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UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN ITS SHIFT AND FIRST DERIVATIVE SHARE THREE VALUES

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Abstract. This paper brings out some improvements as well as generalization results of a paper of X. Qi and L. Yang [17] [Comput. Methods Funct. Theory, 20, 159-178 (2020)], which deals with the uniqueness results of $f'(z)$ and $f(z+c)$. To be more realistic about the obtained results, we exhibit some examples.

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1. Introduction, definitions and results

We assume that the reader is familiar with meromorphic function, standard notations and main results of Nevanlinna's value distribution theory [9, 20]. As usual, the abbreviation "CM" means "counting multiplicities while "IM" stands for "ignoring multiplicities".

Let f and g be two meromorphic functions in the complex plane $\mathbb{C}$. In particular, let z_n, $n = 1, 2, \dots$, be the zeros of $f - a$ with multiplicity $h(n)$. If z_n are also $h(n)$ multiple zeros of $g - a$ at least, then we write $f = a \to g = a$, where $a \in \mathbb{C} \cup \{\infty\}$.

The order of f is denoted by $\sigma(f)$ and is defined by

$$\sigma(f) = \limsup_{r \longrightarrow \infty} \frac{\log^+ T(r, f)}{\log r}.$$

Rubel and Yang [18] first investigated the uniqueness of an entire function concerning its derivative, and proved the following result.

Theorem A. *Let $f(z)$ be a non-constant entire function. If $f(z)$ and $f'(z)$ share two distinct finite values a, b CM, then $f(z) \equiv f'(z)$.*

Mues and Steinmetz [[14], Satz 1] showed the sharing assumption in Theorem A can be replaced by 2 IM. Afterwards, Mues and Steinmetz [15], Gundersen [[6], Thm. 1] improved Theorem A to a non-constant meromorphic function.

Theorem B. *Let $f(z)$ be a non-constant meromorphic function. If $f(z)$ and $f'(z)$ share two distinct finite values a, b CM, then $f(z) \equiv f'(z)$.*

Gundersen [6] has given a counterexample to show the sharing assumption in Theorem B cannot be improved to 1 CM + 1 IM. Further, 2 CM can be replaced by 3 IM, see [5, 14]. Moreover, the results stated above are still true if $f'(z)$ is changed to $f^{(k)}(z)$, where k is a positive integer. For some of related results, the reader is invited to see [[20],Ch. 8].
As a difference analogue, Heittokangas et al. [10, 11] started to consider meromorphic functions sharing values with their shifts. The background for these considerations lies in the parallel difference version to the usual Nevanlinna theory which starts with the papers [4, 7, 8].

Remark A. *We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. We define $S(f)$ is the family of all meromorphic functions $a(z)$ such that $T(r, a) = S(r, f)$ as $r \to \infty$. Here we call $a(z)$ as a small function with respect to f.*

A key results in [10] reads as follows.

Theorem C. *Let $f(z)$ be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period c. If $f(z)$ and $f(z + c)$ share a_1, a_2 CM and a_3 IM, then $f(z) \equiv f(z + c)$.*

Later on, many authors consider the uniqueness of meromorphic functions of finite order concerning their shifts or differences. Some attempts towards relaxing the sharing assumptions can be found in [1, 2, 3, 13, 21].

In real analysis, the time-delay differential equation $f'(x) = f(x - k),\ k > 0$, has been extensively studied. As for a complex variable counterpart, Liu and Dong studied the complex differential-difference equation $f'(z) = f(z + c)$, where c is a non-zero constant, see [12].

In [16], Qi and Yang looked at this complex differential-difference equation from different perspective. That is, "under what sharing values conditions, does $f'(z) = f(z + c)$ hold?" And they investigated the value sharing problem related to $f'(z)$ and $f(z + c)$ as follows.

Theorem D. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z + c)$ share 0, a CM, then $f'(z) \equiv f(z + c)$.*

Then, Qi and Yang [17] posed a list of questions related to Theorem D in the following.

