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UNIQUENESS OF A MEROMORPHIC FUNCTION PARTIALLY AND NORMALLY SHARING SMALL FUNCTIONS WITH ITS DIFFERENT VARIANTS OF GENERALIZED OPERATOR

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Abstract. First of all, in continuation of our previous result related to "2 CM+1 IM" small functions sharing of a meromorphic function of restricted hyper order and its linear shift delay differential operator, in some extend we have been able to answer a question paused by us in [Rendiconti del Circolo Mat. di Palermo, 2021 (Published online)]. As another attempt we improve and extend a result of [Comput. Methods Funct. Theory, 22(2), 197 – 205 (2022)]. Most importantly, we have pointed out a gap in the proof of a recent theorem [Results Math., 76, Article number: 147 (2021)] and citing a proper example we have shown that the result is true only for a particular case. Finally we present the compact version of the same result as an improvement.

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1. Introduction and some useful notations

At the outset we will assume that the readers are familiar with the standard notations and expressions like $m(r,f)$, $N(r,f)$ $(N(r,\infty;f))$, $N(r,\frac{1}{f-a})$ $(N(r,a;f))$, $T(r,f)$ in Nevanlinna theory for meromorphic functions defined on whole complex plane $\mathbb{C}$ (see [10], [19]). In addition, by $S(r,f)$ we mean a quantity satisfies $S(r,f)=o(T(r,f))$ as $r\longrightarrow\infty$ outside of a possible exceptional set E of finite logarithmic measure. We say that $a(z)(\not\equiv\infty)$ is a small function compared to $f(z)$ or slowly moving with respect to $f(z)$ if $T(r,a)=S(r,f)$. We denote by $S(f)$ the set of all small functions compared to $f(z)$ and $\hat{S}(f)$ by $S(f)\cup\{\infty\}$.

Some important terms namely order, hyper-order and ramification index of f will be defined respectively as follows:

$$\rho(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}\ ,\quad \rho_2(f)=\limsup_{r\longrightarrow\infty}\frac{\log\log T(r,f)}{\log r}$$

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$$\text{and} \quad \Theta(a;f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where $a \in \mathbb{C} \cup \{\infty\}$.

The following definitions and notations are required in the sequel.

Definition 1.1. *For some $a \in \mathbb{C}$, we denote by $E(a;f)$, the collection of the zeros of $f - a$, where a zero is counted according to its multiplicity. In addition to this, when $a = \infty$, the above notation implies that we are considering the poles. In the same manner, by $\overline{E}(a;f)$, we denote the collection of the distinct zeros or poles of $f - a$ according as $a \in \mathbb{C}$ or $a = \infty$ respectively.*

If $E(a;f) = E(a;g)$ we say that f and g share the value a CM (counting multiplicities) and if $\overline{E}(a;f) = \overline{E}(a;g)$, then we say that f and g share the value a IM (ignoring multiplicities).

Especially, for $a(z) \in S(f)$, if $f - a(z)$ and $g - a(z)$ share 0 CM (IM), then we will say that f and g share $a(z)$ CM (IM). Let z_0 be a zero of $f - a(z)$ and $g - a(z)$ of multiplicity $p(\geq 0)$ and $q(\geq 0)$ respectively. We denote by $\overline{N}_{\otimes}(r, 0, f - a(z); g - a(z))$, the reduced counting function of common zeros of $f - a(z)$ and $g - a(z)$ with different multiplicities that is $p \neq q$. On the other hand, for $a(z) \in S(f) \cup \{\infty\}$, if $E(0, f - a(z)) \subseteq E(0, g - a(z))$ $(E(0, g - a(z)) \subseteq E(0, f - a(z)))$, then we say that $f(g)$ and $g(f)$ share the small function $a(z)$ CM partially from $f(g)$ to $g(f)$.

Also we denote $N_{=1}(r, f)$ by the counting function of simple poles of f.

For $c \in \mathbb{C} \setminus \{0\}$, we define the shift of $f(z)$ by $f(z + c)$ or f_c and the difference operators of $f(z)$ by

$$\Delta_c f = f(z + c) - f(z), \qquad \Delta_c^k f = \Delta_c(\Delta_c^{k-1} f) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(z + ic),$$

where $k(\geq 2)$ is an integer. Generalization of shifts and derivatives operators, were recently done in [1]. We have defined the operators namely linear shift, shift-differential and differential operator, linear shift delay differential operator as follows:

$$L_1(f(z)) = a_0(z)f(z) + \sum_{i=1}^{k} a_i(z)f(z + c_i), \quad L_2(f(z)) = \sum_{i=1}^{s} b_i(z)f^{(i)}(z + c_i),$$

$$L_3(f(z)) = \sum_{i=1}^{t} d_i(z)f^{(i)}(z), \quad L(f(z)) = L_1(f(z)) + L_2(f(z)) + L_3(f(z)),$$

where $a_i(z)$ $(i = 0, 1, \ldots, k)$; $b_i(z)$ $(i = 1, \ldots, s)$; $d_i(z)$ $(i = 1, \ldots, t) \in S(f)$ and all $c_i's$ are non-zero complex constants. Also by delay-differential operator denoted by $\tilde{L}(f(z))$ and defined by $L_2(f(z)) + L_3(f(z))$. Choosing $c_i = ic$ for $i = 0, 1, \ldots, k$, where c is non-zero complex constant, we denote $L_1(f(z))$ as

$$L_c f \left(= \sum_{j=0}^{k} a_j(z) f(z + jc) (\not\equiv 0) \right)$$

with $a_k(z) \neq 0$ $(k \geq 1)$ and it is called as linear c-shift operator. If we impose the restriction $\sum_{j=0}^{k} a_j(z) = s$ on the coefficients of $L_c f$, then we denote it by $L_c^s f$. By virtue of the definition, all the operators functioning in the following section $\not\equiv 0, f(z)$.

