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GEOMETRIC PROPERTIES OF NORMALIZED LE ROY-TYPE MITTAG-LEFFLER FUNCTIONS

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Abstract. The main focus of the present paper is to establish sufficient conditions for the parameters of the normalized form of the generalized Le Roy-type Mittag-Leffler function have certain geometric properties like close-to-convexity, univalency, convexity and starlikeness inside the unit disc. The results obtained are new and their usefulness is depicted by deducing several interesting corollaries. The results obtained improve some several results available in the literature for the Mittag-Leffler function.

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1. Introduction

To study the asymptotic behavior of the analytic continuation of certain power series, Édouard Le Roy considered the following example [18, Section 6]

$$\sum_{k=0}^{\infty} \frac{z^k}{(k!)^\gamma},\ \gamma > 0, \tag{1.1}$$

when $z \to \infty$ along the real axis. Recently, S. Gerhold [6] and, independently, R. Garra and F. Polito [5] introduced a generalization of (1.1) by

$$F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[\Gamma(\alpha n + \beta)]^\gamma} \qquad (z \in \mathbb{C}, \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0), \tag{1.2}$$

which turns out to be an entire function of the complex variable z for all values of the parameters such that $\Re(\alpha) > 0, \beta \in \mathbb{R}$ and $\gamma > 0$.

Obvious specifications of parameters lead to a set of well-known special functions like the Mittag-Leffler $E_\alpha = F_{\alpha,1}^{(1)}$, two parameter Mittag-Leffler $E_{\alpha,\beta} = F_{\alpha,\beta}^{(1)}$, multi-parameter Mittag-Leffler function ($E_{\alpha,\beta}^{\gamma} = F_{\alpha,\beta}^{(\gamma)}, \gamma \in \mathbb{N}$) and their subsequent special cases.

Various geometric properties has been studied for different classes of special functions such as Mittag-Leffler function, Wright function, hypergeometric functions, Bessel functions, Fox-Wright function and some other related functions are an ongoing part of research in geometric function theory. We refer to some geometric properties of these functions [1, 17, 16, 27, 28, 13, 14, 3, 7, 8, 2, 11, 12] and references therein.

Let $\mathcal{H}$ denote the class of all analytic functions inside the unit disk $\mathcal{D} = \{z : |z| < 1\}$. Suppose that $\mathcal{A}$ is the class of all functions $f \in \mathcal{H}$ which are normalized by $f(0) = f'(0) - 1 = 0$ such that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, for all $z \in \mathcal{D}$.

A function $f \in \mathcal{A}$ is said to be a starlike function (with respect to the origin 0) in $\mathcal{D}$, if f is univalent in $\mathcal{D}$ and $f(\mathcal{D})$ is a starlike domain with respect to 0 in $\mathbb{C}$. This class of starlike functions is denoted by $\mathcal{S}^*$. The analytic characterization of $\mathcal{S}^*$ is given [3] below:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \ \ \forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{S}^*.$$

If $f(z)$ is a univalent function in $\mathcal{D}$ and $f(\mathcal{D})$ is a convex domain in $\mathbb{C}$, then $f \in \mathcal{A}$ is said to be a convex function in $\mathcal{D}$. We denote this class of convex functions by $\mathcal{K}$. This class can be analytically characterized as follows:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ \forall z \in \mathcal{D} \quad \Longleftrightarrow \quad f \in \mathcal{K}.$$

It is well-known that zf' is starlike if and only if $f \in \mathcal{A}$ is convex.

A function $f(z) \in \mathcal{A}$ is said to be close-to-convex in $\mathcal{D}$ if $\exists$ a starlike function $g(z)$ in $\mathcal{D}$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,$$

for all $z \in \mathcal{D}$. The class of all close-to-convex functions is denoted by $\mathcal{C}$.