Question A. *Can the value sharing condition be improved in Theorem D?*

Question B. *Can the condition "$f(z)$ is transcendental" be deleted in Theorem D?*

Corresponding to the questions above, they obtained the following results.

Theorem E. *Let $f(z)$ be a non-constant meromorphic function of finite order, let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a CM, and satisfy $f(z+c) = 0 \to f'(z) = 0$, $f(z+c) = \infty \leftarrow f'(z) = \infty$. Then $f'(z) \equiv f(z+c)$. Further, $f(z)$ is a transcendental entire function.*

Theorem F. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share 0 CM and a IM, then $f'(z) \equiv f(z+c)$.*

Theorem G. *Let $f(z)$ be a transcendental entire function of finite order, and let a, b be two distinct finite values. If $f'(z)$ and $f(z+c)$ share a, b IM, then*

$$T(r, f(z+c)) = O\left(T(r, f')\right), \quad T(r, f') = O\left(T(r, f(z+c))\right),$$

as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure.

In the following, we now pose a list of questions relevant to Theorems E-G such that the conclusions of the theorems are intact.

Question 1.1. *Can the value sharing condition "$f'(z)$ and $f(z+c)$ share a CM" be further improved by "$f'(z)$ and $f(z+c)$ share a IM" in Theorem E, where $a(\neq 0) \in \mathbb{C}$?*

Question 1.2. *Can one further weaker the condition "$f'(z)$ and $f(z+c)$ share 0 CM" by "$f(z+c) = 0 \to f'(z) = 0$" in Theorem F?*

Question 1.3. *What happen if the condition "$f'(z)$ and $f(z+c)$ share a, b IM for a transcendental entire function $f(z)$" be replaced by "$f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$ for transcendental meromorphic function $f(z)$" in Theorem G?*

Corresponding to the questions above, we get the following main results of this paper.

Theorem 1.1. *Let $f(z)$ be a non-constant meromorphic function of finite order, let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM and satisfy $f(z+c) = 0 \to f'(z) = 0$, $f(z+c) = \infty \leftarrow f'(z) = \infty$, then $f'(z) \equiv f(z+c)$ and $f(z)$ is a transcendental entire function of finite order.*

Remark 1.1. *Clearly we see that Theorem E holds for the condition "$f'(z)$ and $f(z+c)$ share a CM" whenever Theorem 1.1 holds for the condition "$f'(z)$ and $f(z+$*

c) share a IM". We know that, in a particular case 'IM' sharing condition becomes 'CM' sharing condition. So Theorem 1.1 is an improvement result of Theorem E.

From Theorem 1.1, it is sufficient to consider the condition that $f(z)$ is an entire function. And then we obtain the following result.

Corollary 1.1. *Let $f(z)$ be a transcendental entire function of finite order, and let $a(\neq 0) \in \mathbb{C}$. If $f'(z)$ and $f(z+c)$ share a IM and satisfy $f(z+c) = 0 \to f'(z) = 0$. Then $f'(z) \equiv f(z+c)$.*

Remark 1.2. *Clearly if we consider "Let $f(z)$ be a transcendental entire function" in Theorem 1.1, then the condition "$f(z+c) = \infty \leftarrow f'(z) = \infty$" doesn't arise. Then this theorem becomes the same as Corollary 1.1. So the proof of Corollary 1.1 follows from the proof of Theorem 1.1.*

Again we observe that Theorem F holds for the condition "$f'(z)$ and $f(z+c)$ share 0 CM" whenever Corollary 1.1 holds for the condition "$f(z+c) = 0 \to f'(z) = 0$". By definition, in a particular situation the condition "$f(z+c) = 0 \to f'(z) = 0$" becomes the condition "$f'(z)$ and $f(z+c)$ share 0 CM". In this sense, Corollary 1.1 is an improvement result of Theorem F.

