Известия НАН Армении, Математика, том 57, н. 6, 2022, стр. 62 – 69.

ON THE IDEAL TRANSFORMS DEFINED BY AN IDEAL

Y. SADEGH, J. A'ZAMI AND S. YAZDANI

https://doi.org/10.54503/0002-3043-2022.57.6-62-69

University of Mohaghegh Ardabili, Ardabil, Iran
E-mails: *y.sadegh@uma.ac.ir; yassin.sadegh@yahoo.com*
azami@uma.ac.ir; jafar.azami@gmail.com
saeedyzdn@uma.ac.ir; saeedyzdn@gmail.com

Abstract. Let R be a commutative Noetherian ring, I an ideal of R and M an R-module. The ambiguous structure of I-transform functor $D_I(-)$ makes the study of its properties attractive. In this paper we gather conditions under which, $D_I(R)$ and $D_I(M)$ appear in certain roles. It is shown, under these conditions that $D_I(R)$ is a Cohen-Macaulay ring, regular ring, $\cdots$ and $D_I(M)$ can be regarded as a Noetherian, flat, $\cdots R$-module. We also present a primary decomposition of zero submodule of $D_I(M)$ and secondary representation of $D_I(M)$.

MSC2020 numbers: 13D45; 14B15; 13E05.
Keywords: Associated primes; ideal transform; local cohomology.

1. Introduction

Throughout this paper, R will always denote a non-trivial commutative Noetherian ring with identity. For an R-module M, the local cohomology modules $H^i_I(M)$, $i = 0, 1, \dots$ of an R-module M with respect to I were introduced by Grothendieck [6]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\cup_{n=1}^{\infty}(0 :_M I^n)$. There is a natural isomorphism

$$H^i_I(M) \cong \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

Recall that for an R-module M, the *cohomological dimension* of M *with respect* to I is defined as

$$\operatorname{cd}(I, M) := \sup\left\{i \in \mathbb{Z} \,:\, H^i_I(M) \neq 0\right\}.$$

The cohomological dimension has been studied by several authors, see for example [6] and [7]. Also, for any proper ideal I of R, the *arithmetic rank* of I denoted by $\operatorname{ara}(I)$, is the least number of elements of R required to generate an ideal which has the same radical as I. For any ideal I of an arbitrary Noetherian ring R, the I-transform functor denoted by $D_I(-)$, is defined as:

$$D_I(-) = \varinjlim_{n\geq 1} \operatorname{Hom}_R(I^n, -).$$

If M is an R-module, then $D_I(M) = \varinjlim_{n\geq 1} \operatorname{Hom}_R(I^n, M)$ is the ideal transform of M with respect to I, or the I-transform of M for short. Recall from [3, Exercise 2.2.3(ii)] that $D_I(R)$ is a commutative ring with identity and also from [3, Exercise 2.2.10] that $\eta : R \to D_I(R)$ is a ring homomorphism. It is well known that the ring $D_I(R)$ has a finitely generated R-algebra structure, whenever the functor $D_I(-)$ is exact. We refer the reader to [3] for more details about ideal transform functor.

For every non-zero R-module M, we denote the set of all zero-divisors of M in R by $Z_R(M)$. Also, for any ideal I of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $V(I)$ and $\{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$ by $\sqrt{I}$. We recall that grade(I, R) is the common length of maximal regular R-sequences in ideal I. For any unexplained notation and terminology we refer the reader to [3] and [8].