2. Background and main results

In 2014, Liu et al. [14] were the first to investigate the uniqueness theorem for a finite order entire function sharing two small functions with its linear shift operator as follows:

Theorem A. [14] *Let f be a non-constant entire function of finite order and $a(z)$, $b(z)$ be two distinct small functions related to $f(z)$, let $L_1(f(z))$ be linear shift operator of $f(z)$ with constant coefficients. If $f(z)$ and $L_1(f(z))$ share $a(z)$, $b(z)$ CM, then $f(z) \equiv L_1(f(z))$.*

After that, in 2017, concerning entire function of finite order, Li et al. [13] tackle the "1 CM+ 1 IM" value sharing problem as follows:

Theorem B. [13] *Let $b \in \mathbb{C} \setminus \{0\}$ and let $f(z)$ be a non constant entire function of finite order. If $f(z)$ and $\Delta_c^k f(z)$ share 0 CM and b IM, then $f(z) \equiv \Delta_c^k f(z)$.*

Theorem C. [13] *Let $f(z)$ be a non constant entire function of finite order. If $f(z)$ and $\Delta_c^k f(z)$ share two distinct complex constants a CM and b IM and if*

$$N\left(r, \frac{1}{f(z) - a}\right) = T(r, f) + S(r, f), \tag{2.1}$$

then $f(z) \equiv \Delta_c^k f(z)$.

Recently, adopting the same procedure of [13], Kaish-Rahaman [12] again proved *Theorem C* but they did not mention it. Also Qi-Yang [17] extended *Theorem B* from finite order entire function to entire function of $\rho_2(f) < 1$ and asked the following question:

Question 2.1. [17] *If the sharing condition in Theorem B is changed into sharing "a CM+ b IM", where a, b are two distinct constants such that $ab \neq 0$, is the result still valid?*

In aspect of the uniqueness result of a meromorphic function f sharing "3 CM values" with its difference operators, Lu-Lu [15], Cui-Chen [6] contributed remarkably. Again we would like to mention that Kaish-Rahaman [12] proved uniqueness result of a meromorphic function f sharing "2 CM values" with its difference operators with the support of the assumption $N(r, f) = S(r, f)$. So the previous results on "3 CM value" sharing are far better than "2 CM value" sharing result in [12]. Unfortunately, again Kaish-Rahaman [12] did not provide any information about [15] and [6]. So considering these facts the paper of Kaish-Rahaman [12] has hardly any value.

After that, Deng et al. [7], Gau et al. [8] investigated the "3 CM small functions" sharing problem for the difference operator or even k-th order difference operator.

In connection with the *Question 2.1*, Qi-Yang [17] also asked the following question:

Question 2.2. [17] *Can the value sharing condition "3 CM" for a meromorphic function with its difference operators be reduced up to "2 CM + 1 IM"?*

It is to be noted that for finite order entire function, *Question 2.1* has already been answered in *Theorem C*. Recently, by the following results, we have answered of *Questions 2.1* and *2.2* in a compact form for a larger class of operators in view of small functions sharing.

Theorem D. (see Theorem 2.1 & Corollary 2.1, [1]) Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z), b(z)$ be two distinct small functions. If $L(f(z))$ and $f(z)$ share $a(z), \infty$ CM and $b(z)$ IM with $\Theta(0; f-a(z))+\Theta(\infty; f) > 0$ and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f-a(z)}\right) = T(r, f) + S(r, f)$,
then $L(f(z)) \equiv f(z)$.

Theorem E. (see Corollaries 2.2 & 2.3, [1]) Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$ and let $b(z)$ $(\not\equiv 0) \in S(f)$. If $L(f(z))$ and $f(z)$ share $a(z)$ CM and $b(z)$ IM and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f-a(z)}\right) = T(r, f) + S(r, f)$,
then $L(f(z)) \equiv f(z)$.

And also in the same paper [1], we asked the following question:

Question 2.3. [1] *Is it possible to remove the condition on ramification index in Theorem D?*

When in *Theorem D*, $a(z) = 0$, $b(z) = 1$ and specifically $L(f(z)) = \Delta_c f(z)$, the condition $\Theta(0; f) + \Theta(\infty; f) > 0$ is no longer required. Recently, by the following theorem, Chen-Xu [3] have been able to prove it.

Theorem F. [3] *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$. If $\Delta_c f(z)$ and $f(z)$ share 0, ∞ CM and 1 IM, then $\Delta_c f(z) \equiv f(z)$.*

In view of partially sharing values, in 2018, Chen [2] investigated the following uniqueness result.

Theorem G. [2] *Let f be a non constant meromorphic function of hyper order $\rho_2(f) < 1$. If $\Delta_c f$ and $f(z)$ share the value 1 CM and satisfy*

$$E(0, f) \subseteq E(0, \Delta_c f) \quad \text{and} \quad E(\infty, \Delta_c f) \subseteq E(\infty, f),$$

then $\Delta_c f \equiv f$.

In this paper we not only resolve the *Questions 2.3* partially as well as we are able the relax the sharing conditions of $a(z)$ and ∞ in view of partially sharing as follows:

Theorem 2.1. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z)$, $b(z)$ be two distinct small functions of f. If $L(f(z))$, $f(z)$ share $b(z)$ IM and satisfy*

$$E(0, f - a(z)) \subseteq E(0, L(f(z)) - a(z)), \quad E(\infty, L(f(z))) \subseteq E(\infty, f) \quad \text{with}$$
$$N_{=1}(r, L(f(z))) = S(r, f),$$

and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f - a(z)}\right) = T(r, f) + S(r, f)$,
then $L(f(z)) \equiv f(z)$.

Consequently we have the following corollary, which is more relaxed with respect to *Theorem E*.

