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K. A. Keryan, A. L. Khachatryan

A Uniqueness Teorem for Orthonormal Spline Series

(Submitted by academician G. G. Gevorkyan 16/I 2023)

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Introduction. It is well known that there are trigonometric series converging almost everywhere to zero and having at least one non-zero coefficient. This also applies to the series in other classical orthogonal systems, for instance, to the series Haar, Walsh and Franklin series.

Uniqueness theorems for almost everywhere convergent or summable trigonometric series were first obtained in the papers [1] and [2], under some additional conditions imposed on the series. Results on uniqueness and restoration of coefficients for series by Haar, generalized Haar systems and regular martingales have been obtained, for instance, in the papers [3-5] and [6].

The following theorem about restoration of coefficients of series by Franklin system was proved in [7]. Specifically, in [7] it was proved that if the Franklin series

$$\sum_{n=0}^{\infty} a_n f_n(x)$$

converges a.e. to a function $f(x)$ and

$$\lim_{\lambda\to\infty}\left(\lambda\cdot\left|\left\{x\in[0,1]:\sup_{k\in\mathbb{N}}|S_k(x)|>\lambda\right\}\right|\right)=0,$$

where $|A|$ denotes the Lebesgue measure of a set A and

$$S_k(x)=\sum_{i=0}^{k} a_i f_i(x)\,,$$

then the coefficients a_n of the Franklin series can be reconstructed by the following formula

$$a_n = \lim_{\lambda\to\infty} \int_0^1 [f(x)]_\lambda \, f_n(x)\, dx,$$

where

$$[f(x)]_\lambda = \begin{cases} f(x), & if \ |f(x)| \le \lambda, \\ 0, & if \ |f(x)| > \lambda. \end{cases}$$

In the multi-dimensional case, in this theorem instead of $S_k(x)$ partial sums one can take $S_{2^k}(x)$ (see [8]) partial sums, or $S_{q_k}(x)$ (see [9]), where q_k is any increasing sequence of natural numbers, for which the ratio q_{k+1}/q_k is bounded. Similar result on uniqueness is also obtained for the multi-dimensional Franklin series (see [10]).

Let $k \ge 2$ be an integer. In this paper we are concerned with orthonormal spline systems of order k corresponding to the k regular partitions. Let $\mathcal{T} = (t_n)_{n=2}^{\infty}$ be a dense sequence of points in the open unit interval $(0,1)$, such that each point occurs at most k times. Moreover, define $t_0 := 0$ and $t_1 := 1$. Such point sequences are called k admissible. For n in the range $-k+2 \le n \le 1$, let $\mathcal{S}_n^{(k)}$ be the space of polynomials of order not exceeding $n+k-1$ (or degree not exceeding $n+k-2$) on the interval $[0,1]$ and $\left\{f_n^{(k)}\right\}_{n=-k+2}^{1}$ be the collection of orthonormal polynomials in $L^2 \equiv L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $n+k-2$.

For $n \ge 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^{n}$ repeated according to their multiplicities and where the knots 0 and 1 have multiplicity k, i.e.

$$\mathcal{T}_n = (0 = \tau_{-k+1}^n = \cdots = \tau_0^n < \tau_1^n \le \cdots \le \tau_{n-1}^n = \cdots = \tau_{n+k-1}^n = 1).$$

In that case, we also define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with points $\mathcal{T}_n$. For each $k \ge 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$, therefore there exists a function $f_n^{(k)} \in \mathcal{S}_n^{(k)}$, that is orthogonal to the space $\mathcal{S}_{n-1}^{(k)}$ and $\left\|f_n^{(k)}\right\|_2 = 1$. Observe that this function $f_n^{(k)}$ is unique up to the sign.

The system of functions $\left\{f_n^{(k)}\right\}_{n=-k+2}^{\infty}$ is called the orthonormal spline system of order k corresponding to the sequence $\mathcal{T}$.

Note that the case $k = 2$ corresponds to orthonormal systems of piecewise linear functions, i.e. general Franklin systems.

Consider a series

$$\sum_{n\in\Lambda} a_n \, f_n(x), \tag{1}$$

where $\Lambda \coloneqq \{n \in \mathbb{Z} \mid n \geq -k + 2\}$.

Let $\{n_q\}$ be an increasing sequence of natural numbers, denote $\Lambda_q \coloneqq \{-k+2, \dots, n_q\}$.

Definition 1. *An admissible sequence $\mathcal{T}$ is called k regular for n_q with a parameter $\gamma > 1$, if*

$$\frac{1}{\gamma} \leq \frac{\left|\Delta_i^{n_q}\right|}{\left|\Delta_{i-1}^{n_q}\right|} \leq \gamma, \quad -k+1 \leq i \leq n_q - 1, \quad q \in \mathbb{N},$$

where $\Delta_i^n \coloneqq [\tau_i^n, \tau_{i+k}^n]$.

