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PERIODIC ORTHONORMAL SPLINE SYSTEMS WITH ARBITRARY KNOTS AS BASES IN $H^1(\mathbb{T})$

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Abstract. We give a simple geometric characterization of sequences of knots for which the corresponding periodic orthonormal spline system of order k is a basis in the atomic Hardy space on the torus $\mathbb{T}$.

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1. Introduction

This paper belongs to a series of papers studying properties of periodic and non-periodic orthonormal spline systems with arbitrary knots. The detailed study of non-periodic orthonormal spline systems started in 1960's with Z. Ciesielski's papers [4, 5] on properties of the Franklin system, which is an orthonormal system consisting of continuous piecewise linear functions with dyadic knots. Next, the results by J. Domsta (1972), cf. [9], made it possible to extend such study to orthonormal spline systems of higher order with dyadic knots. These systems occurred to be bases or unconditional bases in several function spaces like $L^p[0,1]$, $1 \le p < \infty$, $C[0,1]$, $H^p[0,1]$, $0 < p \le 1$, Sobolev spaces $W^{p,k}[0,1]$, they give characterizations of BMO and VMO spaces, and various spaces of smooth functions.

The extension of these results to orthonormal spline systems with arbitrary knots has begun with the case of piecewise linear systems, i.e. general Franklin systems, or orthonormal spline systems of order 2. This was possible due to precise estimates of the inverse to the Gram matrix of piecewise linear B-spline bases with arbitrary knots, as presented in [14]. We would like to mention here two results by G.G. Gevorkyan and A. Kamont. First, each general Franklin system is an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$, cf. [10]. Second, there is a simple geometric characterization of knot sequences for which the corresponding general Franklin system is a basis or an unconditional basis in $H^1[0,1]$, cf. [12]. We note that in

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both of these results, an essential tool for their proof is the association of a so called characteristic interval to each general Franklin function f_n.

The case of splines of higher order is much more difficult. Let us mention that the basic result – the existence of a uniform bound for L^∞-norms of orthogonal projections on spline spaces of order k with arbitrary order (i.e. a bound depending on the order k, but independent of the sequence of knots) – was a long-standing problem known as C. de Boor's conjecture (1973), cf. [2]. The case of $k = 2$ was settled even earlier by Z. Ciesielski [4], the cases $k = 3, 4$ were solved by C. de Boor himself (1968, 1981), cf. [1, 3], but the positive answer in the general case was given by A. Yu. Shadrin [21] in 2001. A much simplified and shorter proof of this theorem was recently obtained by M. v. Golitschek (2014), cf. [22]. An immediate consequence of A.Yu. Shadrin's result is that if a sequence of knots is dense in $[0, 1]$, then the corresponding orthonormal spline system of order k is a basis in $L^p[0, 1]$, $1 \le p < \infty$ and $C[0, 1]$. Moreover, Z. Ciesielski [6] obtained several consequences of Shadrin's result, one of them being some estimate for the inverse to the B-spline Gram matrix. Using this estimate, G.G. Gevorkyan and A. Kamont [12] extended a part of their result from [11] to orthonormal spline systems of arbitrary order and obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is a basis in $H^1[0, 1]$. Further extension required more precise estimates for the inverse of B-spline Gram matrices, of the type known for the piecewise linear case. Such estimates were obtained recently by M. Passenbrunner and A.Yu. Shadrin [19]. Using these estimates, M. Passenbrunner [17] proved that for each sequence of knots, the corresponding orthonormal spline system of order k is an unconditional basis in $L^p[0, 1]$, $1 < p < \infty$. With the help of this result it was obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is an unconditional basis in $H^1[0, 1]$ (see [13]).

Another extension of the previous results can be done for *periodic* orthonormal spline systems with arbitrary knots. In the periodic case K. Keryan [15] proved that for any admissible point sequence the corresponding periodic Franklin system (i.e. periodic piecewise linear system) forms an unconditional basis in $L^p[0, 1]$, $1 < p < \infty$. K. Keryan and M. Passenbrunner [16] obtained an essential estimate for general periodic orthonormal spline functions. Combining the estimate with the methods developed in [10] they proved the unconditionality of periodic orthonormal spline systems in $L^p(\mathbb{T})$, $1 < p < \infty$. A result concerning the basis property of periodic orthonormal spline systems of order 2 in Hardy's atomic space on the torus was carried out by M. Poghosyan and K. Keryan. In the paper [20] they gave a simple geometric characterization of knot sequences for which the corresponding general periodic Franklin system is a basis or unconditional basis in $H^1(\mathbb{T})$.

The main result of the present paper is to give a characterization of those knot sequences for which the corresponding periodic orthonormal spline system of fixed order of smoothness is a basis in $H^1(\mathbb{T})$.

The paper is organized as follows. In Section 2 we give necessary definitions and we formulate the main result of this paper – Theorem 2.1. The proof of the main result is presented in Section 3: in Subsection 3.1 some properties of periodic orthonormal spline systems are provided, then in Subsection 3.2 a lower bound for $H^1(\mathbb{T})$ norm of a function is given, and finally in Subsections 3.3 and Sufficiency it is proved the necessity and sufficiency of k-regularity in Theorem 2.1 correspondingly.

2. Definitions, notation and the main result

We begin with some preliminary notations. The parameter $k \geq 2$ will always be used for the order of the underlying polynomials or splines. We use the notation $A(t) \sim B(t)$ to indicate the existence of two constants $c_1, c_2 > 0$, such that $c_1 B(t) \leq A(t) \leq c_2 B(t)$ for all t, where t denotes all implicit and explicit dependencies that the expressions A and B might have. If the constants c_1, c_2 depend on an additional parameter p, we write this as $A(t) \sim_p B(t)$. Correspondingly, we use the symbols $\lesssim, \gtrsim, \lesssim_p, \gtrsim_p$. For a subset E of the real line, we denote by $|E|$ the Lebesgue measure of E.

