functions. It follows from [12, Theorem 5.1] that

$$\begin{aligned} T(r,g) &= O(T(r,f)) \quad (r \to \infty, r \notin E), \\ T(r,e^{A(z)}) &= O(T(r,f)) \quad (r \to \infty, r \notin E), \\ T(r,e^{B(z)}) &= O(T(r,f)) \quad (r \to \infty, r \notin E), \end{aligned}$$

where E is a set of finite linear measure. Then $A(z)$ and $B(z)$ are two polynomials since the order of $f(z)$ is finite.

Obviously $f(z) \equiv g(z)$ if $e^{A(z)} \equiv 1$ or $e^{B(z)} \equiv 1$ or $e^{B(z)-A(z)} \equiv 1$. Next, we assume that $e^{A(z)} \not\equiv 1$, $e^{B(z)} \not\equiv 1$ and $e^{B(z)-A(z)} \not\equiv 1$. Rewrite (3.2) into the following forms:

$$f(z) = e_1 + (e_2 - e_1)\frac{e^{B(z)} - 1}{e^{C(z)} - 1}, \tag{3.3}$$

or

$$f(z) = e_2 + (e_2 - e_1)\frac{e^{A(z)} - 1}{e^{C(z)} - 1}e^{C(z)}, \tag{3.4}$$

where $C(z) = B(z) - A(z)$. We claim that $\deg A(z) = \deg B(z) = \deg C(z)$. First, we prove $\deg B(z) = \deg C(z)$. Suppose that $\deg B(z) > \deg C(z)$, then from (3.3),

$$T(r,f) = T(r,e^B) + S(r,f), \quad T(r,e^C) = S(r,e^B), \quad N(r,f) = S(r,f),$$

which contradicts with Lemma 2.6. If $\deg B(z) < \deg C(z)$, then

$$T(r,f) = T(r,e^C) + S(r,f), \quad T(r,e^B) = S(r,e^C),$$
$$N(r,\frac{1}{f-e_1}) = N(r,\frac{1}{e^B - 1}) + S(r,f) = S(r,f),$$

which implies $m(r,\frac{1}{f-e_1}) \neq S(r,f)$, and this contradicts with (3.1). So,

$$\deg B(z) = \deg C(z). \tag{3.5}$$

Next we prove $\deg A(z) = \deg C(z)$. Since $A(z) = B(z) - C(z)$, then $\deg A(z) \leq \deg B(z)$. If $\deg A(z) < \deg B(z)$, by (3.4) and (3.5),

$$T(r,f) = T(r,e^C) + S(r,f), \quad T(r,e^A) = S(r,e^C),$$
$$N(r,\frac{1}{f-e_2}) = N(r,\frac{1}{e^A - 1}) + S(r,f) = S(r,f),$$

which also contradicts with (3.1). Therefore $\deg A(z) = \deg C(z)$. Let

$$\deg A(z) = \deg B(z) = \deg C(z) = k > 0. \tag{3.6}$$

The value sharing assumption and the Nevanlinna second fundamental theorem lead to

$$\begin{aligned} T(r,f) &\leq N(r,f) + N(r,\frac{1}{f-e_1}) + N(r,\frac{1}{f-e_2}) + S(r,f) \\ &\leq N(r,g) + N(r,\frac{1}{g-e_1}) + N(r,\frac{1}{g-e_2}) + S(r,f) \\ &\leq 3T(r,g) + S(r,f). \end{aligned} \tag{3.7}$$

Similarly,

$$T(r,g) \leq 3T(r,f) + S(r,g). \tag{3.8}$$

Therefore,

$$S(r,g) = S(r,f) \tag{3.9}$$

By (3.2) and (3.7) to (3.9),

$$T(r,e^A) \leq 4T(r,f) + S(r,f), \tag{3.10}$$
$$T(r,e^B) \leq 4T(r,f) + S(r,f). \tag{3.11}$$