In the case $k = 2$, which corresponds to general Franklin systems, and $n_q = q$ the sequence $\mathcal{T}$ satisfying the above condition is called regular by couples (see [10]). For general n_q we will call such sequence $\mathcal{T}$ regular by couples for n_q.

We will also be interested in sequences $\mathcal{T}$ for which

$$\frac{\left|\Delta_i^{n_q}\right|}{\left|\Delta_j^{n_q+1}\right|} \leq \gamma, \tag{2}$$

for any i, j, q so that $\Delta_i^{n_q} \supset \Delta_j^{n_q+1}$.

Denote by $S_{n_q}(x)$ the n_q-th partial sums of the series (1), that is

$$S_{n_q}(x) \coloneqq \sum_{n=-k+2}^{n_q} a_n\, f_n(x)\,.$$

The main results of the present paper are the following theorems:

Theorem 1. *Let the partition $\mathcal{T}$ be a k regular for n_q with a parameter $\gamma > 1$ and satisfy the condition (2). If the partial sums $S_{n_q}(x)$ converge in measure to a function $f(x)$ and*

$$\lim_{p\to\infty} \left(\lambda_p \cdot \left|\left\{x \in [0,1] \colon \sup_q \left|S_{n_q}(x)\right| > \lambda_p\right\}\right|\right) = 0, \tag{3}$$

for some sequence $\lambda_p \to +\infty$, then for any $n \in \Lambda$

$$a_n = \lim_{p\to\infty} \int_0^1 [f(x)]_{\lambda_p}\, f_n(x)\, dx\,. \tag{4}$$

Theorem 2 shows the necessity of condition (2) in Theorem 1.

Theorem 2. *Let the partition $\mathcal{T}$ be k regular for n_q which doesn't satisfy condition (2) for any γ.*

Then for some subsequence, $\{n_{m_q}\}$ there exists a series, $\sum_{n\in\Lambda} a_n\, f_n(x)$ such that its partial sums $S_{n_{m_q}}(q)$ converge a.e. to some function $f(x)$, and

$$\lim_{p\to\infty}\left(\lambda_p \cdot \left|\left\{x \in [0,1]: \sup_q \left|S_{n_{m_q}}(x)\right| > \lambda_p\right\}\right|\right) = 0,$$

for some sequence $\lambda_p \to +\infty$, but not all the coefficients, a_n, $n \in \Lambda$ are recovered by formulas (4). Particularly

$$a_{-k+2} \neq \lim_{p\to\infty} \int_0^1 [f(x)]_{\lambda_p} f_{-k+2}(x)\, dx\,.$$

In the case $k = 2$, i.e. general Franklin series, we have the following corollary:

Corollary 1. Let the partition $\mathcal{T}$ be regular by couples for n_q with a parameter $\gamma > 1$ and satisfy the condition (2). If the partial sums of the Franklin series corresponding to n_q converge in measure to a function $f(x)$ and the condition (3) takes place, then the coefficients of the Franklin series are recovered by formulas (4).

Yerevan State University
e-mails: karenkeryan@ysu.am, ani.khachatryan@ysu.am

K. A. Keryan, A. L. Khachatryan

A Uniqueness Theorem for Orthonormal Spline Series

In this paper we obtain recovery formulas for coefficients of orthonormal spline series by means of its sum, if the subsequence of partial sums of an orthonormal spline series converge in measure and the majorant of the subsequence of partial sums satisfies some necessary condition, provided that the spline system corresponds to a «regular» sequence. Additionally, it is proved that the regularity of the sequence is essential.

Կ. Ա. Քեռյան, Ա. Լ. Խաչատրյան

Միակության թեորեմ օրթոնորմալ սպլայն շարքերի համար

Ստացել ենք, որ օրթոնորմալ սպլայն շարքերի գործակիցները վերականգնվում են շարքի գումարի միջոցով, եթե շարքի մասնակի գումարների ենթահաջորդականությունը ըստ չափի զուգամետ է, և այդ մասնակի գումարների ենթահաջորդականության մաժորանտը բավարարում է ինչ-որ անհրաժեշտ պայմանի, և որ այդ սպլայն համակարգին համապատասխանող հաջորդականությունը «ռեգուլյար» է: Ավելին, ապացուցված է, որ հաջորդականության ռեգուլյարությունը կարևոր է:

К. А. Керян, А. Л. Хачатрян

Теорема единственности для ортонормированных сплайн рядов

Получены формулы восстановления коэффициентов ортонормированных сплайн рядов с помощью их суммы, если подпоследовательность частичных сумм ортонормированного сплайн ряда сходится по мере, а мажоранта этой подпоследовательности частичных сумм удовлетворяет некоторому необходимому условию, в том случае, если сплайн система соответствует «регулярной» последовательности. Кроме того, доказывается, что регулярность последовательности является существенной.

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