Now let $k \geq 2$ be an integer and $\mathcal{T} := (s_n)_{n=1}^{\infty}$ be a point sequence from the torus $\mathbb{T}$ such that each point occurs at most k times. Such point sequences are called k admissible.

For $n \geq k$, we define $\hat{\mathcal{S}}_n$ to be the space of polynomial splines of order k with grid points $(s_j)_{j=1}^{n} \subset \mathbb{T}$. For each $n \geq k+1$, the space $\hat{\mathcal{S}}_{n-1}$ has codimension 1 in $\hat{\mathcal{S}}_n$ and, therefore, there exists a function $\hat{f}_n \in \hat{\mathcal{S}}_n$ with $\|\hat{f}_n\|_{L^2(\mathbb{T})} = 1$ that is orthogonal to the space $\hat{\mathcal{S}}_{n-1}$. Observe that this function $\hat{f}_n$ is unique up to sign. In addition, let $(\hat{f}_n)_{n=1}^{k}$ be an orthonormal basis for $\hat{\mathcal{S}}_k$. The system of functions $(\hat{f}_n)_{n=1}^{\infty}$ is called *periodic* orthonormal spline system of order k corresponding to the sequence $(s_n)_{n=1}^{\infty}$.

Now we define the atomic Hardy space on $\mathbb{T}$.

Definition 2.1. *A function $a : \mathbb{T} \to \mathbb{R}$ is called a* periodic atom, *if either $a \equiv 1$ or $\exists\, \Gamma \subset \mathbb{T}$ interval such that all these conditions are satisfied:*

(i) $\operatorname{supp} a \subset \Gamma$,

(ii) $\|a\|_{L^\infty(\mathbb{T})} \le |\Gamma|^{-1}$,

(iii) $\int_{\mathbb{T}} a(x)\,\mathrm{d}x = \int_{\Gamma} a(x)\,\mathrm{d}x = 0$.

Definition 2.2. *$H^1(\mathbb{T})$ is the family of all the functions f that has representation*

$$f = \sum_{n=1}^{\infty} c_n a_n$$

for some periodic atoms $(a_n)_{n=1}^{\infty}$ and real scalars $(c_n)_{n=1}^{\infty} \in \ell^1$.

The space $H^1(\mathbb{T})$ becomes a Banach space under the norm

$$\|f\|_{H^1(\mathbb{T})} := \inf \sum_{n=1}^{\infty} |c_n|$$

where inf is taken over all (periodic) atomic representations $\sum c_n a_n$ of f. Now, we introduce regularity conditions in the torus $\mathbb{T}$ for sequence $(s_n)_{n=1}^{\infty}$.

Assume that $n \ge k+1$. Let $(\sigma_j)_{j=0}^{n-1}$ be the ordered sequence of knot points consisting of $(s_j)_{j=1}^{n}$ in $\mathbb{T}$ canonically identified with $[0,1)$:

$$\hat{\mathcal{T}} := \hat{\mathcal{T}}_n = (0 \le \sigma_{n,0} \le \sigma_{n,1} \le \cdots \le \sigma_{n,n-2} \le \sigma_{n,n-1} < 1). \tag{2.1}$$

For the integers $\ell \le k$ and $i \in \mathbb{N}_0$, we define $T_{n,i}^{(\ell)} := [\sigma_{n,i}, \sigma_{n,i+\ell}] \subset \mathbb{T}$ interval. Here we observe index i periodically, i.e. we use the notation of periodic extension of the sequence $(\sigma_j)_{j=0}^{n-1}$, i.e. $\sigma_{rn+j} = r + \sigma_j$ for $j \in \{0, \ldots, n-1\}$ and $r \in \mathbb{Z}$ and in the subindices of the B-spline functions, we take the indices modulo n.

Definition 2.3. *Let $\ell \le k$ and $(s_n)_{n=1}^{\infty}$ be an ℓ-admissible point sequence the in the torus $\mathbb{T}$. Then, this sequence is called* ℓ-regular in torus $\mathbb{T}$ with parameter $\gamma \ge 1$ *if*

$$\frac{|T_{n,i}^{(\ell)}|}{\gamma} \le |T_{n,i+1}^{(\ell)}| \le \gamma |T_{n,i}^{(\ell)}|, \qquad n \ge \ell+1,\ i \in \mathbb{N}_0.$$

Let $\hat{P}_n^{(k)}$ be the orthogonal projection operator onto $\hat{\mathcal{S}}_n$ with respect to the canonical inner product in $L^2(\mathbb{T})$ and $\hat{D}_n^{(k)}$ be its Dirichlet kernel.

The following is the main result of this paper.

Theorem 2.1. *Let $k \ge 1$ and let (s_n) be a k-admissible sequence of knots in $\mathbb{T}$ with the corresponding periodic orthonormal spline system $(\hat{f}_n^{(k)})$ of the order k. Then, $(\hat{f}_n^{(k)})$ is a basis in $H^1(\mathbb{T})$ if and only if (s_n) is k-regular in the torus with some parameter $\gamma \ge 1$*

3. Proof of Theorem 2.1

Since the sequence of knots $(s_n)_{n=1}^{\infty}$ is dense in the torus $\mathbb{T}$, the linear span of the functions $\{\hat{f}_n^{(k)}, n \ge 1\}$ is linearly dense in $C(\mathbb{T})$, which implies its linear density in $H^1(\mathbb{T})$. Therefore, $\{\hat{f}_n^{(k)}, n \ge 1\}$ is a basis in $H^1(\mathbb{T})$ if and only if the partial sum operators $\hat{P}_n^{(k)}$ are uniformly bounded in $H^1(\mathbb{T})$, i.e. there is a constant $C = C(\mathcal{T})$, that only depends on the knot sequence $(s_n)_{n=1}^\infty$, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} = \|\hat{P}_n^{(k)} : H^1(\mathbb{T}) \to H^1(\mathbb{T})\| \le C(\mathcal{T}). \tag{3.1}$$

We show that (3.1) is equivalent to k-regularity of $\mathcal{T}$. This is an immediate consequence of the Propositions 3.1 and 3.2, which contain estimates of norms $\hat{P}_n^{(k)}$ from below and from above, respectively.

