Известия НАН Армении, Математика, том 58, н. 3, 2023, стр. 14 – 20.

KNOTS OF PLANE CURVES. APPLICATIONS TO ODE

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Abstract. In this paper we define knots of plane curve and give a method for counting them. Then the method is applied to study the knots of solutions of a basic system of first order equations. Practical aspects are illustrated in the case of predator-play model (Lotka-Volterra equation).

MSC2020 numbers: 14H50; 34D99.
Keywords: Knots of plane curves; knots of solutions of equations.

A given pair of real functions $(g(t), f(t)) \in C^2[a,b]$, $t \in [a,b]$, we can consider as a curve $\gamma(t)$ in the plane (g,f) and also as a complex curve, i.e. a curve $\gamma(t) := \{(x+iy\} : x = g(t),\ y = f(t)\}$ in the complex plane. Consider a part $\gamma(t_i, t_i')$ of γ corresponding to the interval (t_i, t_i'), $a \le t_i < t_i' \le b$. If $g(t_i) = g(t_i')$ and $f(t_i) = f(t_i')$ and $\gamma(t_i, t_i')$ bounds a simply connected domain we say that $\gamma(t_i, t_i')$ is a *knot* (belonging to or lying on γ). If instead of point t_i we meet an interval σ_i, where g remains the same and f remains the same we count similar intervals as one point t_i. Moving t from a to b we can determine the number N_γ of knots.

Also we define the number of (n,m)-points of the curve $\gamma(t)$: they are those points $a \le t_i(n,m) \le b$, where $g(t_i(n,m)) = n = const$ and $f(t_i(n,m)) = m = const$. Denote by $N_\gamma(n,m)$ the number of (n,m)-points.

Notice that the concept of (n,m)-points is quite similar to that of a-points of analytic functions. Respectively the $(0,0)$-points of curves are similar to the zeros of analytic functions.

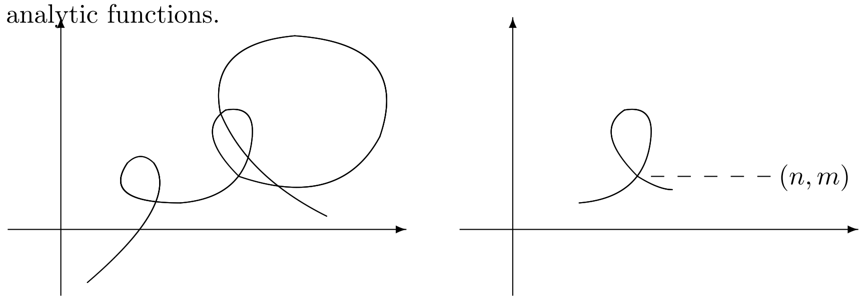

Figure 1. *The closed parts on the left figure are knots in the plane (g,f). The close part on the right figure is one of the curves $\gamma(t_i(n,m), t_{i+1}(n,m))$.*

Comment 1: on applicability of (n, m) -poins. The mentioned above curves can be solutions of different equations arising in physics, chemistry, biology etc. In all similar cases $N_\gamma(n, m)$ indicate how many time situation return to the initial state when $g = n$ and $f = m$. For instance in Lotka-Volterra equation (predator-pray model) n can be the number of foxes and m can be the number of rabbits. Respectively $N_\gamma(n, m)$ indicate how many time during time interval $(a.b)$ we can meet the situation when the number of foxes is equal to n and the number of rabbits is equal to m. Clearly similar interpretations of (n, m)-points can arise in many other sciences (for instance economics).

Observe that for any interval $(t_i(n, m), t_{i+1}(n, m))$ corresponding to curve $\gamma(t_i(n, m), t_{i+1}(n, m))$ is either a knot or implies inside a knot (in the case when this curve has self-intersections. So that the number of (n, m)-points is less than or equal to the number of knots plus 1: i.e. we have

$$N_\gamma(n, m) \leq N_\gamma + 1. \tag{1}$$

In this paper we give a method for estimating N_γ. Due to (1) this gives also estimates for $N_\gamma(n, m)$ [1].

Then we give applications of this method to differential equations.

1. The method for giving upper bounds for the knots.

Passing to the method we assume that $(g(t), f(t)) \in C^2(a, b)$ and that N_γ is finite: no meaning to give upper bounds if $N_\gamma = \infty$. Denote by $\beta(t)$ the tangential angle of $\gamma(t)$ at the point $t \in (a, b)$ (that is the angle formed by the tangent to the curve $\gamma(t)$ at the point $\gamma(t)$ and real axis x $(= g)$.