By Theorem 1.1, we can consider the condition that $f(z)$ is a transcendental meromorphic function and obtain the result as follows.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite order. If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$, then*

$$T(r, f') = T(r, f(z+c)) + S(r, f).$$

Remark 1.3. *We observe that if we consider "Let $f(z)$ be a transcendental entire function" in Theorem 1.2, then the condition "$f(z+c) = \infty \leftarrow f'(z) = \infty$" doesn't arise. Then Theorem 1.2 holds for one finite shared-value 0 whenever Theorem G holds for two finite shared-value a and b. In this sense, Theorem 1.2 is an improvement result of Theorem G. Also we observe that Theorem G holds for an entire function whenever Theorem 1.2 holds for a transcendental meromorphic function. In this sense, Theorem 1.2 is a generalization result of Theorem G.*

Remark 1.4. *To be more realistic about Theorem 1.1 and validity of the conditions in theorem, the following examples are relevant.*

Example 1.1. *Let $f(z) = e^{-z}$, $a \in \mathbb{C} \setminus \{0\}$ and $c = \pi i$. Then $f(z+c) = -e^{-z}$ and $f'(z) = -e^{-z}$. So $f'(z)$ and $f(z+c)$ share $0, \infty$ CM and a CM. Thus $f(z)$ satisfy the conditions of Theorem 1.1. Consequently, $f'(z) \equiv f(z+c)$ follows.*

Example 1.2. *Let* $f(z) = \sin z + \cos z$*,* $b \in \mathbb{C} \setminus \{0\}$ *and* $c = \frac{\pi}{2}$*. Then* $f(z+c) = \cos z - \sin z$ *and* $f'(z) = \cos z - \sin z$*. So* $f'(z)$ *and* $f(z+c)$ *share* $0, \infty$ *CM and a CM. Thus* $f(z)$ *satisfy the conditions of Theorem 1.1. Consequently,* $f'(z) \equiv f(z+c)$ *follows.*

Example 1.3. [12] *We consider the following functions:*

(i) $f(z) = (b_1 z + b_0)e^{ez+B}$*, where* $b_1(\neq 0), b_0, B \in \mathbb{C}$ *and* $c = \frac{1}{e}$*;*

(ii) $f(z) = b_0 e^{Az+B}$*, where* $c = \frac{\ln|A| + i(argA + 2k\pi)}{A}$*,* $k \in \mathbb{Z}$ *and* $A(\neq 0) \in \mathbb{C}$*;*

(iii) $f(z) = g(z)e^{Az+B}$*, where* $g(z)$ *is a transcendental entire function and satisfies* $g'(z) = A\left(g(z+c) - g(z)\right)$ *and* $\sigma(g) < 1$*, where* $A(\neq 0) \in \mathbb{C}$ *and* $c = \frac{\ln|A| + i(argA + 2k\pi)}{A}$.

We observe that the above functions satisfy the conditions of Theorem 1.1. Consequently, $f'(z) \equiv f(z+c)$ *follows.*

Remark 1.5. *The condition "*$f(z+c) = \infty \leftarrow f'(z) = \infty$*" in Theorem 1.1 is sharp and it follows by the following example.*

Example 1.4. [12] *Let* $f(z) = \frac{2}{1-e^{-2z}}$ *and* $c = \pi i$*. Then even* $f'(z)$ *and* $f(z+c)$ *share 1 IM and satisfies* $f(z+c) = 0 \rightarrow f'(z) = 0$*,* $f'(z) \not\equiv f(z+c)$ *follows, since* $f(z+c) = \infty \not\leftarrow f'(z) = \infty$*.*

Remark 1.6. *The condition "*$f(z+c) = 0 \rightarrow f'(z) = 0$*" in Theorem 1.1 is sharp and it follows by the following example.*