Corollary 2.1. *Under the same situation as in* Theorem 2.1 *if $f(z)$ be a transcendental entire function then $L(f(z)) \equiv f(z)$.*

It is to be noted that *Theorem C* provides the answer of *Question 2.1* with one extra supposition on counting function. Continuous efforts are being put in by researchers to remove the condition, but nobody succeeded. Recently, Huang [11] proved the following result which gives the better answer of the *Question 2.1.*

Theorem H. [11] *Let $f(z)$ be a transcendental entire function of finite order. If $f(z)$ and $(\Delta_c^k f(z))^{(n)}$, $n \geq 0$ share two distinct complex constants a CM and b IM then $f(z) \equiv (\Delta_c^k f(z))^{(n)}$.*

Remark 2.1. *Inspecting closely the proof of Theorem H, we can see that there was a fatal error in the proof of Lemma 2.6 (see p. 6, l. 4 from top,* [11]*).*

For the sake of argument, let us think that the *Lemma 2.6* in [11] is correct and consequently that means *Theorem H* is also true for $ab \neq 0$. Then, from the following example we can exhibit an evidence of lacuna in the proof of *Theorem H.*

Example 2.1. *Let $f(z) = e^{2\lambda z} - 2e^{\lambda z} + 2$, where λ is a complex constant. Choose c, k and n (≥ 1) satisfying $e^{\lambda c} = -1$ and $\lambda^n = \frac{1}{(-2)^{k+1}}$. Now,*

$$\begin{aligned} \Delta_c^k f(z) &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(z+ic) = e^{2\lambda z}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{2\lambda ic}\right) \\ &- 2e^{\lambda z}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}e^{\lambda ic}\right) + 2\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\right) \\ &= e^{2\lambda z}\left(e^{2\lambda c}-1\right)^k - 2e^{\lambda z}\left(e^{\lambda c}-1\right)^k. \end{aligned}$$

Putting $e^{\lambda c} = -1$, we have $\Delta_c^k f(z) = (-2)^{k+1}e^{\lambda z}$. So, $(\Delta_c^k f(z))^{(n)} = (-2)^{k+1}\lambda^n e^{\lambda z}$ $= e^{\lambda z}$. Here $f(z)$ and $(\Delta_c^k f(z))^{(n)}$ share 2 CM and 1 IM but $(\Delta_c^k f(z))^{(n)} \neq f(z)$.

In the above example, $N\left(r, \frac{1}{f(z)-2}\right) = N\left(r, \frac{1}{e^{\lambda z}-2}\right) = T(r, e^{\lambda z}) \neq T(r,f) = 2T(r, e^{\lambda z})$ and this does not conform (2.1). Since *Lemma 2.6* is used to deal "$ab \neq 0$", under subcase 2.3 (see p.12, [11]), so the existence of *Theorem H* for the case "$ab \neq 0$" is under question. Thus for $ab \neq 0$, without the aid of supposition (2.1), the existence of *Theorem H* seems to be impossible.

As a result, till now, for the case $ab \neq 0$, Corollary 2.1 is the best possible answer to Question 2.1. We see that it automatically covers the case "$a = 0$" of *Theorem H.* However, in Theorem H the case "$b = 0$" has been resolved conveniently. Hence *Theorem H* is true only when $ab = 0$.

From the above theorem, we see that the only option left to improve *Theorem H* is to manipulate the case $b = 0$. Now we are going to present the next theorem which will significantly extend *Theorem H* for $b = 0$.

Theorem 2.2. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z)$ be a non zero periodic small function of f with period c. If $(L_c^0 f(z))^{(n)}$ $(n \geq 0)$, $f(z)$ share $a(z)$ CM, 0 IM and $N(r,f) = S(r,f)$, then $(L_c^0 f(z))^{(n)} \equiv f(z)$.*

Very recently, concerning shift and k-th derivative of a meromorphic function, Chen-Xu [4] proved the following result.

Theorem I. [4] *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$. If $f^{(k)}(z)$ and f_c share 0, ∞ CM and 1 IM, then $f^{(k)}(z) \equiv f_c$.*

In view of partially sharing Qi-Yang [16] proved the following result:

Theorem J. [16] *Let $f(z)$ be a non constant meromorphic function of finite order and let $b \neq 0 \in \mathbb{C}$. If f', f_c share b CM and satisfy*

$$E(0, f_c) \subseteq E(0, f') \quad \text{and} \quad E(\infty, f') \subseteq E(\infty, f_c),$$

then $f' \equiv f_c$. Further, $f(z)$ is a transcendental entire function.

Remark 2.2. *In Theorem J, the authors showed that when $f' \equiv f_c$ the meromorphic function is ultimately reduces to an entire function. As it is not possible to get such a meromorphic function satisfying $f' \equiv f_c$ so we wonder that why the result carried forward in meromorphic function. Although we are considering meromorphic functions to continue their research and improve their results, still we believe that, in the next theorem, it would have been better to consider the function as an entire function.*

Related to *Theorem I*, we can have the following theorem which will relax and extend the conditions of the shared values of the same theorem from "CM" to "partially CM small functions sharing". The theorem improves *Theorem J* as well.

Theorem 2.3. *Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$ and let $a(z)$, $b(z)$ be two distinct small functions of f. If $\tilde{L}(f(z))$, f_c share $b(z)$ IM and satisfy*

$$E(0, f_c - a(z)) \subseteq E(0, \tilde{L}(f(z)) - a(z)) \quad \text{and} \quad E(\infty, \tilde{L}(f(z))) \subseteq E(\infty, f_c)$$

and one of the following cases is satisfied:
(i) $a(z) \equiv 0$,
(ii) $a(z) \not\equiv 0$ with $N\left(r, \frac{1}{f_c - a(z)}\right) = T(r, f) + S(r, f)$,
then $\tilde{L}(f(z)) \equiv f_c$.