This paper is devided into 3 sections. In the next section we gather some conditions to find affirmative answers to the questions: When is $D_I(M)$ a finitely generated R-module? When is it a flat R-module? See 2.1 and 2.2. Moreover in Theorem 2.3, we show that $D_I(R)$ is a projective R-module in case that I is a non-zero proper ideal of an arbitrary Noetherian domain with $\operatorname{Ann}_R(H^1_I(R)) \neq 0$ and $D_I(-)$ is an exact functor. Theorem 2.4 is a nice result that shows $D_I(R)$ is a Cohen-Macaulay ring under certain conditions. Next, in Theorem 2.5, it is seen that $D_I(R)$ is a regular ring whenever R is regular, $\operatorname{Ann}_R(I)$ is nilpotent and $\eta : R \to D_I(R)$ is a surjective ring homomorphism. In section 3 we present a minimal primary decomposition of zero submodule of $D_I(M)$ in case that M is a finitely generated R-module and ara$(I) = 1$, see 3.1. An R-module M is said to be representable when it has a secondary representation, see [3, Definition 7.2.2]. In 3.2 we show that $D_I(M)$ is representable and $\operatorname{Att}_R(D_I(M)) \subseteq \operatorname{Att}_R(M) \backslash V(I)$ whenever M is a finitely generated representable R-module and cd$(I, R) = 1$.

2. Some results

In this section we begin our investigations with the following Theorem.

Theorem 2.1. *Let R be a Noetherian ring and M be a non-zero finitely generated R-module. Let I be an ideal of R such that $0 \neq \operatorname{Ann}_R(H^1_I(M)) \nsubseteq Z_R(M)$. Then both $H^1_I(M)$ and $D_I(M)$ are Noetherian R-modules.*

Proof. By the assumption, there exists a non-zero element $x \in \operatorname{Ann}_R(H^1_I(M)) \backslash Z_R(M)$. So the exact sequence

$$0 \to M \overset{.x}{\to} M \to \frac{M}{xM} \to 0,$$

induces the long exact sequence

$$0 \to \Gamma_I(M) \stackrel{.x}{\to} \Gamma_I(M) \to \Gamma_I(\frac{M}{xM}) \stackrel{\beta}{\to} H^1_I(M) \stackrel{.x}{\to} H^1_I(M) \to \cdots.$$

Since $x.H^1_I(M) = 0$, it follows that $H^1_I(M)$ is a finitely generated R-module. Now, the exact sequence

$$0 \to \frac{M}{\Gamma_I(M)} \to D_I(M) \to H^1_I(M) \to 0,$$

leads that $D_I(M)$ is a Noetherian R-module. □

Corollary 2.1. *Let R be a Noetherian domain and I an ideal of R with $\mathrm{Ann}_R(H^1_I(R)) \neq 0$. Then $D_I(R)$ is a Noetherian R-module. In particular, it is a Noetherian integral extension of ring R.*

Proof. $D_I(R)$ is a Noetherian R-module by Theorem 2.1 and therefore it is a finitely generated R-module. Since R is a domain it follows that $\eta : R \longrightarrow D_I(R)$ is an injective ring homomorphism. Thus, by outlined Remark after [1, Corollary 5.3], $D_I(R)$ is an integral extension of R. Moreover it is a finitely generated R-algebra and so is a Noetherian ring. □

In the following we denote by $\mathrm{Att}_R(H^1_{\mathfrak{m}}(M))$ the set of all attached prime ideals of $H^1_{\mathfrak{m}}(M)$.

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a non-zero finitely generated R-module with $0 \neq \mathrm{Ann}_R(H^1_{\mathfrak{m}}(M)) \nsubseteq Z_R(M)$. Then $\mathrm{Att}_R(H^1_{\mathfrak{m}}(M)) \subseteq \{\mathfrak{m}\}$.*

Proof. The assertion follows from Theorem 2.1, [3, Theorem 7.1.3] and [3, Corollary 7.2.12].

Lemma 2.1. *Let I be a non-nilpotent proper ideal of the Noetherian ring R and $D_I(-)$ an exact functor. Then $D_I(R)$ is a flat R-module.*

Proof. See [2, Theorem 3.11].