A function $f \in \mathcal{A}$ is said to be uniformly convex (starlike) if for every circular arc γ contained in $\mathcal{D}$ with center $\zeta \in \mathcal{D}$ the image arc $f(\gamma)$ is convex (starlike w.r.t. the image $f(\zeta)$). The class of all uniformly convex (starlike) functions is denoted by UCV (UST) [20]. In [10, 9], A. W. Goodman introduced these classes. Later, F. Rönning [20] introduced a new class of starlike functions $\mathcal{S}_p$ defined by

$$\mathcal{S}_p(\mathcal{D}) := \{f : f(z) = zF'(z), \ F \in UCV\}.$$

The main focus of this paper is to study certain geometric properties including univalency, starlikeness, convexity and close-to-convexity in the open unit disk of the normalized Le Roy-type Mittag-Leffler function defined by

$$\begin{aligned}\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) &= z[\Gamma(\beta)]^{\gamma}\, F^{(\gamma)}_{\alpha,\beta}(z)\\ &= \sum_{k=1}^{\infty}\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^{\gamma} z^k =: z+\sum_{k=2}^{\infty} A_k(\alpha,\beta,\gamma)z^k.\end{aligned} \tag{1.3}$$

Geometric properties of normalized form of Mittag-Leffler function $\mathbb{F}^{(1)}_{\alpha,\beta}(z) := \mathbb{F}_{\alpha,\beta}(z)$ were discussed by Bansal and Prajapat in [1]. Recently, in [17, 16] geometric properties of normalized form of $\mathbb{F}_{\alpha,\beta}(z)$ were studied, which improve some results of [1]. The above results inspire us to study the geometric properties of Le Roy-type Mittag-Leffler function and improve the results available in the literature.

Each of the following definition will be used in our investigation.

Definition 1.1. (Mitrinović and Vasić [15]) A sequence of real numbers $\{a_n\},\ n = 0, 1, 2...$ satisfying the condition

$$2a_{n+1} \leq a_n + a_{n+2},\ n = 0, 1, 2... \tag{1.4}$$

is called convex sequence. Putting $\Delta a_n = a_n - a_{n+1}$ and $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$. Condition (1.4) may be written as $\Delta^2 a_n \geq 0, n = 0, 1, 2...$ It is well known that If $f(x)$ is convex function (of real variable) for $x \geq 0$, then the sequence $a_n = f(n), n = 0, 1, 2...$ is convex.

Definition 1.2. An infinite sequence $\{b_n\}_1^{\infty}$ of complex numbers will be called a subordinating factor sequence if whenever

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{1.5}$$

is analytic, univalent and convex in $\mathcal{D}$, then

$$\left\{\sum_{n=1}^{\infty} a_n b_n z^n : z \in \mathcal{D}\right\} \subseteq f(\mathcal{D}),\ (a_1 = 1). \tag{1.6}$$

For more information on the various geometric properties involving subordination between analytic functions, we refer the reader to the earlier works [3, 23] and also to the references cited therei

2. Useful lemmas

In order to prove our results the following preliminary results will be helpful.

Lemma 2.1. *Let* $\min(\alpha,\gamma) \geq 1, \beta > 0$ *such that* $\alpha + \beta \geq 2$. *Then the following inequality*

$$\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) \leq z + z\theta^{(\gamma)}_{\alpha,\beta}(e^z - 1), \tag{2.1}$$

holds true for all $z > 0$, *where*

$$\theta^{(\gamma)}_{\alpha,\beta} = \left[\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right]^{\gamma}. \tag{2.2}$$

Proof. First, we prove that the sequence

$$x_k := \left\{\frac{\Gamma(k+1)}{[\Gamma(\alpha k+\beta)]^{\gamma}}\right\}_{k\geq 1}, \tag{2.3}$$

is decreasing. Let $\min(\alpha,\gamma) \geq 1$ and $\beta > 0$, then we have

$$\begin{aligned} \frac{x_{k+1}}{x_k} &= \frac{(k+1)[\Gamma(\alpha k+\beta)]^{\gamma}}{[\Gamma(\alpha k+\alpha+\beta)]^{\gamma}} \\ &\leq \frac{(k+1)[\Gamma(\alpha k+\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta+1)]^{\gamma}} = \frac{k+1}{(\alpha k+\beta)^{\gamma}} \leq \frac{k+1}{\alpha k+\beta}. \end{aligned} \tag{2.4}$$

It is easy to proved that the function $\chi(\xi)$ defined by

$$\chi(\xi) = (\alpha-1)\xi + \beta - 1,$$

is non-negative for all $\alpha \geq 1$ such that $\alpha + \beta \geq 2$.