3. Lemmas

In this section, we present some lemmas, which will be needed to proceed further.

Lemma 3.1. [9] *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$ and $c \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f_c}{f}\right) + m\left(r, \frac{f}{f_c}\right) = S(r, f).$$

Lemma 3.2. [9] *Let* $T : [0, +\infty) \to [0, +\infty)$ *be a non-decreasing continuous function, and let* $s \in (0, +\infty)$*. If the hyper-order of* T *is strictly less than* 1*, i.e.,*

$$\limsup_{r \longrightarrow \infty} \frac{\log \log T(r)}{\log r} = \rho_2 < 1,$$

and $\delta \in (0, 1 - \rho_2)$*, then*

$$T(r + s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r *runs to infinity outside of a set of finite logarithmic measures.*

Using this lemma by a simple alteration of the result for finite order meromorphic functions in [5], one can have the following lemma.

Lemma 3.3. *Let* $f(z)$ *be a meromorphic function of* $\rho_2(f) < 1$*, then we have*

$$N(r, f_c) = N(r, f) + S(r, f) \quad \text{and} \quad T(r, f_c) = T(r, f) + S(r, f).$$

Lemma 3.4. [18] *Let* f *be a non-constant meromorphic function,* $a_j \in \hat{S}(f)$*,* $j = 1, 2, \ldots, q$*,* $(q \geq 3)$*. Then for any positive real number* ϵ*, we have*

$$(q - 2 - \epsilon)T(r, f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right), \; r \notin E,$$

where $E \subset [0, \infty)$ *and satisfies* $\int_E d \log \log r < \infty$*.*

Lemma 3.5. [1] *Let* $f(z)$ *be a non constant meromorphic function of* $\rho_2(f) < 1$ *in* $\mathbb{C}$*,* $p \in \mathbb{C}$*. Then for a small function* $b(z)$ *of* f*,*

$$m\left(r, \frac{L(f(z)) + pL(b(z))}{f(z) + p\, b(z)}\right) = S(r, f).$$

Lemma 3.6. *Let* $f(z)$ *be a non constant meromorphic function of* $\rho_2(f) < 1$ *and* $g = L(f(z))$*. If for* $c \in \mathbb{C}$*,* $E(0, f_c - a(z)) \subseteq E(0, g - a(z))$ *and* $E(\infty, g) \subseteq E(\infty, f_c)$ *or* $N(r, f) = S(r, f)$*, then* $S(r, g) = S(r, f)$ *and* $\rho_2(g) = \rho_2(f_c) < 1$*.*

Proof. When $E(\infty, g) \subseteq E(\infty, f_c)$, then by *Lemma 3.3* $N(r, g) \leq N(r, f_c) = N(r, f) + S(r, f)$. So, in view of *Lemma 3.5* we obtain that

$$\begin{aligned} T(r, g) &= m(r, g) + N(r, g) \\ &\leq m(r, f) + m\left(r, \frac{g}{f}\right) + N(r, f) + S(r, f) \\ &\leq T(r, f) + S(r, f). \end{aligned} \tag{3.1}$$

When $N(r,f)=S(r,f)$, we also can establish (3.1). Now by the First fundamental Theorem, *Lemma 3.1* and *Lemma 3.5* we get,

$$\begin{aligned} T(r,f) &\leq T\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f)\\ \leq\ & m\left(r,\frac{g-L(a(z-c))}{f-a(z-c)}\right)+m\left(r,\frac{f-a(z-c)}{f_c-a(z)}\right)+m\left(r,\frac{1}{g-L(a(z-c))}\right)\\ &+N\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f)\\ \leq\ & m\left(r,\frac{1}{g-L(a(z-c))}\right)+N\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f). \end{aligned}$$

Since $E(0,f_c-a(z))\subseteq E(0,g-a(z))$, thereby

$$T(r,f)\leq 2T(r,g)+S(r,f). \tag{3.2}$$

Combining (3.1) and (3.2), it follows that $S(r,f)=S(r,g)$ and $\rho_2(g)=\rho_2(f_c)<1$.
□

Throughout the paper we use the notation of $P(h)$ and its use, which is given in the following lemma.

Lemma 3.7. *For some meromorphic function h, we define*

$$\begin{aligned} P(h) &= \begin{vmatrix} h(z)-a(z) & a(z)-b(z)\\ h'(z)-a'(z) & a'(z)-b'(z)\end{vmatrix}=\begin{vmatrix} h(z)-b(z) & a(z)-b(z)\\ h'(z)-b'(z) & a'(z)-b'(z)\end{vmatrix}\\ &= \begin{vmatrix} h(z)-a(z) & h'(z)-a'(z)\\ h(z)-b(z) & h'(z)-b'(z)\end{vmatrix}\\ &= [a'(z)-b'(z)]h(z)-[a(z)-b(z)]h'(z)+[a(z)b'(z)-a'(z)b(z)], \end{aligned}$$

$a(z)$, $b(z)\in S(f)\cap S(g)$, where f, g are defined in Lemma 3.6. Then $P(f)$, $P(g)$ $\not\equiv 0$ and

$$m\left(r,\frac{P(h)}{h-a(z)}\right)=S(r,h)+S(r,f)=m\left(r,\frac{P(h)}{h-b(z)}\right).$$

Proof. *On the contrary, if $P(f)\equiv 0$, then by a simple integration we have $f(z)$ is small function which shows $T(r,f)=S(r,f)$, that is not possible. Similarly $P(g)\equiv 0$ gives $T(r,g)=S(r,g)$, which also makes a contradiction. So $P(f)$, $P(g)\not\equiv 0$. Now from the construction of $P(h)$, we can easily deduce that*

$$\begin{aligned} m\left(r,\frac{P(h)}{h-a(z)}\right) &= m\left(r,\frac{(h-a(z))(a'(z)-b'(z))-(a(z)-b(z))(h'(z)-a'(z))}{h-a(z)}\right)\\ &= S(r,h)+S(r,f). \end{aligned}$$