Proposition 3.1. *Let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots in the torus $\mathbb{T}$ of multiplicities at most k. Let*

$$M = M_n^{(k)} := \max\left\{ \frac{|T_{n,i}^{(k)}|}{|T_{n,i+1}^{(k)}|}, \frac{|T_{n,i+1}^{(k)}|}{|T_{n,i}^{(k)}|} : \quad 0 \le i \le n-1 \right\}.$$

Then there is a constant $C_k > 0$, depending only on k, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \ge C_k \log M_n^{(k)}.$$

Proposition 3.2. *Let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots in the torus $\mathbb{T}$ of multiplicities at most k. Let γ be such that*

$$\frac{|T_{n,i}^{(k)}|}{\gamma} \le |T_{n,i+1}^{(k)}| \le \gamma |T_{n,i}^{(k)}|, \qquad n \ge k+1, \ i \in \mathbb{N}_0.$$

Then there is a constant $C_{k,\gamma} > 0$ depending only on k and γ, such that

$$\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \le C_{k,\gamma}.$$

Before we begin to prove the Propositions 3.1 and 3.2, we recall some properties of splines and orthogonal projections $\hat{P}_n^{(k)}$.

3.1. **Properties of periodic orthonormal spline systems.** The key result which let us work with periodic orthonormal spline systems of the order k is the periodic version of A. Yu. Shadrin's [21] theorem, i.e. uniform boundedness of L^∞-norms of projections $\hat{P}_n^{(k)}$. The result was obtained by M. Passenbrunner in [18].

Theorem 3.1 ([18])**.** *There exists a constant C_k depending only on the spline order k such that for any sequence $\hat{\mathcal{T}}$ of knots of multiplicity at most k*

$$\|\hat{P}_{\hat{\mathcal{T}}}^{(k)}\|_\infty = \|\hat{P}_{\hat{\mathcal{T}}}^{(k)} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| \le C_k.$$

Clearly, this means that

$$\|\hat{P}_{\hat{\mathcal{T}}}^{(k)}\|_\infty = \sup_{t\in\mathbb{T}} \int_{\mathbb{T}} |\hat{D}_{\hat{\mathcal{T}}}^{(k)}(t,s)| ds \le C_k. \tag{3.2}$$

Now, as before, let $\hat{\mathcal{T}}_n = (0 \le \sigma_0 \le \sigma_1 \le \cdots \le \sigma_{n-2} \le \sigma_{n-1} < 1)$ be a sequence of knots in the torus $\mathbb{T}$ of multiplicities at most k. By $\hat{N}_{n,i}^{(k)}$, $i = 0, \ldots, n-1$ we denote the L^∞-normalized periodic B-spline basis of $\hat{\mathcal{S}}_n^{(k)}$. These functions are nonnegative, linearly independent and form a partition of unity, i.e. $\sum_{i=0}^{n-1} \hat{N}_{n,i}^{(k)}(t) = 1$ for each $t \in \mathbb{T}$. Moreover, $\operatorname{supp} \hat{N}_{n,i}^{(k)} = [\sigma_i, \sigma_{i+k}]$ and $\|\hat{N}_{n,i}^{(k)}\|_{L^1(\mathbb{T})} = \frac{|T_{n,i}^{(k)}|}{k}$. Corresponding to this basis, there exists a biorthogonal basis of $\hat{\mathcal{S}}_n^{(k)}$, which is denoted by $(\hat{N}_{n,i}^{(k)*})_{i=0}^{n-1}$.

Let $\hat{G}_n^{(k)} = [(\hat{N}_{n,i}^{(k)}, \hat{N}_{n,j}^{(k)}), 0 \le i, j \le n-1]$ be the Gram matrix for the system $\{\hat{N}_{n,i}^{(k)}, i = 0, \dots, n-1\}$, and let $A_n^{(k)} = [\hat{a}_{i,j} = (\hat{N}_{n,i}^{(k)*}, \hat{N}_{n,j}^{(k)*}), 0 \le i, j \le n-1]$. the following is a equivalent version of Theorem 3.1 (See [18], Section 3, Remark 3.2):

Theorem 3.2 ([18]). *Let $n \ge 2k$. Then, there exists a constant $q \in (0,1)$ depending only on the spline order k such that*

$$|\hat{a}_{ij}| \lesssim_k \frac{q^{\hat{d}(i,j)}}{\max(|\operatorname{supp} \hat{N}_{n,i}^{(k)}|, |\operatorname{supp} \hat{N}_{n,j}^{(k)}|)}, \qquad 0 \le i, j \le n-1,$$

where $\hat{d}$ is the periodic distance function on $\{0, \dots, n-1\}$.

In particular, since $\hat{D}_n^{(k)}(t,s) = \sum_{i,j=0}^{n-1} \hat{a}_{i,j} \hat{N}_{n,i}^{(k)}(t) \hat{N}_{n,j}^{(k)}(s)$, Theorem 3.2 implies that

$$|\hat{D}_n^{(k)}(t,s)| \lesssim_k \sum_{i,j=0}^{n-1} \frac{q^{\hat{d}(i,j)}}{\max(|\operatorname{supp} \hat{N}_{n,i}^{(k)}|, |\operatorname{supp} \hat{N}_{n,j}^{(k)}|)} \hat{N}_{n,i}^{(k)}(t) \hat{N}_{n,j}^{(k)}(s). \tag{3.3}$$

If the setting of the parameters k and n is clear from the context, we will omit k and n and write $\hat{N}_i$ instead of $\hat{N}_{n,i}^{(k)}$.

3.2. **A lower bound for $H^1(\mathbb{T})$ norm of a function.** In order to prove Proposition 3.1 we will need the periodic version of the claim used in [12] (cf. page 7, estimate (3.4)).