Theorem 2.2. *Let R be a Noetherian domain and I an ideal with $\mathrm{cd}(I, R) = 1$. If M is an R-module of finite projective dimension d and $\mathrm{Ass}_R\, M = \{0\}$, then $D_I(M)$ is a flat R-module.*

Proof. We proceed by induction on d. If $d = 0$, then M is projective and so, it is a direct summand of a free module. Thus the assertion follows from Lemma 2.1 and [3, Exercise 3.4.5]. Now assume that $d \geq 1$, and that

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \stackrel{\varepsilon}{\to} M \to 0,$$

is a projective resolution of M. Applying the exact functor $D_I(-)$ to the exact sequence

$$0 \to \ker\varepsilon \to P_0 \xrightarrow{\varepsilon} M \to 0,$$

we obtain the following exact sequence

$$0 \to D_I(\ker\varepsilon) \to D_I(P_0) \xrightarrow{\varepsilon} D_I(M) \to 0.$$

Since $\mathrm{pd}_R(\ker\varepsilon) = d-1$, by the inductive hypothesis, one can say $D_I(\ker\varepsilon)$ is a flat R-module. Moreover, $D_I(P_0)$ is flat because P_0 is a projective R-module. Thus for every ideal J of R we have $JD_I(P_0) \cong J\otimes_R D_I(P_0)$ and $JD_I(\ker\varepsilon) \cong J\otimes_R D_I(\ker\varepsilon)$. On the other hand, from hypothesis and [5, Proposition 2.10], we find that $Z_R(D_I(M)) = 0$. This guarantees the exactness of the bottom row, in the following commutative diagram.

$$\begin{array}{ccccccc} J \otimes_R D_I(\ker\varepsilon) & \longrightarrow & J \otimes_R D_I(P_0) & \longrightarrow & J \otimes_R D_I(M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ JD_I(\ker\varepsilon) & \longrightarrow & JD_I(P_0) & \longrightarrow & JD_I(M) & \longrightarrow & 0 \end{array}$$

Hence we have $JD_I(M) \cong J\otimes_R D_I(M)$. But $JD_I(M) \subseteq D_I(M)$. This means

$$J\otimes_R D_I(M) \to R\otimes_R D_I(M)$$

is an injective homomorphism and therefore $D_I(M)$ is a flat R-module. □

Lemma 2.2. *Let R be a Noetherian local ring and I a proper non-zero ideal of R. Then the I-transform functor $D_I(-)$ is exact if and only if* $\mathrm{cd}(I, R) \leq 1$.

Proof. It follows from [2, Lemma 3.2].

Theorem 2.3. *Let R be a Noetherian domain and I a non-zero proper ideal of R such that the I-transform functor $D_I(-)$ is exact. If $\mathrm{Ann}_R(H^1_I(R)) \neq 0$, then $D_I(R)$ is a projective R-module.*

Proof. In case that $(R, \mathfrak{m})$ is a Noetherian local ring, the assertion is clear by Corollary 2.1 and Lemma 2.1. Suppose that R is not local and assume the contrary that there exists an R-module M such that $\mathrm{Ext}^1_R(D_I(R), M) \neq 0$. Hence there exists a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $(\mathrm{Ext}^1_R(D_I(R), M))_\mathfrak{p} \neq 0$. By Corollary 2.1, $D_I(R)$ is a finitely generated R-module. Thus by [8, Exercise 7.7] and [3, Exercise 4.3.5, iii], $\mathrm{Ext}^1_{R_\mathfrak{p}}(D_{IR_\mathfrak{p}}(R_\mathfrak{p}), M_\mathfrak{p}) \neq 0$ and so $D_{IR_\mathfrak{p}}(R_\mathfrak{p}) \neq 0$. In case that $I \nsubseteq \mathfrak{p}$, we have $D_{IR_\mathfrak{p}}(R_\mathfrak{p}) = R_\mathfrak{p}$ because $IR_\mathfrak{p} = R_\mathfrak{p}$. This means that $D_{IR_\mathfrak{p}}(R_\mathfrak{p})$ is a projective $R_\mathfrak{p}$-module.

Now consider the case that $I \subseteq \mathfrak{p}$. Since $D_I(-)$ is an exact functor, it follows that $\mathrm{cd}(I, R) \leq 1$ by Lemma 2.2. Moreover, it is clear that $\mathrm{cd}(IR_\mathfrak{p}, R_\mathfrak{p}) \leq \mathrm{cd}(I, R)$.