And by (3.3), we have

$$T(r,f) \leq T(r,e^B) + T(r,e^C) + S(r,f). \tag{3.12}$$

The above equation together with (3.6), (3.10) and (3.11), gives

$$S(r,f) = S(r,e^A) = S(r,e^B) = S(r,e^C). \tag{3.13}$$

For convenience, we define $\overline{f} = f(z+1)$, $\underline{f} = f(z-1)$. Substituting (3.3) into (1.2), we obtain

$$\begin{aligned}
&\{(e_2-e_1)(\frac{e^{\overline{B}}-1}{e^{\overline{C}}-1} - \frac{e^{\underline{B}}-1}{e^{\underline{C}}-1})[e_1 + (e_2-e_1)\frac{e^B-1}{e^C-1}] \\
&+a(z)(e_2-e_1)\frac{(B'-C')e^Be^C - B'e^B + C'e^C}{(e^C-1)^2}\} \\
&\cdot \sum_{n=0}^{2} b_n[e_1 + (e_2-e_1)\frac{e^B-1}{e^C-1}]^n = \sum_{m=0}^{3} a_m[e_1 + (e_2-e_1)\frac{e^B-1}{e^C-1}]^{m+1}.
\end{aligned}$$

Multiplying both sides of the last equality by $(e^{\overline{C}}-1)(e^{\underline{C}}-1)(e^C-1)^4$, we get

$$\begin{aligned}
&\{(e_2-e_1)[(e^{\overline{B}}-1)(e^{\underline{C}}-1) - (e^{\underline{B}}-1)(e^{\overline{C}}-1)][e_1(e^C-1)^2 \\
&+(e_2-e_1)(e^B-1)(e^C-1)] + a(z)(e_2-e_1)[(B'-C')e^Be^C - B'e^B + C'e^C] \\
&\cdot(e^{\overline{C}}-1)(e^{\underline{C}}-1)\} \sum_{n=0}^{2} b_n[e_1(e^C-1) + (e_2-e_1)(e^B-1)]^n(e^C-1)^{2-n} \\
(3.14)&= (e^{\overline{C}}-1)(e^{\underline{C}}-1)\sum_{m=0}^{3} a_m[e_1(e^C-1) + (e_2-e_1)(e^B-1)]^{m+1}(e^C-1)^{3-m}.
\end{aligned}$$

We denote:

$$\overline{B} = B + s_1, \quad \underline{B} = B + s_2,$$
$$\overline{C} = C + t_1, \quad \underline{C} = C + t_2,$$

s_i, t_i $(i = 1, 2)$ are polynomials of degree at most $k-1$. Then, (3.14) can be rewritten as

$$\sum_{\mu=0}^{4}\sum_{\nu=0}^{6} M_{\mu,\nu}e^{\mu B+\nu C} = 0, \tag{3.15}$$

where $M_{\mu,\nu}$ is either 0 or a polynomial in $a(z)$, a_m, b_n, e_1, e_2 and $e^{s_i(z)}$, $e^{t_i(z)}$. We get

$$M_{0,0} = \Phi(z, e_2) \neq 0, \tag{3.16}$$

where $\Phi(z,f) = ((f(z+1)-f(z-1))f(z)+a(z)f'(z))\sum_{n=0}^{2} b_nf^n(z) - \sum_{m=0}^{3} a_mf^{m+1}(z)$. Let B_1, C_1 be the highest degree terms of B and C respectively. We claim that there exist some $1 \leq \mu_0 \leq 4$, $1 \leq \nu_0 \leq 6$ such that

$$\mu_0B_1 + \nu_0C_1 = 0 \tag{3.17}$$

or

$$\mu_0 B_1 - \nu_0 C_1 = 0. \tag{3.18}$$

Otherwise, by (3.15) and Lemma 2.4, $M_{\mu,\nu}$=0 for all $0 \le \mu \le 4$, $0 \le \nu \le 6$, which contradicts with (3.16). Set $\alpha = \frac{B_1}{C_1}$, then α is a rational number and

$$|\alpha| \in \{1,2,3,4,5,6,\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{5}{3},\frac{1}{4},\frac{3}{4},\frac{5}{4}\}.$$

Moreover, $\deg A(z) = k$, therefore $\alpha \neq 1$. On the other hand, from (1.2) we obtain

$$(\bar{f} - \underline{f} - a_3 f)f^3 = H(z,f),$$

$H(z,f)$ is a differential difference polynomial of f, with degree at most 3. Set

$$\begin{aligned} G &= \overline{f} - \underline{f} - a_3 f \\ &= (e_2 - e_1)\frac{e^{\overline{B}} - 1}{e^{\overline{C}} - 1} - (e_2 - e_1)\frac{e^{\underline{B}} - 1}{e^{\underline{C}} - 1} - a_3 e_1 - a_3(e_2 - e_1)\frac{e^{B} - 1}{e^{C} - 1}. \end{aligned}$$