Proposition 3.3 ([12]). *Define $\Phi(x) := \max(0, 1/2 - |x/4|)$ and $\Phi_\epsilon(x) = \frac{1}{\epsilon}\Phi(\frac{x}{\epsilon})$, for $x \in [0,1]$. Then, there is a constant $C > 0$ such that*

$$\|f\|_{H^1[0,1]} \ge C\|f^*\|_{L^1[0,1]}, \quad \text{where} \quad f^*(x) = \sup_{\epsilon > 0} \Big|\int_0^1 \Phi_\epsilon(x-t) f(t)dt\Big|.$$

Using this proposition we prove the following.

Lemma 3.1. *Define the 1-periodic functions $\hat{\Phi}(x) := \max(0, 1/2 - |x/4|)$ and $\hat{\Phi}_\epsilon(x) = \frac{1}{\epsilon}\hat{\Phi}(\frac{x}{\epsilon})$, for $x \in \mathbb{T}$. Then, for some constant $c > 0$ the following holds,*

$$\|f\|_{H^1(\mathbb{T})} \ge c\|f^{**}\|_{L^1(\mathbb{T})}, \quad \text{where} \quad f^{**}(x) = \sup_{\epsilon > 0} \Big|\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) f(t)dt\Big|.$$

Proof. Let f be a function from $H^1(\mathbb{T})$. Then, there exists a sequence of periodic atoms $(\hat{a}_i)_{i=1}^\infty$ and coefficients $(\lambda_i)_{i=1}^\infty$ such that,

$$f = \sum_{i=1}^\infty \lambda_i \hat{a}_i, \quad \text{and} \quad \sum_{i=1}^\infty |\lambda_i| \le 2\|f\|_{H^1(\mathbb{T})}. \tag{3.4}$$

Now by (3.4), we get

$$\begin{aligned}
\|f^{**}\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} \sup_{\epsilon>0} |\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) \sum_{i=1}^{\infty} \lambda_i \hat{a}_i(t) dt| dx \\
&\leq \sum_{i=1}^{\infty} |\lambda_i| \int_{\mathbb{T}} \sup_{\epsilon>0} |\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) \hat{a}_i(t) dt| dx \\
&\leq \sum_{i=1}^{\infty} |\lambda_i| \int_{\mathbb{T}} \sup_{0<\epsilon<1/16} |\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) \hat{a}_i(t) dt| dx \\
&+ \sum_{i=1}^{\infty} |\lambda_i| \int_{\mathbb{T}} \sup_{1/16\leq\epsilon} |\int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) \hat{a}_i(t) dt| dx =: \Sigma_1 + \Sigma_2
\end{aligned}$$

First we estimate Σ_2. We have that

$$\hat{\Phi}_\epsilon(x) \leq \frac{1}{2\epsilon}, \quad x \in \mathbb{T},$$

and $\|\hat{a}_i\|_{L^1(\mathbb{T})} \leq 1$ so we get the following,

$$\Sigma_2 \leq \sum_{i=1}^{\infty} 8|\lambda_i| \lesssim \|f\|_{H^1(\mathbb{T})}.$$

Now, let $\Gamma_j \subset \mathbb{T}$ be the interval that contains the support of the periodic atom $\hat{a}_j$. Define $J := \{i : |\Gamma_i^c| \geq 1/4\}$ and split Σ_1 into 2 sums. The first sums over all the indices from the set J and the second one sums over indices of J^c. Let's denote the sums by $\Sigma_{1,J}$ and Σ_{1,J^c}, respectively.

Observe $\Sigma_{1,J}$. Fix an arbitrary $i \in J$ and identify tours $\mathbb{T}$ with $[0,1)$ in such a way that 0 coincides with the the center of Γ_i^c. We have that $0 < \epsilon < \frac{1}{16}$ and $|\Gamma_i^c| \geq \frac{1}{4}$. Hence, we get that $\hat{\Phi}_\epsilon(\cdot) = \Phi_\epsilon(\cdot)$. Consequently, by Proposition 3.3 we get

$$\begin{aligned}
\Sigma_{1,J} &= \sum_{i\in J} |\lambda_i| \int_{\mathbb{T}} \sup_{0<\epsilon<1/16} |\int_{\mathbb{T}} \Phi_\epsilon(x-t) \hat{a}_i(t) dt| dx \\
&\leq \sum_{i\in J} |\lambda_i| \|\hat{a}_i^*\|_{L^1[0,1]} \lesssim \sum_{i\in J} |\lambda_i| \|\hat{a}_i\|_{H^1[0,1]} \leq \|f\|_{H^1(\mathbb{T})}.
\end{aligned}$$

The last inequality comes from $\|\hat{a}_i\|_{H^1[0,1]} \leq 1$. This is true because by the right identification of $\mathbb{T}$ with $[0,1)$, i.e. the starting point 0 is not Γ_i, we made sure that $\hat{a}_i$ is an atom on $[0,1)$.

Consider Σ_{1,J^c}. For all $i \in J^c$ we have $|\Gamma_i|^{-1} \leq 4/3$ and $\|\hat{a}_i\|_{L^\infty(\mathbb{T})} \leq |\Gamma_i|^{-1}$. Thus,

$$\begin{aligned}
\Sigma_{1,J^c} &\leq \sum_{i\in J^c} \frac{|\lambda_i|}{|\Gamma_i|} \int_{\mathbb{T}} \sup_{0<\epsilon<1/16} \int_{\mathbb{T}} \hat{\Phi}_\epsilon(x-t) dt dx \\
&\leq \frac{4}{3} \sum_{i\in J^c} |\lambda_i| \lesssim \|f\|_{H^1(\mathbb{T})}
\end{aligned}$$

Combining all above we get the desired result, i.e.