Let P be a partition $\{A = t_0^*, t_1^*, ..., t_{N_P}^* = B\}$ of an interval (A, B). According to definition the *total variation* $\mathrm{V}_A^B(\beta)$ of our (angle) function $\beta(t)$, defined on an interval $t \in (A, B)$ is the quantity

$$\mathrm{V}_A^B(\beta) := \sup_{\{P\}} \sum_{k=0}^{N_P - 1} \left|\beta(t_{k+1}^*) - \beta(t_k^*)\right|,$$

where the supremum runs over the set of all partitions P.

Notice that due to geometric meaning for any closed curve defined on an interval $t \in (A, B)$ we have $\mathrm{V}_A^B(\beta) \geq \pi$, since $\mathrm{V}_A^B(\beta)$ is determined by all increments $\left|\beta(t_{k+1}^*) - \beta(t_k^*)\right|$ on this closed curve. Denote by $\gamma(t_i, t_i')$ the part of the curve γ which corresponds to (t_i, t_i'). It is a closed curve which starts and ends at the same point (i.e. $g(t_i) = g(t_i')$ and $f(t_i) = f(t_i')$). Hence we have $\mathrm{V}_{t_i}^{t_i'}(\beta) \geq \pi$.

[1] The method wasn't formulated earlier. However was utilized in the proofs in [1]

Summing up by i we get

$$N_\gamma \le \frac{1}{\pi}\sum_{i=1}^{N_\gamma} \mathrm{V}_{t_i}^{t_i'}(\beta).$$

and since all intervals (t_i, t_i') lie in (a,b) we obtain the following

Theorem 1. *For any pair* $(g(t), f(t)) \in C^2[a,b]$ *we have*

$$N_\gamma \le \frac{1}{\pi}\mathrm{V}_a^b(\beta). \tag{2}$$

Due to inequality (1) inequality (2) implies

Theorem 2. *For any pair* $(g(t), f(t)) \in C^2[a,b]$ *and any pair of real numbers* (n,m) *we have*

$$N_\gamma(n,m) \le \frac{1}{\pi}\mathrm{V}_a^b(\beta) + 1. \tag{3}$$

Assuming that the function $\frac{d}{dt}\beta(t)$ is continuos in (a,b) we have also $\mathrm{V}_a^b(\beta) = \int_a^b \left|\frac{d}{dt}\beta(t)\right| dt$. Taking into account that $\beta(t)$ is equal to the argument of the complex point $\frac{d}{dt}g(t) + i\frac{d}{dt}f(t)$ (i.e. $\beta(t) := \arg(g'(t) + if'(t))$) we get

$$\mathrm{V}_a^b(\beta) = \int\limits_a^b \left|\frac{d}{dt}\arg(g'(t) + if'(t))\right| dt$$

and since

$$\frac{d}{dt}\arg(g'(t) + if'(t)) = \frac{d}{dt}\arctan\frac{f'(t)}{g'(t)} = \frac{f''(t)g'(t) - g''(t)f'(t)}{(g'(t))^2 + (f'(t))^2}$$

we get

$$\mathrm{V}_a^b(\beta) = \int\limits_a^b \frac{|f''(t)g'(t) - g''(t)f'(t)|}{(g'(t))^2 + (f'(t))^2} dt.$$

Thus in this case we have (due to Theorem 1)

$$N_\gamma \le \int_a^b \left|\frac{d}{dt}\beta(t)\right| dt = \frac{1}{\pi}\int\limits_a^b \frac{|f''(t)g'(t) - g''(t)f'(t)|}{(g'(t))^2 + (f'(t))^2} dt \tag{2'}$$

instead of (2) and

$$N_\gamma(n,m) \le \int_a^b \left|\frac{d}{dt}\beta(t)\right| dt + 1 = \frac{1}{\pi}\int\limits_a^b \frac{|f''(t)g'(t) - g''(t)f'(t)|}{(g'(t))^2 + (f'(t))^2} dt + 1 \tag{3'}$$

instead of (3).