Example 1.5. *Let* $f(z) = k_1 e^{k_2 z} - 1$ *and* $e^{k_2 c} = 2k_2$*, where* $k_1, k_2 \in \mathbb{C} \setminus \{0\}$*. Then even* $f'(z)$ *and* $f(z+c)$ *share 1 IM and satisfies* $f(z+c) = \infty \leftarrow f'(z) = \infty$*,* $f'(z) \not\equiv f(z+c)$ *follows, since* $f(z+c) = 0 \not\rightarrow f'(z) = 0$*.*

Remark 1.7. *The condition "*$f'(z)$ *and* $f(z+c)$ *share a IM,* $a \in \mathbb{C} \setminus \{0\}$*" in Theorem 1.1 is sharp and it follows by the following example.*

Example 1.6. *Let* $f(z) = \frac{1}{2}e^{4z - \frac{1}{2}}$ *and* $e^{4c} = 5$*. Then even* $f'(z)$ *and* $f(z+c)$ *satisfy* $f(z+c) = 0 \rightarrow f'(z) = 0$*,* $f(z+c) = \infty \leftarrow f'(z) = \infty$*,* $f'(z) \not\equiv f(z+c)$ *follows, since* $f'(z)$ *and* $f(z+c)$ *does not share a IM.*

Remark 1.8. *Clearly, the function* $f(z)$ *is of finite order in Theorem 1.1 is sharp by the following example.*

Example 1.7. *Let* $f(z)$ *be a transcendental meromorphic function of infinite order such that* $f(z+c) = \frac{e^z - 1}{z + e^z - e^{e^z}}$ *and* $f'(z) = \frac{\left(e^{e^z} - z\right)(e^z - 1)}{z + e^z - e^{e^z}}$*. Clearly,* $f'(z) - 1 = e^z\left(f(z+c) - 1\right)$ *and* $f'(z) = \left(e^{e^z} - z\right) f(z+c)$*. So* $f'(z)$ *and* $f(z+c)$ *share 1 CM* *and satisfy* $f(z+c)=0 \to f'(z)=0$, $f(z+c)=\infty \leftarrow f'(z)=\infty$. *Clearly, the conclusion of Theorem 1.1 doesn't hold in this situation, i.e.,* $f'(z) \not\equiv f(z+c)$.

2. Some Lemmas

The following are relevant lemmas of this paper and are used in the sequel.

Lemma 2.1. [19] *Let* $f(z)$ *be a non-constant meromorphic function, and let* $a_i(z)$ *be meromorphic functions such that* $T(r,a_i)=S(r,f)$ *for* $i=0,1,2,\ldots,n$, *where* $a_n(z)\not\equiv 0$. *Then we have*

$$T\left(r, a_nf^n+a_{n-1}f^{n-1}+\ldots+a_1f+a_0\right)=nT(r,f)+S(r,f).$$

Lemma 2.2. [4, 7] *Let* $f(z)$ *be a non-constant meromorphic function of finite order* σ, *and let* $c\in\mathbb{C}\setminus\{0\}$ *be fixed. Then for each* $\varepsilon>0$, *we have*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.3. [10] *Let* $f(z)$ *be a non-constant meromorphic function of finite order, and let* $c\in\mathbb{C}\setminus\{0\}$ *be fixed. Then we have*

$$N(r,0;f(z+c))\leq N(r,0;f(z))+S(r,f), \tag{2.1}$$

$$N(r,\infty;f(z+c))\leq N(r,\infty;f)+S(r,f),$$

$$\overline{N}(r,0;f(z+c))\leq \overline{N}(r,0;f(z))+S(r,f) \tag{2.2}$$

$$\text{and}\quad \overline{N}(r,\infty;f(z+c))\leq \overline{N}(r,\infty;f)+S(r,f). \tag{2.3}$$

Lemma 2.4. *[*[4], *Lem. 5.1] Let* $f(z)$ *be a non-constant meromorphic function of finite order* σ, *and let* $c\in\mathbb{C}\setminus\{0\}$ *be fixed. Then for each* $\varepsilon>0$, *we have*

$$T\left(r,f(z+c)\right)=T(r,f)+O\left(r^{\sigma-1+\varepsilon}\right)+O(\log r).$$

Furthermore, if $f(z)$ *is a transcendental meromorphic function with finite order, then we have*

$$T(r,f(z+c))=T(r,f)+S(r,f).$$

Remark 2.1. *By Lemma 2.4, we conclude for a non-constant meromorphic function* $f(z)$ *of finite order that,* $S(r,f)=S(r,f(z+c))$.