Hence by using again Lemma 2.2, the $IR_{\mathfrak{p}}$-transform functor $D_{IR_{\mathfrak{p}}}(-)$ is exact. Since R is domain, it follows that the Noetherian ring $R_{\mathfrak{p}}$ is domain and so $IR_{\mathfrak{p}}$ is a non-nilpotent proper ideal of $R_{\mathfrak{p}}$. Hence by Lemma 2.1, the $R_{\mathfrak{p}}$-module $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is flat. Therefore $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a non-zero projective $R_{\mathfrak{p}}$-module because $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ is a Noetherian local ring and $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a finitely generated $R_{\mathfrak{p}}$-module.
As it is seen, in both cases above $D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a projective $R_{\mathfrak{p}}$-module which contradict the fact that $\mathrm{Ext}^1_{R_{\mathfrak{p}}}(D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}), M_{\mathfrak{p}}) \neq 0$. Thus for every R-module M we must have $\mathrm{Ext}^1_R(D_I(R), M) = 0$, i.e., $D_I(R)$ is a projective R-module. $\square$

Theorem 2.4. *Let R be a Noetherian domain of dimension d and I be an ideal of R such that $I \subseteq J(R)$. Let $\mathrm{Ann}_R(H^1_I(R)) \neq 0$ and $H^i_I(R) = 0$ for each $1 < i < d$. Then $D_I(R)$ is a Noetherian Cohen-Macaulay ring.*

Proof. It follows from Corollary 2.1 that $D_I(R)$ is a Noetherian ring and it is integral over R. Consequently, we have $\dim D_I(R) = \dim R$. By [3, Corollary 2.2.10, iv] and [3, Theorem 4.2.1], one has $\Gamma_{ID_I(R)}(D_I(R)) = H^1_{ID_I(R)}(D_I(R)) = 0$. Moreover, one can find by [3, Corollary 2.2.10, v] and [3, Theorem 4.2.1] that $H^i_{ID_I(R)}(D_I(R)) = 0$ for every $1 < i < d$. Hence by view of [3, Theorem 6.2.7] we have $d \leq \mathrm{grade}(ID_I(R), D_I(R))$. On the other hand $\mathrm{grade}(ID_I(R), D_I(R)) \leq \dim D_I(R)$. These yield $\mathrm{grade}(ID_I(R), D_I(R)) = d$. Now let $\mathfrak{n} \in \mathrm{Max}(D_I(R))$. It follows from [1, Corollary 5.8] that $\mathfrak{m} := \mathfrak{n}^c$ is a maximal ideal of R. Since $I \subseteq \mathfrak{m}$, we have $ID_I(R) \subseteq \mathfrak{n}$ because of $ID_I(R) \subseteq \mathfrak{m}D_I(R) = \mathfrak{n}^{ce} \subseteq \mathfrak{n}$. Therefore

$$\mathrm{grade}(ID_I(R), D_I(R)) \leq \mathrm{grade}(\mathfrak{n}, D_I(R)) \leq \mathrm{grade}(\mathfrak{n}(D_I(R))_{\mathfrak{n}}, (D_I(R))_{\mathfrak{n}})$$
$$= \mathrm{depth}(D_I(R))_{\mathfrak{n}} \leq \dim(D_I(R))_{\mathfrak{n}} \leq \dim D_I(R) = d.$$

Thus for every $\mathfrak{n} \in \mathrm{Max}(D_I(R))$ we have $\mathrm{depth}(D_I(R))_{\mathfrak{n}} = \dim(D_I(R))_{\mathfrak{n}} = d$. $\square$

Let I be an ideal of R such that $\mathrm{Ann}_R(I)$ is nilpotent. Then $IR_{\mathfrak{p}} \neq 0$ for every prime ideal $\mathfrak{p}$ of R, because $\mathrm{Ann}_R(I) \subseteq \mathfrak{p}$. In the following, we show that under certain assumptions, $D_I(R)$ is a regular ring. Recall that a Noetherian ring R is regular, if $R_{\mathfrak{p}}$ is a regular local ring for every prime ideal $\mathfrak{p}$ of R. For more details about regular local rings see [4, Section 2.2].