2. Applications in ODE: the knots of solutions of a basic system of equations.

Many phenomena in physics, technics, biology, economics are described by the differential equations

$$\left\{ \begin{array}{l} y' = F_1(x,y) \\ x' = F_2(x,y) \end{array} \right. . \tag{4}$$

The solutions are the curves $\gamma(t) := (x(t), y(t))$, $t \in [T_1, T_2]$, that may have knots and (n,m)-points.

The equation (4) was studied in details in Poincaré theory when F_1, F_2 are linear polynomials. Numerous studies are devoted to the cases when F_1, F_2 are second and third order polynomials.

As far as we know the knots and (n,m)-points weren't consider before.

Below we study N_γ and $N_\gamma(n,m)$ for the solutions of (4) with very general coefficients.

Assume that the values x and y lie in the closure $\bar{D}$ of a given domain D and $F_1(x,y), F_2(x,y) \in C^2(\bar{D})$, where $\bar{D}$ is the closure of D.

Let $\gamma := (x(t), y(t))$, $t \in [T_1, T_2]$, be a part of an integral curve of (4) lying in $\bar{D}$. We will refer simply γ as a solution of (4), [3].

The next result gives upper bounds for N_γ and $N_\gamma(n,m)$ of the integral curves of (4).

Theorem 3. *For any solution $(x(t), y(t)) \in C^2[T_1, T_2]$ of equation (4) with $F_1(x,y)$, $F_2(x,y) \in C^2(\bar{D})$ we have*

$$N_\gamma \le c(F_1, F_2, D)\,|T_2 - T_1|\,, \tag{5}$$

where $c(F_1, F_2, D)$ is a finite constant depending only on F_1, F_2 and D.

Due to (1), for any (n,m) we have also

$$N_\gamma(n,m) \le c(F_1, F_2, D)\,|T_2 - T_1| + 1. \tag{6}$$

Comment 2. The constant $c(F_1, F_2, D)$ is equal to the maximum of

$$\frac{1}{\pi}\left\{ \left|(F_1(x,y))'_x\right| + \frac{\left|(F_1(x,y))'_y - (F_2(x,y))'_x\right|}{2} + \left|(F_2(x,y))'_y\right| \right\}$$

for $(x,y) \in \bar{D}$.

Proof of Theorem 3. First we show that for our solutions $\gamma(t) := (x(t), y(t))$, $t \in [T_1, T_2]$ the function $\frac{d}{dt}\beta(t)$ is continuos in $[T_1, T_2]$. In the above part we showed that

$$\frac{d}{dt}\beta(t) = \frac{d}{dt}\arg(x'(t) + iy'(t)) = \arctan\frac{y'(t)}{x'(t)} = \frac{y''(t)x'(t) - x''(t)y'(t)}{(x'(t))^2 + (y'(t))^2}. \tag{7}$$

Since

$$y''(t) = (F_1(x,y))'_x\, x' + (F_1(x,y))'_y\, y' =$$

$$(F_1(x,y))'_x F_2(x,y) + (F_1(x,y))'_y F_1(x,y)$$

and

$$x''(t) = (F_2(x,y))'_x x' + (F_2(x,y))'_y y' =$$

$$(F_2(x,y))'_x F_2(x,y) + (F_2(x,y))'_y F_1(x,y)$$

we have

$$\frac{y''x' - x''y'}{(x')^2 + (y')^2} =$$

$$\frac{\left[(F_1(x,y))'_x F_2(x,y) + (F_1(x,y))'_y F_1(x,y)\right] F_2(x,y)}{F_2^2(x,y) + F_1^2(x,y)} -$$

$$\frac{\left[(F_2(x,y))'_x F_2(x,y) + (F_2(x,y))'_y F_1(x,y)\right] F_1(x,y)}{F_2^2(x,y) + F_1^2(x,y)} =$$

$$\frac{(F_1(x,y))'_x F_2^2(x,y) - (F_2(x,y))'_y F_1^2(x,y)}{F_2^2(x,y) + F_1^2(x,y)} +$$

$$\frac{\left[(F_1(x,y))'_y - (F_2(x,y))'_x\right] F_1(x,y) F_2(x,y)}{F_2^2(x,y) + F_1^2(x,y)\,.} \tag{8}$$