From Lemma 2.2, we have

$$m(r,G) = S(r,f).$$

Set

$$G_1 = \frac{e^{\overline{B}} - 1}{e^{\overline{C}} - 1} - \frac{e^{\underline{B}} - 1}{e^{\underline{C}} - 1} - a_3\frac{e^{B} - 1}{e^{C} - 1}. \tag{3.19}$$

Obviously,

$$m(r,G_1) = S(r,f). \tag{3.20}$$

G_1 can be rewritten into the form:

$$\begin{aligned} G_1 &= \frac{A_1 e^{B_1} - 1}{A_2 e^{C_1} - 1} - \frac{D_1 e^{B_1} - 1}{D_2 e^{C_1} - 1} - a_3\frac{E_1 e^{B_1} - 1}{E_2 e^{C_1} - 1} \\ &= \frac{F_1 - F_2 - F_3}{(A_2 e^{C_1} - 1)(D_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)}, \end{aligned} \tag{3.21}$$

where $F_1 = (A_1 e^{B_1} - 1)(D_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)$, $F_2 = (D_1 e^{B_1} - 1)(A_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)$, $F_3 = a_3(E_1 e^{B_1} - 1)(A_2 e^{C_1} - 1)(D_2 e^{C_1} - 1)$, A_1, A_2, D_1, D_2, E_1, E_2 are small functions of e^{B_1} and e^{C_1}. We discuss the following three cases.

Case 1. Suppose that $0 < \alpha < 1$, then

$$f = e_1 + (e_2 - e_1)\frac{e^{B} - 1}{e^{C} - 1} = e_1 + (e_2 - e_1)\frac{H_1 e^{\alpha C_1} - 1}{H_2 e^{C_1} - 1},$$

where H_1, H_2 are small functions of e^{C_1}. The numerator and denominator of f may cancel some common items, for example, when $\alpha = \frac{1}{2}$ and $H_1 = H_2 = 1$. Even so, by standard Valiron-Mohon'ko identity, we can still get $T(r,f) = N(r,f) + S(r,f)$ since α is rational. Then $m(r,f) = S(r,f)$.

Next, we prove that

$$T(r,e^A)=T(r,e^B)+S(r,f)=T(r,e^C)+S(r,f). \tag{3.22}$$

Let the greatest common factor of $e^{B(z)}-1$ and $e^{C(z)}-1$ be $D(z)$, then

$$e^{B(z)}-1=D(z)B_1(z),\quad e^{C(z)}-1=D(z)C_1(z),$$

where $B_1(z)$, $C_1(z)$ and $D(z)$ are entire functions. So (3.3) can be rewritten as

$$f(z)=e_1+(e_2-e_1)\frac{B_1(z)}{C_1(z)}.$$

Since

$$\begin{aligned} T(r,f) &= m(r,\frac{1}{f-e_1})+N(r,\frac{1}{f-e_1})+S(r,f)=N(r,\frac{1}{B_1})+S(r,f),\\ T(r,f) &= m(r,f)+N(r,f)+S(r,f)=N(r,\frac{1}{C_1})+S(r,f), \end{aligned}$$

we have

$$N(r,\frac{1}{B_1})=N(r,\frac{1}{C_1}).$$

By also considering that

$$T(r,e^B)=N(r,\frac{1}{e^B-1})+S(r,f)=N(r,\frac{1}{B_1})+N(r,\frac{1}{D})+S(r,f),$$

and

$$T(r,e^C)=N(r,\frac{1}{e^C-1})+S(r,f)=N(r,\frac{1}{C_1})+N(r,\frac{1}{D})+S(r,f),$$

we obtain:

$$T(r,e^C)=T(r,e^B)+S(r,f).$$

Similarly, by (3.4) we can prove

$$T(r,e^C)=T(r,e^A)+S(r,f).$$

So (3.22) holds.

If (3.17) holds, then $\deg(\mu_0B+\nu_0C)<k$. By (3.6), $e^{\mu_0B+\nu_0C}$ is a small function of e^A and $f(z)$. Then by (3.6), (3.13) and (3.22),

$$T(r,e^{\mu_0B+\nu_0C}\cdot e^{-\mu_0A})=T(r,e^{-\mu_0A})+S(r,f)=\mu_0T(r,e^A)+S(r,f).$$

On the other hand,

$$T(r,e^{\mu_0B+\nu_0C}\cdot e^{-\mu_0A})=T(r,e^{(\mu_0+\nu_0)C})=(\mu_0+\nu_0)T(r,e^A)+S(r,f).$$

From the above two equations, we get $\nu_0=0$, which contradicts with $1\le\nu_0\le 6$.