Theorem 2.5. *Let R be a Noetherian regular ring and I an ideal of R such that $\mathrm{Ann}_R(I)$ is nilpotent. Then $D_I(R)$ is a regular ring, provided $\eta : R \to D_I(R)$ is a surjective ring homohorphism.*

Proof. The assertion follows immediately in case that $\Gamma_I(R) = 0$ or I is a nilpotent ideal of R. Thus we may assume that $\Gamma_I(R) \neq 0$ and I is not nilpotent. Also, it should be mentioned that $D_I(R)$ is a Noetherian ring because R is a

Noetherian ring and $\eta : R \to D_I(R)$ is a surjective ring homohorphism. Now let $\mathfrak{q} \in \operatorname{Spec}(D_I(R))$ and $\mathfrak{p} := \mathfrak{q}^c$. Then the canonical map $\bar{\eta} : R_\mathfrak{p} \to (D_I(R))_\mathfrak{q}$ by $\bar{\eta}(\frac{r}{s}) = \frac{\eta(r)}{\eta(s)}$ for every $\frac{r}{s} \in R_\mathfrak{p}$, is a surjective ring homomorphism. Let $\frac{r}{s} \in \ker \bar{\eta}$. There exists $v \in R \setminus \mathfrak{p}$ such that $\eta(r)\eta(v) = 0$. Thus $rv \in \ker \eta = \Gamma_I(R)$. Hence $\frac{rv}{s} \in \Gamma_{IR_\mathfrak{p}}(R_\mathfrak{p})$. But $\operatorname{grade}(IR_\mathfrak{p}, R_\mathfrak{p}) \geq 1$ because $R_\mathfrak{p}$ is a domain and $IR_\mathfrak{p} \neq 0$ due to $\operatorname{Ann}_R(I) \subseteq \mathfrak{p}$. Therefore in view of [3, Exercise 1.3.9, (iii)], $\Gamma_{IR_\mathfrak{p}}(R_\mathfrak{p}) = 0$ and we get $\frac{r}{s} = 0$. This leads to $R_\mathfrak{p} \cong (D_I(R))_\mathfrak{q}$. □

3. Primary decomposition and secondary representation

Let R be a Noetherian ring and M a finitely generated R-module. It is well-known that every proper submodule of M has a primary decomposition. In the following Theorem, for an ideal I with $\operatorname{ara}(I) = 1$ we find a minimal primary decomposition of the zero submodule of $D_I(M)$.

Theorem 3.1. *Let R be a Noetherian ring and I an ideal of R with* $\operatorname{ara}(I) = 1$. *Let M be a finitely generated R-module and $0 = \bigcap_{i=1}^{t} N_i$ be a minimal primary decomposition of* 0 *in M. Then the zero submodule of $D_I(M)$ has a minimal primary decomposition in the form $0 = \bigcap_{i=1}^{s} D_I(N_i)$, $s \leq t$.*

Proof. Since $\operatorname{ara}(I) = 1$ there exists $x \in R$ such that $\sqrt{Rx} = \sqrt{I}$. By [3, Proposition 2.2.23] and [3, Theorem 2.2.19] we have

$$0 = D_I(0) \cong D_{Rx}(\bigcap_{i=1}^{t} N_i) \cong (\bigcap_{i=1}^{t} N_i)_x \cong \bigcap_{i=1}^{t} (N_i)_x \cong \bigcap_{i=1}^{t} D_{Rx}(N_i) = \bigcap_{i=1}^{t} D_I(N_i). \tag{3.1}$$

From the above intersection let us remove all $D_I(N_j), 1 \leq j \leq t$, such that

$$\bigcap_{\substack{i=1 \\ i \neq j}}^{t} D_I(N_i) \subseteq D_I(N_j). \tag{3.2}$$

Then we will get a minimal primary decomposition in the form $0 = \bigcap_{i=1}^{s} D_I(N_i), s \leq t$, provided that we show every $D_I(N_i), 1 \leq i \leq s$, is a primary submodule of $D_I(M)$.