Remark 2.2. *By Lemmas 2.2 and 2.4, we can see that, if* $f(z)$ *is a transcendental meromorphic function with finite order, then*

$$N(r,\infty;f)=N(r,\infty;f(z+c))+S(r,f) \tag{2.4}$$

$$\text{and}\quad N(r,0;f)=N(r,0;f(z+c))+S(r,f). \tag{2.5}$$

Lemma 2.5. *[[9], p. 55] Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Then we have*

$$T\left(r, f^{(k)}\right) \leq (1+o(1))(k+1)T(r,f),$$

as $r \to \infty$ possibly outside some exceptional set of finite linear measure.

Remark 2.3. *By Remark 2.1 and Lemma 2.5, we conclude for a non-constant meromorphic function $f(z)$ of finite order that $S(r,f') \leq S(r,f) = S(r, f(z+c))$.*

We now introduce some relevant notations for this paper. Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}_0\left(r, a; f'(z) \mid f(z+c) = a\right)$ the reduced counting function of common a-points of $f'(z)$ and $f(z+c)$ of different multiplicities, whereas

$$N_E\left(r, a; f'(z) \mid f(z+c) = a\right)$$

denotes the counting function of common a-points of $f'(z)$ and $f(z+c)$ of equal multiplicities.

Again we denote by $N\left(r, a; f'(z) \mid f(z+c) \neq a\right)$ the counting function of a-points of $f'(z)$ which are not the a-points of $f(z+c)$.

In the following, we define

$$H(z) = \frac{f'(z)}{f(z+c)}. \tag{2.6}$$

Lemma 2.6. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let $H(z)$ be defined in (2.6). If $f'(z)$ and $f(z+c)$ satisfy $f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$, then the following conclusions occur:*

(i) *$H(z)$ is an entire function and $T\left(r, H\right) = S(r,f)$,*
(ii) *$N_0\left(r, 0; f'(z) \mid f(z+c) = 0\right) + N\left(r, 0; f'(z) \mid f(z+c) \neq 0\right) = S(r,f)$,*
(iii) *$N_0\left(r, \infty; f(z+c) \mid f'(z) = \infty\right) + N\left(r, \infty; f(z+c) \mid f'(z) \neq \infty\right) = S(r,f)$,*
(iv) *$T\left(r, f'\right) = T(r, f(z+c)) + S(r,f)$,*
(v) *$S(r,f) = S(r, f(z+c)) = S(r, f')$ and*
(vi) *$\overline{N}(r, \infty; f(z+c)) = \overline{N}(r, \infty; f') = S(r,f)$.*

Proof. Clearly from $f(z+c) = 0 \to f'(z) = 0$ and $f(z+c) = \infty \leftarrow f'(z) = \infty$ it follows that H has zeros only and so H is an entire function. But the zeros of $f'(z)$ and $f(z+c)$ of equal multiplicities are not zeros of $H(z)$. Since $f(z)$ is a non-constant meromorphic function of finite order, in view of Logarithmic Derivative Lemma and by Lemma 2.2, we get

$$m\left(r, H\right) \leq m\left(r, \frac{f'(z)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f).$$

Then we see that

$$T\,(r,H)=N\,(r,\infty;H)+m\,(r,H)=S(r,f).$$

This completes the proof (i). Note that

$$\begin{aligned}N_0\,(r,0;f'(z)\mid f(z+c)=0)+N\,(r,0;f'(z)\mid f(z+c)\neq 0)\leq\\ \leq N(r,0;H)\leq T(r,H)=S(r,f).\end{aligned}$$