This in turn implies that the sequence $(x_k)_{k\geq 1}$ monotonically decreases. Therefore, for $z > 0$ we get

$$\begin{aligned} \frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} &= 1 + \sum_{k=1}^{\infty} \frac{[\Gamma(\beta)]^{\gamma}\Gamma(k+1)}{[\Gamma(\alpha k+\beta)]^{\gamma}} \frac{z^k}{k!} \\ &\leq 1 + \theta^{(\gamma)}_{\alpha,\beta} \sum_{k=1}^{\infty} \frac{z^k}{k!} = 1 + \theta^{(\gamma)}_{\alpha,\beta}(e^z - 1). \end{aligned}$$

This proves (2.1). □

Lemma 2.2. (Ozaki [19]) *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k z^k$. *If* $1 \leq 2A_2 \leq ... \leq nA_n \leq (n+1)A_{n+1} \leq ... \leq 2$, *or* $1 \geq 2A_2 \geq ... \geq nA_n \geq (n+1)A_{n+1} \geq ... \geq 0$, *then* f *is close-to-convex with respect to* $-\log(1-z)$.

Lemma 2.3. [11] *Let* $f \in \mathcal{A}$ *and* $|(f(z)/z) - 1| < 1$ *for each* $z \in \mathcal{D}$, *then* f *is univalent and starlike in* $\mathcal{D}_{1/2} = \{z : |z| < 1/2\}$.

Lemma 2.4. [12] *Let* $f \in \mathcal{A}$ *and* $|f'(z) - 1| < 1$ *for each* $z \in \mathcal{D}$, *then* f *is convex in* $\mathcal{D}_{1/2} = \{z : |z| < 1/2\}$.

Lemma 2.5. [24] *If $f \in \mathcal{A}$ and satisfy*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M, \quad z \in \mathcal{D},$$

where M is a solution of the equation $\cos M = M$, *then* $\Re(f'(z)) > 0$.

Lemma 2.6. [25] *Assume that $f \in \mathcal{A}$.*

(1) *If* $\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}$, *then* $f \in UCV(\mathcal{D})$.

(2) *If* $\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{2}$, *then* $f \in \mathcal{S}_p(\mathcal{D})$.

Lemma 2.7. (Féjer [4]). *If $A_n \geq 0$, $\{nA_n\}$ and $\{nA_n - (n+1)A_{n+1}\}$ both are nonincreasing, then the function $f(z) = z + \sum\limits_{n=2}^{\infty} A_n z^n$ is in $\mathcal{S}^*$.*

Lemma 2.8. (Féjer [4]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_1 = 1$, and that for $n \geq 2$ the sequence $\{a_n\}$ is a convex decreasing, i.e.*

$$a_1 - a_2 \geq \cdots \geq a_k - a_{k+1} \geq \cdots \geq 0.$$

Then

$$\Re\left(\sum_{n=1}^{\infty} a_n z^{n-1}\right) > 1/2, \quad z \in \mathcal{D}. \tag{2.5}$$

Lemma 2.9. (Wilf [29]). *The sequence $\{b_n\}_1^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \; z \in \mathcal{D}. \tag{2.6}$$

3. Main results

Theorem 3.1. *Let $\min(\alpha, \gamma) \geq 1$ and $\beta > 0$ such that $\alpha + \beta \geq 2$. Then the following assertions hold true:*

(a). *If $(e-1)[\Gamma(\beta)]^{\gamma} < [\Gamma(\alpha+\beta)]^{\gamma}$, then the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is starlike in $\mathcal{D}_{1/2}$.*

(b). *If $2(e-1)[\Gamma(\beta)]^{\gamma} < [\Gamma(\alpha+\beta)]^{\gamma}$ and $\beta \geq 2$, then the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex in $\mathcal{D}_{1/2}$.*

Proof. (a) In view of (2.1) and straightforward calculation would yield

$$\begin{aligned}\left|\frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{z} - 1\right| &\leq \sum_{k=1}^{\infty} A_{k+1}|z|^k \\ &\leq \mathbb{F}_{\alpha,\beta}^{(\gamma)}(1) - 1 \\ &\leq \theta_{\alpha,\beta}^{(\gamma)}(e-1),\end{aligned} \tag{3.1}$$

for all $z \in \mathcal{D}$. Hence, under the given hypotheses we obtain

$$\left| \frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z} - 1 \right| < 1,\ z \in \mathcal{D},$$

and consequently the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike in $\mathcal{D}_{1/2}$ by the means of Lemma 2.3.
(b) A simple computation becomes

$$\begin{aligned} \left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1 &= \sum_{k=1}^{\infty} (k+1) A_{k+1} z^k \\ &= \sum_{k=1}^{\infty} \frac{y_k z^k}{k!}, \end{aligned} \tag{3.2}$$

where $(y_k)_k$ is defined by

$$y_k = \frac{[\Gamma(\beta)]^{\gamma} \Gamma(k+2)}{[\Gamma(\alpha k + \beta)]^{\gamma}},\ k \geq 1. \tag{3.3}$$