In order to avoid inaccuracies, it is harmless to assume that the minimal primary decomposition $0 = \bigcap_{i=1}^{t} N_i$ is sorted in the following sense:

For every $1 \leq j \leq s$, $D_I(N_j)$ doesn't satisfy condition (3.2) and $D_I(N_j)$ satisfies condition (3.2) for every $j \geq s + 1$.

Since $\operatorname{cd}(I,R) \leq \operatorname{ara}(I) = 1$, it follows from [3, Lemma 6.3.1] that $D_I(-)$ is an exact functore and so

$$D_I\left(\frac{M}{N_i}\right) \cong \frac{D_I(M)}{D_I(N_i)}.$$

Hence in order to show that $D_I(N_i)$ is a primary submodule of $D_I(M)$ it is enough to show that the map $D_I(\frac{M}{N_i}) \xrightarrow{.r} D_I(\frac{M}{N_i})$ is either injective or nilpotent homomorphism for every $r \in R$. This is clear the fact because $D_I(-)$ is an R-linear functore. □

Before we discuss about representability of $D_I(M)$, we need the following preparative Lemma.

Lemma 3.1. *Let R be a Noetherian ring and I an ideal of R with $\operatorname{cd}(I,R) = 1$. Suppose that M_1, M_2 are submodules of a finitely generated R-module M such that $M = M_1 + M_2$. Then*

$$D_I(M) \cong D_I(M_1) + D_I(M_2).$$

Proof. First note that because $\operatorname{cd}(I,R) = 1$, by [3, Lemma 6.3.1] we find that $D_I(-)$ is an exact functor. Moreover, it is clear that I is not nilpotent. Therefore $D_I(R)$ is a flat R-module by Lemma 2.1. So by applying the functor $D_I(R) \otimes_R -$ to the exact sequence

$$0 \longrightarrow M_1 \cap M_2 \longrightarrow M_1 \oplus M_2 \longrightarrow M_1 + M_2 \longrightarrow 0,$$

we obtain the following exact sequence

$$0 \longrightarrow D_I(R) \otimes_R (M_1 \cap M_2) \longrightarrow D_I(R) \otimes_R (M_1 \oplus M_2) \longrightarrow D_I(R) \otimes_R (M_1 + M_2) \longrightarrow 0.$$

On the other hand, the following sequence is also exact

$$0 \longrightarrow (D_I(R) \otimes_R M_1) \cap (D_I(R) \otimes_R M_2) \longrightarrow (D_I(R) \otimes_R M_1) \oplus (D_I(R) \otimes_R M_2)$$

$$\longrightarrow (D_I(R) \otimes_R M_1) + (D_I(R) \otimes_R M_2) \longrightarrow 0.$$

Hence by view of [8, Theorem 7.4] and [1, Proposition 2.14], we have

$$D_I(R) \otimes_R (M_1 + M_2) \cong (D_I(R) \otimes_R M_1) + (D_I(R) \otimes_R M_2).$$

This fact together with [3, Exercise 6.1.9] concludes

$$D_I(M) = D_I(M_1 + M_2) \cong D_I(M_1) + D_I(M_2).$$

This completes the proof. □

Theorem 3.2. *Let R be a Noetherian ring and I an ideal of R with $\operatorname{cd}(I,R) = 1$. Suppose that the finitely generated R-module M is representable. Then, so is $D_I(M)$ and moreover $\operatorname{Att}_R(D_I(M)) \subseteq \operatorname{Att}_R(M) \setminus V(I)$.*