Similarly, we get

$$N_0\,(r,\infty;f(z+c)\mid f'(z)=\infty)+N\,(r,\infty;f(z+c)\mid f'(z)\neq\infty)=S(r,f).$$

This completes the proofs (ii) and (iii). Now using (2.6), the conclusion (i) and the first fundamental theorem of Nevanlinna, we get

$$\begin{aligned}&& T\,(r,f')\leq T(r,H)+T(r,f(z+c))=T(r,f(z+c))+S(r,f)\\ \text{and} && T\,(r,f(z+c))\leq T\left(r,\frac{1}{H}\right)+T(r,f')\\ &\leq& T(r,H)+T(r,f')+S(r,f)=T(r,f')+S(r,f).\end{aligned}$$

Hence the conclusion (iv) follows. So we use this result whenever needed in the following. By Remark 2.3 and the conclusion (iv), we get $S(r,f)=S(r,f(z+c))=S(r,f')$. Hence the conclusion (v) follows. From the assumption, we have

$$N(r,\infty;f')\leq N(r,\infty;f(z+c))+S(r,f). \tag{2.7}$$

Note that

$$N(r,\infty;f')=N(r,\infty;f)+\overline{N}(r,\infty;f) \tag{2.8}$$

$$\text{i.e.,}\qquad N(r,\infty;f')\geq N(r,\infty;f) \tag{2.9}$$

By (2.4) and (2.9), we get

$$N(r,\infty;f')\geq N(r,\infty;f(z+c))+S(r,f). \tag{2.10}$$

Then from (2.4), (2.8) and (2.7), (2.10), we get respectively

$$\begin{aligned}&N(r,\infty;f')=N(r,\infty;f(z+c))+\overline{N}(r,\infty;f)+S(r,f)\\ \text{and}\qquad &N(r,\infty;f')=N(r,\infty;f(z+c))+S(r,f).\end{aligned}$$

Consequently we get

$$\overline{N}(r,\infty;f')=\overline{N}(r,\infty;f)=S(r,f). \tag{2.11}$$

Note that

$$\overline{N}_E\,(r,\infty;f(z+c)\mid f'(z)=\infty)\leq\overline{N}(r,\infty;f')=S(r,f).$$

Then using the conclusion (iii), we get

$$\begin{aligned}\overline{N}(r,\infty;f(z+c)) = \;& \overline{N}_0\left(r,\infty;f(z+c)\mid f'(z)=\infty\right)+\overline{N}_E\left(r,\infty;f(z+c)\mid f'(z)=\infty\right)\\ &+\overline{N}\left(r,\infty;f(z+c)\mid f'(z)\neq\infty\right)=S(r,f).\end{aligned} \tag{2.12}$$

So from (2.11) and (2.12), we get

$$\overline{N}(r,\infty;f')=\overline{N}(r,\infty;f(z+c))=S(r,f).$$

Hence the result (vi) follows. □

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of finite order, let $H(z)$ be defined in (2.6) and $a\in\mathbb{C}\setminus\{0\}$. If $f'(z)$ and $f(z+c)$ share a IM and satisfy $f(z+c)=0\to f'(z)=0$ and $f(z+c)=\infty\leftarrow f'(z)=\infty$. Then $f'(z)\equiv f(z+c)$ and further $f(z)$ is a transcendental entire function of finite order.*

Proof. From (2.6), we get

$$H(z)-1=\frac{f'(z)-f(z+c)}{f(z+c)}. \tag{2.13}$$

Now the two possibilities may arise, i.e., either $H(z)\not\equiv 1$ or $H(z)\equiv 1$. So we consider two cases separately in the following.

Case 1. Suppose $H(z)\equiv 1$. Consequently from (2.13), we get

$$f'(z)\equiv f(z+c). \tag{2.14}$$

Then by the same argument of proof used in Theorem E, we get that $f(z)$ is a transcendental entire function.