Similarly, if (3.18) holds, then $\deg(\mu_0B-\nu_0C)<k$. We have

$$T(r,e^{\mu_0B-\nu_0C}\cdot e^{-\mu_0A})=T(r,e^{-\mu_0A})+S(r,f)=\mu_0T(r,e^A)+S(r,f),$$

and

$$T(r,e^{\mu_0B-\nu_0C}\cdot e^{-\mu_0A})=T(r,e^{(\mu_0-\nu_0)C})=(\mu_0-\nu_0)T(r,e^A)+S(r,f),$$

which also deduce a contradiction.

Case 2. Suppose that $\alpha < 0$. Set $\beta = -\alpha$. Substituting $B_1 = -\beta C_1$ into (3.21) and multiplying both numerator and denominator by $e^{\beta C_1}$, we have

$$G_1 = \frac{I_1 - I_2 - I_3}{(A_2 e^{C_1} - 1)(D_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)e^{\beta C_1}},$$

where $I_1 = (A_1 - e^{\beta C_1})(D_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)$, $I_2 = (D_1 - e^{\beta C_1})(A_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)$, $I_3 = a_3(E_1 - e^{\beta C_1})(A_2 e^{C_1} - 1)(D_2 e^{C_1} - 1)$. From standard Valiron-Mohon'ko identity, we have

$$\begin{aligned} T(r, G_1) &= (3+\beta)T(r, e^{C_1}) + S(r, e^{C_1}), \\ N(r, G_1) &= 3T(r, e^{C_1}) + S(r, e^{C_1}), \end{aligned}$$

which implies that $m(r, G_1) = \beta T(r, e^{C_1}) + S(r, f) \neq S(r, f)$. Even if the numerator and denominator of G_1 can cancel some items, we can still get the same conclusion, which contradicts with (3.20).

Case 3. Suppose that $\alpha > 1$. Substituting $B_1 = \alpha C_1$ into (3.21), then

$$G_1 = \frac{J_1 - J_2 - J_3}{(A_2 e^{C_1} - 1)(D_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)}, \tag{3.23}$$

where $J_1 = (A_1 e^{\alpha C_1} - 1)(D_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)$, $J_2 = (D_1 e^{\alpha C_1} - 1)(A_2 e^{C_1} - 1)(E_2 e^{C_1} - 1)$, $J_3 = a_3(E_1 e^{\alpha C_1} - 1)(A_2 e^{C_1} - 1)(D_2 e^{C_1} - 1)$. On the other hand, substituting B, C to (3.19). When $k > 1$, we can see that $e^{\overline{B} + \underline{C} + C} - e^{\underline{B} + \overline{C} + C} - a_3 e^{B + \overline{C} + \underline{C}} \neq 0$. In this situation, (3.23) gives:

$$\begin{aligned} T(r, G_1) &= (\alpha + 2)T(r, e^{C_1}) + S(r, e^{C_1}), \\ N(r, G_1) &= 3T(r, e^{C_1}) + S(r, e^{C_1}), \end{aligned}$$

which implies that $m(r, G_1) = (\alpha - 1)T(r, e^{C_1}) + S(r, f) \neq S(r, f)$. Even if the numerator and denominator of G_1 can cancel some items, we can still get this conclusion, which contradicts with (3.20). When $k = 1$, without loss of generality, we assume that $B = \alpha l z + c_0$, $C = lz + d_0$. If $e^{\overline{B} + \underline{C} + C} - e^{\underline{B} + \overline{C} + C} - a_3 e^{B + \overline{C} + \underline{C}} = e^{c_0 + 2d_0}(e^{(\alpha-1)l} - e^{(1-\alpha)l} - a_3)e^{(\alpha+2)lz} \neq 0$, using the same method as above, we have $m(r, G_1) \neq S(r, f)$, which contradicts with (3.20). If $e^{(\alpha-1)l} - e^{(1-\alpha)l} - a_3 = 0$, then substituting B and C into $\Phi(z, f) = 0$. After combining similar terms, we can get

$$\sum_{p=0}^{4} \sum_{q=0}^{6} M_{p,q} e^{(p\alpha + q)lz} = 0,$$

where $M_{p,q}$ is either 0 or a polynomial in $a(z)$, a_m, b_n, e_1, e_2 and e^{c_0}, e^{d_0}. By Lemma 2.4, we obtain $M_{p,q} = 0$, which contradicts with (3.16). In conclusion, the theorem holds. $\square$

Proof of Theorem 1.2. Taking the Nevanlinna characteristic function of both sides of (1.3), we get

$$4T(r,f) \leq 2T(r,f) + \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + S(r,f),$$

namely,

$$2T(r,f) \leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + S(r,f),$$

which implies $m(r,f) = S(r,f)$.