Proof. We may assume $D_I(M) \neq 0$. Let

$$M = M_1 + M_2 + \cdots + M_t \qquad \text{with } M_j \qquad \mathfrak{p}_j - \text{secondary} \, (1 \leq j \leq t),$$

be a minimal secondary representation of M. Then it follows from 3.1 that

$$D_I(M) \cong D_I(M_1) + D_I(M_2) + \cdots + D_I(M_t).$$

Note that, it may $D_I(M_j) = 0$ for some $1 \leq j \leq t$. Putting $T := \Sigma_{i=1}^n D_I(M_i), n \leq t$, where each $D_I(M_i)$, $1 \leq i \leq n$ is not zero, we have $D_I(M) \cong T$. So it is enough to show that T is representable; i.e., $\Sigma_{i=1}^n D_I(M_i)$ is a secondary representation of T. This is clearly the case because $D_I(-)$ is an R-linear functor. Therefore

$$\mathrm{Att}_R(D_I(M)) = \mathrm{Att}_R(T) \subseteq \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n\},$$

where $\mathfrak{q}_i = \sqrt{\mathrm{Ann}_R(D_I(M_i))} \in \mathrm{Spec}\, R$.

Now, let $1 \leq i \leq n$ and $r \in \mathfrak{q}_i$ be arbitrary. Then there exists $l \in \mathbb{N}$ such that $r^l D_I(M_i) = 0$. Since M_i is secondary, either $M_i = rM_i$ or $r \in \sqrt{\mathrm{Ann}_R(M_i)} := \mathfrak{p}$. It is not false to assume that the index of $\mathfrak{p}$ is i. In other words one can assume $\mathfrak{p} \in \mathrm{Att}_R(M)$ is exactly $\mathfrak{p}_i$ itself. (This is possible by rearranging the elements of set $\mathrm{Att}_R(M)$). If $r \notin \mathfrak{p}_i$, then we have $M_i = r^l M_i$. Consequently $D_I(M_i) = r^l D_I(M_i) = 0$ which contradicts the fact that $D_I(M_i) \neq 0$. Hence $\mathfrak{q}_i \subseteq \mathfrak{p}_i$. On the other hand, it is obvious that $\mathfrak{p}_i \subseteq \mathfrak{q}_i$. These yield that $\mathfrak{q}_i = \mathfrak{p}_i$ for all $1 \leq i \leq n$ and therefore $\mathrm{Att}_R(D_I(M)) \subseteq \mathrm{Att}_R(M)$. Finally we claim that $I \nsubseteq \mathfrak{q}_i$ for every $1 \leq i \leq n$. Otherwise, by [3, Corollary 2.2.10] we find $\Gamma_{\mathfrak{q}_i}(D_I(M_i)) \subseteq \Gamma_I(D_I(M_i)) = 0$. But $D_I(M_i)$ is an $\mathrm{Ann}_R(D_I(M_i))$-torsion module. Hence $D_I(M_i) = 0$ which is a contradiction. □

Список литературы

[1] M. F. Atiyah, I. G. Macdonald, Introduction to Commutative Algebra, Addison- Wesley (1969).

[2] K. Bahmanpour, “Exactness of ideal transforms and annihilators of top local cohomology modules”, J. Korean Math. Soc, **52**(6), 1253 – 1270 (2015).

[3] M. P. Brodmann, R. Y. Sharp, Local Cohomology: An Algebraic Introduction With Geometric Applications, Cambridge University Press, Cambridge (1998).

[4] W. Bruns, J. Herzog, Cohen-Macaulay Rings, Cambridge University Press (1998).

[5] L. Chu, V. H. Jorge Pe‘rez and P. Lima, “Ideal transforms and local cohomology defined by a pair of ideals”, Journal of algebra and its applications, 17, no. 10, 1850200, 1 – 20 (2015).

[6] A. Grothendieck, Local Cohomology, Lecture Notes in Mathematics, **41**, Springer, Berlin (1967).

[7] R. Hartshorne, “Cohomological dimension of algebraic varieties”, Ann. Math, **45**, 63 – 70 (1968).

[8] H. Matsumura, Commutative Ring Theory, Cambridge Univ. Press, Cambridge, UK (1986).

Поступила 09 декабря 2021

После доработки 05 апреля 2022

Принята к публикации 10 апреля 2022