Case 2. Suppose $H(z)\not\equiv 1$. Then $H(z)-1$ has zeros. Since $f'(z)$ and $f(z+c)$ share a IM, from (2.13) we get

$$\overline{N}(r,a;f'(z))=\overline{N}(r,a;f(z+c))\le N(r,1;H)\le T(r,H)=S(r,f). \tag{2.15}$$

We claim that $\overline{N}_E(r,0;f'(z)\mid f(z+c)=0)\neq S(r,f)$. If not, suppose $\overline{N}_E(r,0;f'(z)\mid f(z+c)=0)=S(r,f)$. Then from the conclusion (ii) of Lemma 2.6, we deduce that

$$\begin{aligned}\overline{N}(r,0;f'(z)) \;&=\; \overline{N}_0\left(r,0;f'(z)\mid f(z+c)=0\right)+\overline{N}_E(r,0;f'(z)\mid f(z+c)=0)\\ &\quad+\overline{N}\left(r,0;f'(z)\mid f(z+c)\neq 0\right)=S(r,f).\end{aligned}$$

So by the second fundamental theorem of Nevanlinna, we get

$$T(r,f')\le\overline{N}(r,\infty;f')+\overline{N}(r,0;f')+\overline{N}(r,a;f')+S(r,f')=S(r,f'),$$

which is impossible. Hence $\overline{N}_E(r,0;f'(z)\mid f(z+c)=0)\neq S(r,f)$. We now consider

$$H_1(z)=\frac{f''(z)}{f'(z)-a}-\frac{f'(z+c)}{f(z+c)-a}. \tag{2.16}$$

Then the two possibilities may arise, i.e., either $H_1(z) \not\equiv 0$ or $H_1(z) \equiv 0$. First we suppose $H_1(z) \equiv 0$. Then on integration from (2.16), we get

$$f'(z) - a \equiv d\left(f(z+c) - a\right),$$

where $d \in \mathbb{C} \setminus \{0\}$. Let z_0 be a common zero of $f'(z)$ and $f(z+c)$. Then $f'(z_0) = 0$ and $f(z_0 + c) = 0$. So $d = 1$ and then $f'(z) \equiv f(z+c)$. Thus again the conclusions follow.

Next we suppose $H_1(z) \not\equiv 0$. Now by the conclusion (vi) of Lemma 2.6 and (2.15), we get

$$N(r, \infty; H_1) = \overline{N}(r, \infty; f') + \overline{N}(r, a; f') + \overline{N}(r, \infty; f(z+c)) + \overline{N}(r, a; f(z+c)) = S(r, f).$$

Also $m(r, H_1) = S(r, f)$. Therefore $T(r, H_1) = S(r, f)$. Now from (2.6) by differentiation, we get

$$f''(z) = Hf'(z+c) + H'f(z+c).$$

Putting the above value in (2.16), we get

$$\begin{aligned} H_1 &= \frac{Hf'(z+c) + H'f(z+c)}{Hf(z+c) - a} - \frac{f'(z+c)}{f(z+c) - a} \\ &= \frac{H'f^2(z+c) - aHf'(z+c) - aH'f(z+c) + af'(z+c)}{Hf^2(z+c) - af(z+c) - aHf(z+c) + a^2}, \end{aligned}$$

i.e.,
$$\begin{aligned} &HH_1f^2(z+c) - aH_1f(z+c) - aHH_1f(z+c) + a^2H_1 \\ &= H'f^2(z+c) - aHf'(z+c) - aH'f(z+c) + af'(z+c) \end{aligned}$$

i.e.,
$$\begin{aligned} &a(H-1)f'(z+c) = (H' - HH_1)f^2(z+c) + (aH_1 + aHH_1 - aH')f(z+c) - a^2H_1 \\ &Af'(z+c) = Bf^2(z+c) + Cf(z+c) + D, \end{aligned}$$