Similarly,

$$m\left(r,\frac{P(h)}{h-b(z)}\right)=S(r,h)+S(r,f).$$

□

Lemma 3.8. *Let f and g be two non constant meromorphic functions as defined in Lemma 3.6. Set*

$$H_f^g = \frac{P(g)}{(g-a(z))(g-b(z))} - \frac{P(f)}{(f-a(z))(f-b(z))},$$

$a(z)$, $b(z) \in S(f) \cap S(g)$. Then the following occurs:
(i) $m(r, H_f^g) = S(r,f)$.
(ii)

$$T(r, H_f^g) \quad \leq \quad \overline{N}_{\otimes}(r, 0, f-a(z); g-a(z)) + \overline{N}_{\otimes}(r, 0, f-b(z); g-b(z)) + S(r,f).$$

(iii) Let $N(r,f) = N(r,g) = S(r,g)$. If $H_f^g \equiv 0$, then either $g \equiv f$ or

$$2T(r,f) \leq \overline{N}\left(r, \frac{1}{f-a(z)}\right) + \overline{N}\left(r, \frac{1}{f-b(z)}\right) + S(r,f).$$

Proof. (i)

$$m(r, H_f^g) = m\left(r, \frac{1}{a(z)-b(z)}\left(\left(\frac{P(g)}{g-a(z)} - \frac{P(g)}{g-b(z)}\right) - \left(\frac{P(f)}{f-a(z)} - \frac{P(f)}{f-b(z)}\right)\right)\right).$$

Now we apply *Lemma 3.7* to obtain

$$m(r, H_f^g) = S(r,f).$$

(ii) Rewriting H_f^g we have

$$H_f^g = \left(\frac{g'-b'(z)}{g-b(z)} - \frac{g'-a'(z)}{g-a(z)}\right) - \left(\frac{f'-b'(z)}{f-b(z)} - \frac{f'-a'(z)}{f-a(z)}\right).$$

Clearly

$$N(r, H_f^g) \quad \leq \quad \overline{N}_{\otimes}(r, 0, f-a(z); g-a(z)) + \overline{N}_{\otimes}(r, 0, f-b(z); g-b(z)) + S(r,f)$$

and so in view of (i), (ii) holds.
(iii) Now $H_f^g \equiv 0$ implies

$$\frac{\begin{vmatrix} f(z)-a(z) & f'(z)-a'(z) \\ f(z)-b(z) & f'(z)-b'(z) \end{vmatrix}}{(f-a(z))(f-b(z))} = \frac{\begin{vmatrix} g(z)-a(z) & g'(z)-a'(z) \\ g(z)-b(z) & g'(z)-b'(z) \end{vmatrix}}{(g-a(z))(g-b(z))}.$$

Integrating we have

$$\frac{f-a(z)}{f-b(z)} = A\frac{g-a(z)}{g-b(z)},$$

where A is a non zero constant. If $A=1$ then $g \equiv f$. So let $A \neq 1$. Proceeding in a similar way as in page 15 of [1] we have

$$f - \frac{Ab(z)-a(z)}{A-1} = \frac{(a(z)-b(z))A}{A-1}.\frac{f-b(z)}{g-b(z)}.$$

Let $d(z) = \frac{Ab(z)-a(z)}{A-1}$. As $A \neq 0$, 1 and $a(z) \neq b(z)$, so $d(z) \neq a(z)$, $b(z)$. From the above equation it is obvious that any zero of $f - d(z)$ must be a zero of at least one of $a(z)-b(z)$ or $d(z)-b(z)$. Therefore $\overline{N}\left(r,\frac{1}{f-d(z)}\right)=S(r,f)$. So, by *Lemma 3.4* we obtain

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a(z)}\right)+\overline{N}\left(r,\frac{1}{f-b(z)}\right)+\overline{N}\left(r,\frac{1}{f-d(z)}\right)\\ &+ S(r,f)\overline{N}\left(r,\frac{1}{f-a(z)}\right)+\overline{N}\left(r,\frac{1}{f-b(z)}\right)+S(r,f).\end{aligned}$$

□

4. Proofs of the theorems

The following proof of the **Theorem 2.1** is based on some ideas from Chen-Xu [4].

Proof of Theorem 2.1. Set $g=L(f(z))$. Since $E(0,f-a(z))\subseteq E(0,g-a(z))$, $E(\infty,g)\subseteq E(\infty,f)$, so

$$\frac{g-a(z)}{f-a(z)}=\gamma(z), \tag{4.1}$$

where $\gamma(z)$ is a meromorphic function such that $N(r,\gamma(z))=S(r,f)$.

First suppose $a(z)\not\equiv 0$ with $N\left(r,\frac{1}{f-a(z)}\right)=T(r,f)+S(r,f)$. By *Lemma 3.5* and then applying the First Fundamental Theorem we have,

$$\begin{aligned}T(r,\gamma(z)) &= m(r,\gamma(z))+S(r,f)\\ &\leq m\left(r,\frac{g-L(a(z))}{f-a(z)}\right)+m\left(r,\frac{L(a(z))-a(z)}{f-a(z)}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{f-a(z)}\right)+S(r,f)=T(r,f)-N\left(r,\frac{1}{f-a(z)}\right)+S(r,f),\end{aligned}$$

which implies

$$T(r,\gamma(z))=S(r,f). \tag{4.2}$$

If $a(z)\equiv 0$, then in view of *Lemma 3.5* from (4.1) we automatically get (4.2).