$$\|f^{**}\|_{L^1(\mathbb{T})} \lesssim \|f\|_{H^1(\mathbb{T})}.$$

3.3. **Necessity of k-regularity: proof of Proposition 3.1.** Since $\hat{P}_n^{(k)}$ is a projection onto $\hat{\mathcal{S}}_n^{(k)}$, it follows that $\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \geq 1$. Therefore we can assume that $M \geq M(k)$, where $M(k) \geq 2$ will be specified later. Let u be such that $M = \frac{|T_{n,u}^{(k)}|}{|T_{n,u+1}^{(k)}|}$ (clearly, the case of $M = \frac{|T_{n,u+1}^{(k)}|}{|T_{n,u}^{(k)}|}$ is analogous). As $M \geq 2$, it follows that

$$|T_{n,u}^{(1)}| \geq (M-1)|T_{n,u+1}^{(k)}| \geq \frac{M}{2}|T_{n,u+1}^{(k)}|.$$

Here we identified the torus $\mathbb{T}$ with $[0,1)$ in such a way that the starting point 0 is not in the intervals $T_{n,u}^{(1)}$ and $T_{n,u+1}^{(k)}$. Now let $\phi_1(\cdot) = \frac{k}{|T_{n,u+1}^{(k)}|}\hat{N}_{u+1}(\cdot)$ and $\phi_2(\cdot) = \phi_1(\cdot + |T_{n,u+1}^{(k)}|)$. Then supp $\phi_1 = [\sigma_{u+1}, \sigma_{u+k+1}]$, supp $\phi_2 = [\sigma_{u+1} - |T_{n,u+1}^{(k)}|, \sigma_{u+1}] \subset [\sigma_u, \sigma_{u+1}]$ and $\|\phi_1\|_1 = \|\phi_2\|_1 = 1$. Put $\phi = \phi_1 - \phi_2$. Then $\int_{\mathbb{T}} \phi(x)dx = 0$, supp$\phi = \Gamma = [\sigma_{u+1} - |T_{n,u+1}^{(k)}|, \sigma_{u+k+1}]$ and $\|\phi\|_\infty \leq \frac{2k}{|\Gamma|}$, so $\|\phi\|_{H^1(\mathbb{T})} \leq 2k$. We need to estimate from below $\|\hat{P}_n^{(k)}\phi\|_{H^1(\mathbb{T})}$. For this, we use Lemma 3.1.

At first, consider $\hat{P}_n^{(k)}\phi_2$. We observe $\hat{D}_n^{(k)}$: the kernel of the projection $\hat{P}_n^{(k)}$. Note that $\hat{D}_n^{(k)}$ is a polynomial of degree at most $k-1$ on $T_{n,u}^{(1)}$, and by comparison of different norms of polynomials of fixed degree (cf. e.g. Theorem 2.6 of Chapter 4 in [8])

$$\int_{T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|dt \sim_k |T_{n,u}^{(1)}| \max_{t \in T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|,$$

and the constants in the above equivalences depend only on k. As supp $\phi_2 \subset T_{n,u}^{(1)}$, we find

$$\begin{aligned} |\hat{P}_n^{(k)}\phi_2(x)| &= |\int_{\mathbb{T}} \hat{D}_n^{(k)}(t,x)\phi_2(t)dt| \leq \int_{T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)||\phi_2(t)|dt \\ &\leq \max_{t \in T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|\|\phi_2\|_{L^1(\mathbb{T})} \leq \frac{C_k}{|T_{n,u}^{(1)}|} \int_{T_{n,u}^{(1)}} |\hat{D}_n^{(k)}(t,x)|dt. \end{aligned}$$

Combining 3.2 and the last sequence of inequalities we get $|\hat{P}_n^{(k)}\phi_2(x)| \leq \frac{C_k}{|T_{n,u}^{(1)}|}$, and consequently

$$(\hat{P}_n^{(k)}\phi_2)^{**} \leq \frac{C_k}{|T_{n,u}^{(1)}|}. \tag{3.5}$$

Now, we estimate $(\hat{P}_n^{(k)}\phi_1)^{**}$ from below . Clearly, since $\hat{P}_n^{(k)}$ is a projection onto $\hat{\mathcal{S}}_n^{(k)}$, we have $\hat{P}_n^{(k)}\phi_1 = \phi_1$. Take $x \in T_{n,u}^{(1)}$, $x \leq \sigma_{u+1} - |T_{n,u+1}^{(k)}|$ and let $\epsilon(x) = \sigma_{u+k+1} - x$. Then $\hat{\Phi}_{\epsilon(x)}(x-t) = \hat{\Phi}_{\epsilon(x)}(t-x) \geq \frac{1}{4\epsilon(x)}$ for $t \in$ supp ϕ_1, and consequently

$$\phi_1^{**}(x) \geq \int_0^1 \hat{\Phi}_{\epsilon(x)}(t-x)\phi_1(t)dt \geq \frac{1}{4\epsilon(x)}.$$

Combining this with (3.5) we find that

$$(\hat{P}_n^{(k)}\phi)^{**}(x) \geq \phi_1^{**}(x) - (\hat{P}_n^{(k)}\phi_2)^{**}(x) \geq \frac{1}{4\epsilon(x)} - \frac{C_k}{|T_{n,u}^{(1)}|} \quad \text{for} \quad x \in [\sigma_u, \sigma_{u+1} - |T_{n,u+1}^{(k)}|].$$

Then, as $|T_{n,u}^{(1)}| \geq \frac{M}{2}|T_{n,u+1}^{(k)}|$, we have for $M \geq 32C_k$

$$(\hat{P}_n^{(k)}\phi)^{**}(x) \geq \frac{1}{8\epsilon(x)} \quad \text{for} \quad x \in [\sigma_{u+1} - \frac{|T_{n,u}^{(1)}|}{16C_k}, \sigma_{u+1} - |T_{n,u+1}^{(k)}|]. \tag{3.6}$$