Similar to Theorem 1.1, we can proof (3.1) – (3.4), (3.10) – (3.12) also hold. Evidently, when $e^{A(z)} \equiv 1$ or $e^{B(z)} \equiv 1$ or $e^{B(z)-A(z)} \equiv 1$, $f(z) \equiv g(z)$. We still need to consider the case when $e^{A(z)} \not\equiv 1$, $e^{A(z)} \not\equiv 1$ and $e^{B(z)-A(z)} \not\equiv 1$. Using the same method in the proof of Theorem 1.1, we will get (3.22). (3.10) – (3.12) and (3.22) give

$$\deg A(z) = \deg B(z) = \deg C(z) = k > 0.$$

Substituting (3.3) into (1.3), we obtain

$$\begin{aligned}
&\{[e_1 + (e_2 - e_1)\frac{e^{\overline{B}} - 1}{e^{\overline{C}} - 1}][e_1 + (e_2 - e_1)\frac{e^{\underline{B}} - 1}{e^{\underline{C}} - 1}][e_1 + (e_2 - e_1)\frac{e^{B} - 1}{e^{C} - 1}] \\
&+a(z)(e_2 - e_1)\frac{(B' - C')e^B e^C - B'e^B + C'e^C}{(e^C - 1)^2}\} \\
&\cdot \sum_{n=0}^{3} b_n[e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1}]^n = \sum_{m=0}^{4} a_m[e_1 + (e_2 - e_1)\frac{e^B - 1}{e^C - 1}]^{m+1}.
\end{aligned} \tag{3.24}$$

Multiplying both sides of (3.24) by $(e^{\overline{C}} - 1)(e^{\underline{C}} - 1)(e^C - 1)^5$, we get

$$\begin{aligned}
&\{[e_1(e^{\overline{C}} - 1) + (e_2 - e_1)(e^{\overline{B}} - 1)][e_1(e^{\underline{C}} - 1) + (e_2 - e_1)(e^{\underline{B}} - 1)] \\
&[e_1(e^C - 1) + (e_2 - e_1)(e^B - 1)](e^C - 1) + a(z)(e_2 - e_1) \\
&[(B' - C')e^B e^C - B'e^B + C'e^C](e^{\overline{C}} - 1)(e^{\underline{C}} - 1)\} \\
&\cdot \sum_{n=0}^{3} b_n[e_1(e^C - 1) + (e_2 - e_1)(e^B - 1)]^n (e^C - 1)^{3-n} = (e^{\overline{C}} - 1)(e^{\underline{C}} - 1) \\
&\cdot \sum_{m=0}^{4} a_m[e_1(e^C - 1) + (e_2 - e_1)(e^B - 1)]^{m+1}(e^C - 1)^{4-m}.
\end{aligned} \tag{3.25}$$

(3.25) can be rewritten as

$$\sum_{\mu=0}^{6}\sum_{\nu=0}^{7} M_{\mu,\nu} e^{\mu B + \nu C} = 0,$$

where $M_{\mu,\nu}$ is either 0 or a polynomial in $a(z)$, a_m, b_n, e_1, e_2 and $e^{s_i(z)}$, $e^{t_i(z)}$. And

$$M_{0,0} = \Psi(z, e_2) \neq 0. \tag{3.26}$$

We claim that $\deg(\mu B + \nu C) = \deg(\mu B - \nu C) = k$ for $1 \leq \mu \leq 6$ and $1 \leq \nu \leq 7$. Otherwise, if some $1 \leq \mu_0 \leq 6$ and $1 \leq \nu_0 \leq 7$ make $\deg(\mu_0 B + \nu_0 C) < k$ or $\deg(\mu_0 B - \nu_0 C) < k$, then we use the same method in the proof of Theorem 1.1 and obtain $\nu_0 = 0$, which is a contradiction. Thus,

$$T(r, M_{\mu,\nu}) = S(r, e^{\pm(\mu B + \nu C)}), \quad T(r, M_{\mu,\nu}) = S(r, e^{\pm(\mu B - \nu C)}),$$

where $0 \leq \mu \leq 6$, $0 \leq \nu \leq 7$ are not simultaneously zero. By Lemma 2.4, we get $M_{\mu,\nu} = 0$, which contradicts with (3.26). $\square$

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