Let z_0 be a zero of $f-b(z)$ such that it is not a zero of $b(z)-a(z)$. Since g and f share $b(z)$ IM, so z_0 is also a zero of $g-b(z)$. Therefore from (4.1) we have,

$$\gamma(z_0)=\frac{b(z_0)-a(z_0)}{b(z_0)-a(z_0)}=1,$$

which yields all zeros of $f-b(z)$ are zeros of $\gamma(z)-1$ as long as they are not zeros of $b(z)-a(z)$. Suppose $g\not\equiv f$. So $\gamma(z)\not\equiv 1$. Therefore we can write

$$\begin{aligned}\overline{N}\left(r,\frac{1}{g-b(z)}\right) &= \overline{N}\left(r,\frac{1}{f-b(z)}\right)\\ &\leq N\left(r,\frac{1}{\gamma(z)-1}\right)+N\left(r,\frac{1}{b(z)-a(z)}\right)\\ &\leq T(r,\gamma(z))+S(r,f)=S(r,f).\end{aligned} \tag{4.3}$$

Set $d_{\gamma}(z)=a(z)-\frac{a(z)-b(z)}{\gamma(z)}$. Then it is obvious that $d_{\gamma}(z)\not\equiv a(z)$ as well as $\not\equiv b(z)$. Rewriting (4.1) we have,

$$g-b(z)=\gamma(z)[f-d_{\gamma}(z)].$$

Therefore,

$$\overline{N}\left(r,\frac{1}{f-d_{\gamma}(z)}\right)=\overline{N}\left(r,\frac{1}{g-b(z)}\right)+S(r,f)=S(r,f). \tag{4.4}$$

Let us consider the same auxiliary function H_{f}^{g} as defined in *Lemma 3.8*. Since $E(0,f-a(z))\subseteq E(0,g-a(z))$ and f, g share $b(z)$ IM, so by *Lemma 3.8*

$$\begin{aligned} T(r,H_{f}^{g}) &\leq \overline{N}\left(r,\frac{1}{f-b(z)}\right)+\overline{N}_{\otimes}(r,0,f-a(z);g-a(z))+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f-b(z)}\right)+N\left(r,\frac{1}{\gamma(z)}\right)+S(r,f)\end{aligned}$$

Using (4.2) and (4.3) we obtain that

$$T(r,H_{f}^{g})=S(r,f). \tag{4.5}$$

Now we consider two cases:

Case 1: $H_{f}^{g}\equiv 0$. Since $g\not\equiv f$, so proceeding in a similar way as in case (iii) of *Lemma 3.8* we have $\overline{N}\left(r,\frac{1}{f-d(z)}\right)=S(r,f)$. Also, one can easily check that $d(z)\not\equiv d_{\gamma}(z)$. So, by *Lemma 3.4*, from (4.3) and (4.4) we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,\frac{1}{f-b(z)}\right)+\overline{N}\left(r,\frac{1}{f-d_{\gamma}(z)}\right)+\overline{N}\left(r,\frac{1}{f-d(z)}\right)+S(r,f)\\ &= S(r,f),\end{aligned}$$

which is a contradiction.

Case 2: $H_{f}^{g}\not\equiv 0$. Here $d_{\frac{1}{\gamma}}(z)=a(z)-(a(z)-b(z))\gamma(z)$. Then it is obvious that $d_{\frac{1}{\gamma}}(z)\not\equiv a(z)$ as well as $\not\equiv b(z)$. Rewriting (4.1) we have,

$$f-b(z)=\frac{1}{\gamma(z)}[g-d_{\frac{1}{\gamma}}(z)].$$

Therefore,

$$\overline{N}\left(r,\frac{1}{g-d_{\frac{1}{\gamma}}(z)}\right)=\overline{N}\left(r,\frac{1}{f-b(z)}\right)+S(r,f)=S(r,f). \tag{4.6}$$

Since $E(\infty,g)\subseteq E(\infty,f)$ with $N_{=1}(r,g)=S(r,f)$, so in view of (4.5) we get

$$\overline{N}(r,g)\leq N\left(r,\frac{1}{H_{f}^{g}}\right)+S(r,f)=S(r,f).$$

Now, by *Lemma 3.4*, (4.3) and (4.6) we obtain

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g-b(z)}\right)+\overline{N}\left(r,\frac{1}{g-d_{\frac{1}{\gamma}}(z)}\right)+S(r,g)\\ &= S(r,f),\end{aligned}$$

which is a contradiction. Hence $g \equiv f$ holds. □

Proof of Theorem 2.2. Suppose $g_1 = (L_c^0 f(z))^{(n)}$ and $g_1 \not\equiv f$. Since f and g_1 share $a(z)$ CM, so there exist a meromorphic function $h(z)$ such that

$$\frac{g_1 - a(z)}{f - a(z)} = h(z). \tag{4.7}$$

Here $h \not\equiv 0, 1$. Clearly $N(r, f) = S(r, f)$ with *Lemma 3.3*, implies $N(r, g_1) = S(r, f)$, which yields $N(r, h) = S(r, f)$. As f and g_1 share $a(z)$ CM and 0 IM, so by *Lemma 3.4* and then by applying *Lemma 3.5* we have

$$\begin{aligned}
T(r, f) &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - a(z)}\right) + S(r, f)\\
&\leq N\left(r, \frac{1}{g_1 - f}\right) + S(r, f) \leq T\left(r, \frac{1}{g_1 - f}\right) + S(r, f)\\
&\leq T(r, g_1 - f) + S(r, f) \leq m(r, g_1 - f) + S(r, f)\\
&\leq m(r, f) + m\left(r, \frac{g_1 - f}{f}\right) + S(r, f)\\
&\leq m(r, f) + m\left(r, \frac{g_1}{f}\right) + S(r, f) \leq T(r, f) + S(r, f).
\end{aligned}$$