Using again $|T_{n,u}^{(1)}| \geq \frac{M}{2}|T_{n,u+1}^{(k)}|$, we get from the last inequality

$$\begin{aligned} \|(\hat{P}_n^{(k)}\phi)^{**}\|_{L^1(\mathbb{T})} &\geq \int_{\sigma_{u+1} - \frac{|T_{n,u}^{(1)}|}{16C_k}}^{\sigma_{u+1} - |T_{n,u+1}^{(k)}|} (\hat{P}_n^{(k)}\phi)^{**}(x)dx \\ &\geq \int_{2|T_{n,u+1}^{(k)}|}^{\frac{|T_{n,u}^{(1)}|}{16C_k} + |T_{n,u+1}^{(k)}|} \frac{1}{8u}du \geq \frac{1}{8}\log M - C_{k,1}. \end{aligned}$$

Fix $M(k) \geq 32C_k$ and such that $\frac{1}{16}\log M(k) \geq C_{k,1}$; then for $M \geq M(k)$ we have $\frac{1}{8}\log M - C_{k,1} \geq \frac{1}{16}\log M$. As $\|\phi\|_{H^1(\mathbb{T})} \leq 2k$, by Lemma 3.1 we get $\|\hat{P}_n^{(k)}\|_{H^1(\mathbb{T})} \geq C_k \log M$. •

3.4. **Sufficiency of k-regularity: proof of Proposition 3.2.** The idea of the proof is analogous to the idea of the proof of Proposition 3.2 in [12]. We recall the mesh (2.1) obtained by the canonical identification of $\mathbb{T}$ with $[0,1)$

$$\hat{\mathcal{T}}_n = (0 \leq \sigma_{n,0} \leq \sigma_{n,1} \leq \cdots \leq \sigma_{n,n-2} \leq \sigma_{n,n-1} < 1).$$

Let $n \geq 2k$ and η be a periodic atom. It is enough to show that

$$\|\hat{P}_n^{(k)}\eta\|_{H^1(\mathbb{T})} \lesssim_{k,\gamma} 1.$$

For this, we find a suitable atomic decomposition of $\hat{P}_n^{(k)}\eta$.

If $\eta \equiv 1$, then also $\hat{P}_n^{(k)}\eta \equiv 1$ and it is a periodic atom.

Now, let $\int_{\mathbb{T}} \eta(t)dt = 0$, and let $\Gamma \subset \mathbb{T}$ be an interval such that supp $\eta \subset \Gamma$, $\|\eta\|_{L^\infty(\mathbb{T})} \leq \frac{1}{|\Gamma|}$. Let

$$\mathcal{G} = \{0 \leq i \leq n-1 : \operatorname{supp} \hat{N}_i \cap \Gamma = \emptyset\}.$$

Put

$$\psi_i = \hat{N}_i \cdot \hat{P}_n^{(k)}\eta \quad \text{for} \quad i \in \mathcal{G}, \quad \text{and} \quad \psi = \hat{P}_n^{(k)}\eta - \sum_{i \in \mathcal{G}} \psi_i. \tag{3.7}$$

We check that the collection $\{\psi, \psi_i, i \in \mathcal{G}\}$ gives a desired (periodic) atomic decomposition of $\hat{P}_n^{(k)}$. Clearly, $\hat{P}_n^{(k)} = \psi + \sum_{i\in\mathcal{G}} \psi_i$. For $i \in \mathcal{G}$ the supports of $\hat{N}_i$ and η are disjoint. Since $\hat{P}_n^{(k)}$ is an orthogonal projection onto $\hat{\mathcal{S}}_n^{(k)}$ we have for $i \in \mathcal{G}$

$$\int_{\mathbb{T}} \psi_i(t)dt = \int_{\mathbb{T}} \hat{N}_i(t) \cdot \hat{P}_n^{(k)}\eta(t)dt = \int_{\mathbb{T}} \hat{N}_i(t) \cdot \eta(t)dt = 0.$$

Since $\int_{\mathbb{T}} \eta(t) = 0$, we have also $\int_{\mathbb{T}} \hat{P}_n^{(k)} \eta(t)dt = 0$, which implies $\int_{\mathbb{T}} \psi(t)dt = 0$.

Now, we estimate $\|\psi_i\|_\infty$ for $i \in \mathcal{G}$. Let $0 \le m \le n-1$ be the unique index such that σ_m is not in Γ, but σ_{m+1} is in Γ. Next, we let $0 \le l \le n-1$ be the unique index such that σ_{l-1} is in Γ, but σ_l is not in Γ, Then $\mathcal{G} = \{0 \le i \le n-1 : i \le m-k\} \cup \{0 \le i \le n-1 : i \ge l\} =: \mathcal{G}_1 \cup \mathcal{G}_2$.

Consider the case $i \in \mathcal{G}_1$. Note that supp $\psi_i \subset$ supp $\hat{N}_i = [\sigma_i, \sigma_{i+k}]$. Recall that (cf. formulae in the Section 3.1)

$$\hat{P}_n^{(k)} \eta(t) = \sum_{j_1,j_2=0}^{n-1} \hat{a}_{j_1,j_2} \int_{\mathbb{T}} \hat{N}_{j_1}(u)\eta(u)du\, \hat{N}_{j_2}(t). \tag{3.8}$$

By Theorem 3.2, we have the estimate

$$|\hat{a}_{j_1,j_2}| \lesssim_k \frac{q^{\hat{d}(j_1,j_2)}}{\max(|\operatorname{supp} \hat{N}_{j_1}|, |\operatorname{supp} \hat{N}_{j_2}|)},$$

where $0 < q < 1$, depends only on the order k. Note that if $t \in$ supp ψ_i and j_2 is such that $\hat{N}_{j_2}(t) \ne 0$, then for those indices j_2 we have

$$|T_{n,j_2}^{(k)}| \sim_{k,\gamma} |T_{n,i}^{(k)}|,$$

by the k-regularity. Therefore, for j_2 such that $\hat{N}_{j_2}(t) \ne 0$

$$|\hat{a}_{j_1,j_2}| \lesssim_{k,\gamma} \frac{q^{\hat{d}(j_1,j_2)}}{|T_{n,i}^{(k)}|}.$$