Taking into account that

$$\frac{F_2^2(x,y)}{F_2^2(x,y) + F_1^2(x,y)} \le 1,\ \ \frac{F_1^2(x,y)}{F_2^2(x,y) + F_1^2(x,y)} \le 1 \tag{9}$$

and

$$\frac{|F_1(x,y) F_2(x,y)|}{F_2^2(x,y) + F_1^2(x,y)} \le \frac{1}{2} \tag{10}$$

we obtain that $\frac{d}{dt}\beta(t)$ is finite: since due to (7)-(10) we have

$$\left|\frac{d}{dt}\beta(t)\right| \le \left|(F_1(x,y))'_x\right| + \frac{\left|(F_1(x,y))'_y - (F_2(x,y))'_x\right|}{2} + \left|(F_2(x,y))'_y\right| \tag{11}$$

and since $F_1(x,y), F_2(x,y) \in C^2(\bar{D})$. Consequently we obtain also that $\frac{d}{dt}\beta(t)$ is continuos for $t \in [T_1, T_2]$.

Hence we can apply inequalities $(2')$ and $(3')$. This yields

$$N_\gamma \le \int_a^b \left|\frac{d}{dt}\beta(t)\right| dt \text{ and } N_\gamma(n,m) \le \int_a^b \left|\frac{d}{dt}\beta(t)\right| dt + 1,$$

and applying inequality (11) we obtain Theorem 3 with

$$c(F_1, F_2, D) :=$$

$$\max_{(x,y)\in\bar{D}} \left\{ \left|(F_1(x,y))'_x\right| + \frac{\left|(F_1(x,y))'_y - (F_2(x,y))'_x\right|}{2} + \left|(F_2(x,y))'_y\right| \right\}.$$

3. Practical aspects: illustration in the case of predator-play model.

This model describing the dynamics of biological systems in which two species interact, one as a predator (for example, foxes), the other as prey (for example, rabbits). The interaction is described by known Lotka-Volterra equation

$$\left\{ \begin{array}{l} x' = \alpha x - \beta xy \\ y' = \delta xy - \theta y \end{array} \right. , \tag{12}$$

where x $(= x(t))$ is the number of preys, y $(= y(t))$ is the number of predators, α, β, δ, θ are positive constants.

We consider $x(t)$, $y(t)$ in arbitrary time interval $t \in [T_1, T_2]$.

With this notations $N_\gamma(n,m)$ indicate how many time during time interval $[T_1, T_2]$ we can meet the situation when the number of foxes is equal to n and the number of rabbits is equal to m. In other words how many time initial situation can return to the same state.

Usually experts have an idea what is the maximal number of prey (denote by X) and maximal number of predators (denote by Y) in a given area. So that it is natural to take as a domain D the rectangle $\{(x,y) : 0 \le x \le X,\ 0 \le y \le Y\}$: obviously any curve $x(t)$, $y(t)$ lies in D. With similar D we prove the following

Theorem 4. *For any solution $(x(t), y(t)) \in C^2[T_1, T_2]$ of equation (12) and any pair (n,m) we have*

$$N_\gamma(n,m) \le \frac{1}{\pi}\left[\frac{3}{2}(\beta+\delta)(Y+X)+\alpha+\theta\right]|T_2-T_1|+1. \tag{13}$$

Proof. The inequality (12) is a particular case of (4) so that we can apply inequality (6) of Theorem 3. We obtain: for any solution $(x(t), y(t)) \in C^2[T_1, T_2]$ of equation (12) and any pair (n,m) we have

$$N_\gamma(n,m) \le c_{LV}(D)\,|T_2-T_1|+1, \tag{14}$$

where $c_{LV}(D)$ is a finite constant depending only on the equation and D.

It remains to estimate the constant $c_{LV}(D)$.

Remembering that all constants in (12) are positive, due to Comment 2 we have

$$\pi c_{LV}(D) =$$

$$\left|(\alpha x-\beta xy)'_x\right| + \frac{1}{2}\left[\left|(\alpha x-\beta xy)'_y-(\delta xy-\theta y)'_x\right|\right] + |\delta xy-\theta y|'_y \le$$

$$\alpha+\beta y+\frac{1}{2}\left[|\beta x-\delta y|\right]+\delta x+\theta \le \frac{3}{2}(\beta+\delta)(y+x)+\alpha+\theta.$$

Taking into account that (x,y) belong to the rectangle D we obtain

$$\pi c_{LV}(D) \le \frac{3}{2}(\beta+\delta)(Y+X)+\alpha+\theta$$

and substituting this into (14) we obtain Theorem 4.

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Поступила 20 июля 2022

После доработки 20 июля 2022

Принята к публикации 15 декабря 2022