Therefore

$$\begin{aligned}
T(r, f) &= \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - a(z)}\right) + S(r, f)\\
&= T(r, g_1 - f) + S(r, f) = N\left(r, \frac{1}{g_1 - f}\right) + S(r, f)
\end{aligned} \tag{4.8}$$

and so

$$m\left(r, \frac{1}{g_1 - f}\right) = S(r, f).$$

From (4.7), in view of *Lemma 3.5* using (4.8) we can obtain

$$\begin{aligned}
T(r, h) &= m(r, h) + N(r, h) = m(r, h) + S(r, f)\\
&\leq m\left(r, \frac{g_1 - (L_c^0 a(z))^{(n)}}{f - a(z)}\right) + m\left(r, \frac{1}{f - a(z)}\right) + S(r, f)\\
&\leq m\left(r, \frac{1}{f - a(z)}\right) + S(r, f) = m\left(r, \frac{h - 1}{g_1 - f}\right) + S(r, f)\\
&\leq m(r, h) + m\left(r, \frac{1}{g_1 - f}\right) + S(r, f) \leq T(r, h) + S(r, f).
\end{aligned}$$

So,

$$T(r, h) = m\left(r, \frac{1}{f - a(z)}\right) + S(r, f). \tag{4.9}$$

Since f and g_1 share 0 IM, from (4.7) we can say that all zeros of f are 1-point of $h(z)$ or zeros of $a(z)$. Hence, in view of (4.9) we can write

$$\overline{N}\left(r,\frac{1}{f}\right) \leq N\left(r,\frac{1}{h-1}\right)+S(r,f) \leq T(r,h)+S(r,f) = m\left(r,\frac{1}{f-a(z)}\right)+S(r,f). \tag{4.10}$$

By the First Main Theorem and then using (4.8), (4.10) we get

$$\begin{aligned} m\left(r,\frac{1}{f-a(z)}\right)+N\left(r,\frac{1}{f-a(z)}\right) &= T(r,f)+S(r,f)\\ &= \overline{N}\left(r,\frac{1}{f-a(z)}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{f-a(z)}\right)+m\left(r,\frac{1}{f-a(z)}\right)+S(r,f), \end{aligned}$$

that yields

$$N\left(r,\frac{1}{f-a(z)}\right)=\overline{N}\left(r,\frac{1}{f-a(z)}\right)+S(r,f)$$

and consequently in view of (4.10) we have

$$m\left(r,\frac{1}{f-a(z)}\right)=\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)=T(r,h)+S(r,f). \tag{4.11}$$

Now we consider two auxiliary functions as follows:

$$\alpha(z)=\frac{P(f)(g_1-f)}{f(f-a(z))}, \qquad \beta(z)=\frac{P(g_1)(g_1-f)}{g_1(g_1-a(z))},$$

where $P(f)$ is as defined in *Lemma 3.7* together with $b(z)\equiv 0$. Here $\alpha(z)$ as well as $\beta(z)\not\equiv 0$. As f and g_1 share $a(z)$ CM, 0 IM and $N(r,f)=N(r,g_1)=S(r,f)$, so $N(r,\alpha(z))=S(r,f)$ and $N(r,\beta(z))=S(r,f)$. In a similar way as in page 11 of [1] we can easily have $m(r,\alpha(z))=S(r,f)$. Thus,

$$T(r,\alpha(z))=S(r,f). \tag{4.12}$$

Following the same logic of construction of the auxiliary function H_f^g in *Lemma 3.8*, here we define $H_f^{g_1}$ with $b(z)\equiv 0$. Since f and g_1 share $a(z)$ CM, 0 IM, from *Lemma 3.8* we have

$$T(r,H_f^{g_1})\leq \overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \tag{4.13}$$

As $a(z)$ is periodic small function with period c, so $(L_c^0a(z))^{(n)}=0$ which in view of *Lemma 3.5* gives $m\left(r,\frac{g_1}{f-a(z)}\right)=m\left(r,\frac{g_1-(L_c^0a(z))^{(n)}}{f-a(z)}\right)=S(r,f)$. Rewriting (4.7) and then using (4.9) we can obtain

$$
\begin{aligned}
(4.14) \qquad m\left(r, \frac{g_1 - f}{g_1 - a(z)}\right) &= m\left(r, \frac{h-1}{h}\right)\\
&\leq T(r,h) + O(1) = m\left(r, \frac{1}{f - a(z)}\right) + S(r,f)\\
&\leq m\left(r, \frac{g_1}{f - a(z)}\right) + m\left(r, \frac{1}{g_1}\right) + S(r,f)\\
&\leq m\left(r, \frac{1}{g_1}\right) + S(r,f).
\end{aligned}
$$

Now we distinguish in two cases on the consideration of $H_f^{g_1}$.

Case 1. Suppose $H_f^{g_1} \equiv 0$. Then by *Lemma 3.8* in view of (4.8) we get a contradiction.

Case 2. Next suppose $H_f^{g_1} \not\equiv 0$. Since $N(r,f) = S(r,f)$ and $N(r,g_1) = S(r,f)$, so from (4.8), we can write

$$
\begin{aligned}
T(r,f) = m(r,f) + S(r,f) &= m(r, g_1 - f) + S(r,f) = m\left(r, \frac{H_f^{g_1}(g_1 - f)}{H_f^{g_1}}\right) + S(r,f)\\
&= m\left(r, \frac{\alpha(z) - \beta(z)}{H_f^{g_1}}\right) + S(r,f)\\
&\leq m\left(r, \alpha(z) - \beta(z)\right) + m\left(r, \frac{1}{H_f^{g_1}}\right) + S(r,f).
\end{aligned}
$$