Moreover, the number of the indices j_2 such that $N_{j_2}(t) \ne 0$ doesn't exceed $2k-1$. Thus, (3.8) gives

$$|\psi_i(t)| \le |\hat{P}_n^{(k)} \eta(t)| \lesssim_{k,\gamma} \frac{1}{|T_{n,i}^{(k)}|} \sum_{j_1=0}^{n-1} q^{\hat{d}(i,j_1)} \Big| \int_{\mathbb{T}} \hat{N}_{j_1}(u)\eta(u)du \Big|.$$

Next, note that if j_1 is such that supp $\hat{N}_{j_1} \cap \operatorname{supp}\eta \ne \emptyset$ then $m-k+1 \le j_1 \le l-1$. Moreover, we have $\int_{\mathbb{T}} |\hat{N}_{j_1}(u)\eta(u)|du \le 1$. Therefore the above inequality implies

$$|\psi_i(t)| \lesssim_{k,\gamma} \frac{1}{|T_{n,i}^{(k)}|} \sum_{j_1=m-k+1}^{l-1} q^{\hat{d}(i,j_1)} \lesssim_k \frac{q^{\min\{\hat{d}(i,l),\hat{d}(i,m)\}}}{|T_{n,i}^{(k)}|}.$$

Now, put $\alpha_i = \|\psi_i\|_{L^\infty(\mathbb{T})} |T_{n,i}^{(k)}|$ and $\psi_i = \alpha_i \tilde{\psi}_i$. Clearly, supp $\tilde{\psi}_i$ = supp $\psi_i \subset [\sigma_i, \sigma_{i+k}]$ and $\|\tilde{\psi}_i\|_{L^\infty(\mathbb{T})} \le \frac{1}{|T_{n,i}^{(k)}|}$, so $\tilde{\psi}_i$ is a periodic atom. Since

$$0 \le \alpha_i \lesssim_{k,\gamma} q^{\min\{\hat{d}(i,l),\hat{d}(i,m)\}},$$

we finally get

$$\sum_{0 \le i \le m-k} \psi_i = \sum_{0 \le i \le m-k} \alpha_i \tilde{\psi}_i \quad \text{with} \quad \sum_{0 \le i \le m-k} \alpha_i \lesssim_{k,\gamma} 1. \tag{3.9}$$

Analogously, for $i \geq l$ we get $\psi_i = \alpha_i \tilde{\psi}_i$ with $\tilde{\psi}_i$ a periodic atom and $0 \leq \alpha_i \lesssim_{k,\gamma} q^{\min\{\hat{d}(i,l),\hat{d}(i,m)\}}$, and consequently

$$\sum_{l \leq i \leq n-1} \psi_i = \sum_{l \leq i \leq n-1} \alpha_i \tilde{\psi}_i \quad \text{with} \quad \sum_{l \leq i \leq n-1} \alpha_i \lesssim_{k,\gamma} 1. \tag{3.10}$$

It remains to consider ψ. Since the functions $\hat{N}_j$, $0 \leq j \leq n-1$, are a partition of unity, we have $\psi = \hat{P}_n^{(k)}\eta \cdot \sum_{j=m-k+1}^{l-1} \hat{N}_j$. Let $\tilde{\Gamma} = [\sigma_{m-k+1}, \sigma_{l+k-1}]$. Note that $\operatorname{supp} \psi \subset \tilde{\Gamma}$. We will show that $\|\psi\|_{L^\infty(\mathbb{T})} \lesssim_{k,\gamma} \frac{1}{|\tilde{\Gamma}|}$.

At first, consider the case when Γ contains at least one support of $\hat{N}_j$. Then by the k-regularity $|\Gamma| \sim_{k,\gamma} |\tilde{\Gamma}|$. Using $\int_{\mathbb{T}} |\hat{D}_n^{(k)}\eta(t,s)|ds \lesssim_k 1$ (cf. 3.1) we get

$$|\psi(t)| \leq |\hat{P}_n^{(k)}\eta(t)| \leq \int_{\mathbb{T}} |\hat{D}_n^{(k)}\eta(t,s)||\eta(s)|ds \leq \frac{1}{|\Gamma|} \int_{\mathbb{T}} |\hat{D}_n^{(k)}(t,s)|ds \lesssim_{k,\gamma} \frac{1}{|\tilde{\Gamma}|}.$$

In the other case, i.e. when Γ does not contain any B-spline support, it follows by the k-regularity that $|T_{n,m}^{(k)}| \sim_{k,\gamma} |\tilde{\Gamma}|$. We again use formula (3.8) to estimate $\|\psi\|_\infty$. If j_1 is such that $(\hat{N}_{j_1}, \eta) \neq 0$, then $m-k+1 \leq j_1 \leq l-1$, so by the k-regularity $|T_{n,m}^{(k)}| \sim_{k,\gamma} |T_{n,j_1}^{(k)}|$. This and Theorem 3.2 imply that $|\hat{a}_{j_1,j_2}^{(k)}| \lesssim_{k,\gamma} \frac{1}{|T_{n,m}^{(k)}|}$. As $|(\hat{N}_{j_1}, \eta)| \leq 1$, we get $\sum_{j_1=0}^{n-1} |\hat{a}_{j_1,j_2}||(\hat{N}_{j_1}, \eta)| = \sum_{j_1=m-k+1}^{m+k} |\hat{a}_{j_1,j_2}||(\hat{N}_{j_1}, \eta)| \lesssim_{k,\gamma} \frac{1}{|T_{n,m}^{(k)}|}$. As the functions $\hat{N}_{j_2}$, $0 \leq j_2 \leq n-1$ are a partition of unity, we get for $t \in \tilde{\Gamma}$

$$|\psi(t)| \leq |\hat{P}_n^{(k)}\eta(t)| \lesssim_{k,\gamma} \frac{1}{|T_{n,m}^{(k)}|} \lesssim_{k,\gamma} \frac{1}{|\tilde{\Gamma}|}.$$