where $A = a(H-1)$, $B = H' - HH_1$, $C = aH_1 + aHH_1 - aH'$ and $D = -a^2H_1$. Now suppose $B \not\equiv 0$. Note that A, B, C and D are small functions of $f(z+c)$. Then using Lemmas 2.1 and 2.6, we get

$$\begin{aligned} 2T(r, f(z+c)) &\leq T(r, f'(z+c)) + S(r, f(z+c)) \\ &\leq T\left(r, \frac{f'(z+c)}{f(z+c)}\right) + T(r, f(z+c)) + S(r, f(z+c)) \\ &= N\left(r, \infty; \frac{f'(z+c)}{f(z+c)}\right) + T(r, f(z+c)) + S(r, f(z+c)) \\ &= \overline{N}(r, \infty; f(z+c)) + T(r, f(z+c)) + S(r, f(z+c)) \\ &= T(r, f(z+c)) + S(r, f(z+c)), \end{aligned}$$

i.e., $T(r, f(z+c)) \leq S(r, f(z+c))$, which is impossible. So $B \equiv 0$ and then by (2.16), we get

$$\frac{H'}{H} \equiv H_1 = \frac{f''(z)}{f'(z) - a} - \frac{f'(z+c)}{f(z+c) - a}.$$

On integration, we have

$$H \equiv k\frac{f'(z)-a}{f(z+c)-a},$$

where $k \in \mathbb{C} \setminus \{0\}$. So by (2.6), we have

$$H(z) = \frac{f'(z)}{f(z+c)} \equiv k\frac{f'(z)-a}{f(z+c)-a}.$$

Now let z_0 be an a-point of $f(z+c)$ of multiplicity p, where $p \in \mathbb{N}$. Since $f'(z)$ and $f(z+c)$ share a IM, we conclude that z_0 be also an a-point of $f'(z)$ of multiplicity q, where $q \in \mathbb{N}$ and $f'(z_0) = a$. Since H is an entire function, we get that $p \leq q$. If $p < q$, then z_0 must be a zero of $f'(z)$. Consequently, we get $f'(z_0) = 0$. Since $a \in \mathbb{C} \setminus \{0\}$, simultaneously $f'(z_0) = 0$ and $f'(z_0) = a$ are impossible. So the only possibility is $p = q$. This shows that $f'(z)$ and $f(z+c)$ share a CM. Now it is clear that $f'(z)$ and $f(z+c)$ satisfy all conditions of Theorem E and consequently the claimed conclusions arise. □

3. Proofs of the main theorems

Proof of Theorem 1.1. Here $f(z)$ is a non-constant meromorphic function of finite order. So the following two cases separately occur.

Case 1. Let $f(z)$ be a non-constant rational function. Then $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)(\not\equiv 0)$ are two mutually prime polynomials. Following the same argument of proof used in Theorem E, we get $Q(z) \equiv$ constant $= k$, say, where $k \in \mathbb{C} \setminus \{0\}$.

As $f(z)$ is a non-constant rational function, $P(z)$ is a non-constant polynomial. Then $f(z) = \frac{1}{k}P(z)$. Furthermore, we have $f'(z) = \frac{1}{k}P'(z)$ and $f(z+c) = \frac{1}{k}P(z+c)$. As $f(z+c) = 0 \to f'(z) = 0$, we see that $P(z+c) = 0 \to P'(z) = 0$. Therefore $P'(z) \equiv P(z+c)p(z)$, where $p(z)(\not\equiv 0)$ is a polynomial of $\deg(p(z)) \geq 0$. But this contadicts the fact that $\deg(P'(z)) < \deg(P(z+c)p(z))$. So $f(z)$ is not a non-constant rational function.

Case 2. Let $f(z)$ be a transcendental meromorphic function of finite order. The next part of the theorem follows from the Lemma 2.7. □

Proof of Theorem 1.2. The proof of the theorem follows from the conclusion (iv) of Lemma 2.6. □

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