Now, using (4.12), (4.13), (4.14) and *Lemmas 3.7 and 3.5* we can have

$$
\begin{aligned}
T(r,f) &\leq m(r,\beta) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f)\\
&\leq m\left(r, \frac{P(g_1)}{g_1}\right) + m\left(r, \frac{g_1 - f}{g_1 - a(z)}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f)\\
&\leq m\left(r, \frac{1}{g_1}\right) + \overline{N}\left(r, \frac{1}{g_1}\right) + S(r,f)\\
&\leq T(r,g_1) + S(r,f) \leq m(r,g_1) + S(r,f)\\
&\leq m(r,f) + m\left(r, \frac{g_1}{f}\right) + S(r,f) = m(r,f) + S(r,f) \leq T(r,f) + S(r,f).
\end{aligned}
$$

Noting that $N(r,\beta(z)) = S(r,f)$, also from the above we see that i.e.,

$$
(4.15) \qquad T(r,f) = T(r,\beta) + \overline{N}\left(r, \frac{1}{f}\right) + S(r,f)
$$

and

$$
(4.16) \qquad T(r,g_1) = T(r,f) + S(r,f).
$$

Now our claim is $T(r,\beta(z)) = S(r,f)$. Putting $b(z) \equiv 0$, using (4.16), in a similar manner as used in Page 12-14 of [1] we can easily establish our claim. Therefore, (4.15) yields

$$T(r,f)=\overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \tag{4.17}$$

Using (4.17), from (4.8), we get $\overline{N}\left(r,\frac{1}{f-a(z)}\right)=S(r,f)$. As f and g_1 share $a(z)$ CM, so $\overline{N}\left(r,\frac{1}{g_1-a(z)}\right)=S(r,f)$. Again according to the sharing hypothesis of f and g_1 using (4.16) from (4.8) we have

$$\begin{aligned}T(r,g_1) &= \overline{N}\left(r,\frac{1}{g_1}\right)+\overline{N}\left(r,\frac{1}{g_1-a(z)}\right)+S(r,f)\\ &= \overline{N}\left(r,\frac{1}{g_1}\right)+S(r,f)\leq N\left(r,\frac{1}{g_1}\right)+S(r,f)\leq T(r,g_1)+S(r,f),\end{aligned}$$

that implies

$$T(r,g_1)=N\left(r,\frac{1}{g_1}\right)+S(r,f)\ \text{ and so }\ m\left(r,\frac{1}{g_1}\right)=S(r,f).$$

Now from (4.17), using (4.11) and (4.14) we have

$$T(r,f)=\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)=m\left(r,\frac{1}{f-a(z)}\right)+S(r,f)\leq m\left(r,\frac{1}{g_1}\right)+S(r,f).$$

Hence from the above two lines we get $T(r,f)=S(r,f)$, a contradiction. Hence $g_1\equiv f$. □

Proof of Theorem 2.3. Set $g_2=\tilde{L}(f(z))$. Since $E(0,f_c-a(z))\subseteq E(0,g_2-a(z))$, $E(\infty,g_2)\subseteq E(\infty,f_c)$, so

$$\frac{g_2-a(z)}{f_c-a(z)}=\gamma_1(z), \tag{4.18}$$

where $\gamma_1(z)$ is a meromorphic function such that $N(r,\gamma_1(z))=S(r,f)$.
First suppose $a(z)\not\equiv 0$ with $N\left(r,\frac{1}{f_c-a(z)}\right)=T(r,f)+S(r,f)$. Now, by *Lemma 3.5* and then applying the First Fundamental Theorem we have,

$$\begin{aligned}T(r,\gamma_1(z)) &= m(r,\gamma_1(z))+S(r,f)\\ &\leq m\left(r,\frac{g_2-\tilde{L}(a(z-c))}{f_c-a(z)}\right)+m\left(r,\frac{\tilde{L}(a(z-c))-a(z)}{f_c-a(z)}\right)+S(r,f)\\ &\leq m\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f)=T(r,f)-N\left(r,\frac{1}{f_c-a(z)}\right)+S(r,f),\end{aligned}$$

that implies

$$T(r,\gamma_1(z))=S(r,f). \tag{4.19}$$

If $a(z)\equiv 0$, then in view of *Lemma 3.5* from (4.18) we automatically get (4.19). By similar argument as used in *Theorem 2.1*, we can have

$$\overline{N}\left(r,\frac{1}{f_c-b(z)}\right)=\overline{N}\left(r,\frac{1}{g_2-b(z)}\right)=S(r,f). \tag{4.20}$$

Clearly $d_{\gamma_1}(z) \not\equiv a(z)$ as well as $\not\equiv b(z)$. Rewriting (4.18) we have,

$$g_2 - b(z) = \gamma_1(z)[f_c - d_{\gamma_1}(z)].$$

Therefore,

$$\overline{N}\left(r, \frac{1}{f_c - d_{\gamma_1}(z)}\right) = \overline{N}\left(r, \frac{1}{g_2 - b(z)}\right) + S(r,f) = S(r,f). \tag{4.21}$$

Here as usual we can define $H_{f_c}^{g_2}$ like *Lemma 3.8.* Since f_c, g_2 share $b(z)$ IM and $E(0, f_c - a(z)) \subseteq E(0, g_2 - a(z))$, by the similar argument as used in *Theorem 2.1* we can get

$$T(r, H_{f_c}^{g_2}) = S(r,f). \tag{4.22}$$

Now we consider two cases:

Case 1: $H_{f_c}^{g_2} \equiv 0$. Then proceeding in a similar manner as in *Case 1* of *Theorem 2.1* we can reach up to a contradiction.

Case 2: $H_{f_c}^{g_2} \not\equiv 0$. Since $E(\infty, g_2) \subseteq E(\infty, f_c)$ and g_2 has no simple poles, so in view of (4.22) we get

$$\overline{N}(r, g_2) \le N\left(r, \frac{1}{H_{f_c}^{g_2}}\right) = S(r,f).$$

After that, following *Case 2* of *Theorem 2.1* we can again reach up to a contradiction.

□

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