It follows from these considerations that $\psi = \alpha\tilde{\psi}$, where $\tilde{\psi}$ is a periodic atom and $0 \leq \alpha \lesssim_{k,\gamma} 1$. Putting together this fact, (3.7), (3.9) and (3.10) we get for periodic atoms $\tilde{\psi}, \tilde{\psi}_i$

$$\hat{P}_n^{(k)}\eta = \sum_{0 \leq i \leq m-k} \alpha_i \tilde{\psi}_i + \alpha\tilde{\psi} + \sum_{l \leq i \leq n-1} \alpha_i \tilde{\psi}_i,$$

where $\alpha, \alpha_i \geq 0$ and $\sum_{0 \leq i \leq m-k} \alpha_i + \alpha + \sum_{l \leq i \leq n-1} \alpha_i \lesssim_{k,\gamma} 1$. This is the desired atomic decomposition of $\hat{P}_n^{(k)}\eta$.

Next we consider the case when $n < 2k$. Let $f \in H^1(\mathbb{T})$, then

$$\begin{aligned} \|\sum_{m=1}^{n} (f, \hat{f}_m)\hat{f}_m\|_{H^1(\mathbb{T})} &\leq \sum_{m=1}^{n} \|f\|_{L^1(\mathbb{T})} \|\hat{f}_m\|_{L^\infty(\mathbb{T})} \|\hat{f}_m\|_{H^1(\mathbb{T})} \\ &\leq \|f\|_{H^1(\mathbb{T})} \sum_{m=1}^{2k} \|\hat{f}_m\|_{L^\infty(\mathbb{T})} \|\hat{f}_m\|_{H^1(\mathbb{T})} \leq C_k \|f\|_{H^1(\mathbb{T})}. \end{aligned}$$

This concludes the proof of the proposition.

Список литературы

[1] C. de Boor, “On the convergence of odd-degree spline interpolation”, J. Approx. Theory, **1**, 452 – 463 (1968).

[2] C. de Boor, “The quasi-interpolant as a tool in elementary polynomial spline theory”, Approximation Theory (Austin, TX, 1973), Academic Press, New York, 269 – 276 (1973).
[3] C. de Boor, “On a max-norm bound for the least-squares spline approximant”, Approximation and Function Spaces (Gda´nsk, 1979), North-Holland, Amsterdam, 163 – 175 (1981).
[4] Z. Ciesielski, “Properties of the orthonormal Franklin system”, Studia Math. **23**, 141 -– 157 (1963).
[5] Z. Ciesielski, “Properties of the orthonormal Franklin system, II”, Studia Math. **27**, 289 -– 323 (1966).
[6] Z. Ciesielski, “Orthogonal projections onto spline spaces with arbitrary knots”, Function Spaces (Poznan, 1998), Lecture Notes in Pure Appl. Math. **213**, Dekker, New York, 133 – 140 (2000).
[7] R. R. Coifman and G. Weiss, “Extensions of Hardy spaces and their use in analysis”, Bull. Amer. Math. Soc., **83**, no. 4, 569 – 645 (1977).
[8] R. A. DeVore and G. G. Lorentz, Constructive Approximation, Springer (1993).
[9] J. Domsta, ”Theorem on B-splines”, Studia Math. **41**, 291 -– 314 (1972).
[10] G. G. Gevorkyan and A. Kamont, “Unconditionality of general Franklin systems in $L^p[0,1], 1 < p < \infty$”, Studia Math. **164**, 161 – 204 (2004).
[11] G. G. Gevorkyan and A. Kamont, “General Franklin systems as bases in $H^1[0,1]$”, Studia Math. **167**, 259 – 292 (2005).
[12] G. G. Gevorkyan and A. Kamont, “Orthonormal spline systems with arbitrary knots as bases in $H^1[0,1]$”, East J. Approx. **14**, 161 – 182 (2008).
[13] G. Gevorkyan, A. Kamont, K. Keryan and M. Passenbrunner, “Unconditionality of orthogonal spline systems in H^1”, Studia Mathematica, **226**, no. 2, 123 – 154 (2015).
[14] B. S. Kashin and A. A. Saakyan, Orthogonal Series, Transl. Math. Monogr. **75**, Amer. Math. Soc., Providence, RI (1989).
[15] K. Keryan, “Unconditionality of general periodic spline systems in $L^p[0,1]$, $1 < p < \infty$”, J. Contemp. Math. Anal., **40** (1): 13 – 55 (2005).
[16] K. Keryan and M. Passenbrunner, “Unconditionality of periodic orthonormal spline systems in L^p”, Studia Mathematica, **248** (1), 57 – 91 (2019).
[17] M. Passenbrunner, “Unconditionality of orthogonal spline systems in L^p”, Studia Math. **222**, 51 – 86 (2014).
[18] M. Passenbrunner, “Orthogonal projectors onto spaces of periodic splines”, Journal of Complexity **42**, 85 – 93 (2017).
[19] M. Passenbrunner and A. Shadrin, “On almost everywhere convergence of orthogonal spline projections with arbitrary knots”, J. Approx. Theory **180**, 77 – 89 (2014).
[20] M. P. Poghosyan and K. A. Keryan, “General periodic Franklin system as a basis in $H^1[0,1]$”, Izv. Nats. Akad. Nauk Armenii, Mat. **40**, no. 1, 61 – 84 (2005).
[21] A. Yu. Shadrin, “The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: a proof of de Boor’s conjecture”, Acta Math.. **187**, 59 – 137 (2001).
[22] M. v. Golitschek, “On the L^∞-norm of the orthogonal projector onto splines. A short proof of A. Shadrin’s theorem”, J. Approx. Theory **181**, 30 – 42 (2014).

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