We define the function $\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi)$ by

$$\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi) = \frac{\Gamma(\xi+2)}{[\Gamma(\alpha\xi + \beta)]^{\gamma}},\ \xi > 0.$$

Therefore

$$(\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi))' = \tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi)[\psi(\xi+2) - \alpha\gamma\psi(\alpha\xi+\beta)]. \tag{3.4}$$

Under the given conditions, we deduce that $\psi(\alpha\xi+\beta) \geq \psi(\xi+2)$ and consequently the function $\tilde{f}^{(\gamma)}_{\alpha,\beta}(\xi)$ is decreasing on $[1, \infty)$. This implies that the sequence $(y_k)_{k\geq 1}$ monotonically decreases for all $\alpha \geq 1, \beta \geq 2$ and $\gamma \geq 1$. Therefore

$$\left| \left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1 \right| < \sum_{k=1}^{\infty} \frac{y_1}{k!} = y_1(e-1). \tag{3.5}$$

This implies that

$$\left| \left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1 \right| < 1,\ \ z \in \mathcal{D}.$$

Hence, the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$ by Lemma 2.4. This completes the proof of Theorem 3.1. □

On setting $\alpha = 1$ and $\gamma = 2$ in Theorem 3.1, we get the following results as follows:

Corollary 3.1. *The following assertions hold true:*
(a). If $\beta > \sqrt{e-1}$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is starlike in $\mathcal{D}_{1/2}$.
(b). If $\beta \geq 2$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$.

Remark 3.1. Theorem 3.1, indicates that the function $\mathbb{F}_{1,\beta}(z)$ is convex in $\mathcal{D}_{1/2}$ if $\beta \geq 2$. It concludes that our result improve the result proved in [1, Theorem 2.4 (b)].

Theorem 3.2. *Suppose that $\alpha, \beta, \gamma > 0$ such that $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$, then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$, and consequently it is univalent in $\mathcal{D}$.*

Proof. To prove that $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$, it is sufficient to prove, in view of Lemma 2.2, that the sequence $\{kA_k\}_{k\geq 1}$ is decreasing. A simple computation gives

$$\begin{aligned} kA_k - (k+1)A_{k+1} &= k\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^\gamma - (k+1)\left[\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\right]^\gamma \\ &= \frac{k[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha k+\beta)]^\gamma}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - \frac{k+1}{k}\right]. \end{aligned}$$

By using the fact that the function $z \mapsto \frac{\Gamma(z+a)}{\Gamma(z)}$, $a > 0$ is increasing we deduce that the sequence

$$\left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma\right\}_{k\geq 1},$$

is increasing provided that $\alpha > 0$ and $\gamma > 0$, on the other hand the sequence $\{\frac{k+1}{k}\}_{k\geq 1}$ is decreasing sequence. This implies that the sequence

$$\{v_k\}_{k\geq 1} := \left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - \frac{k+1}{k}\right\}_{k\geq 1}$$

is increasing and consequently

$$v_k \geq v_1 = \frac{[\Gamma(\alpha+\beta)]^\lambda - 2[\Gamma(\beta)]^\lambda}{[\Gamma(\beta)]^\lambda},$$

which is non-negative under the given hypotheses. Hence

$$kA_k - (k+1)A_{k+1} \geq 0$$

for all $k \geq 1$. This completes the proof of the Theorem 3.2. □

Corollary 3.2. *For $\alpha \geq 1, \gamma > 0$ and $\beta \geq 2^{1/\gamma}$, then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$, close-to-convex with respect to starlike function $-\log(1-z)$ in $\mathcal{D}$.*

Theorem 3.3. *Let $\alpha > 0, \beta > 0, \gamma > 0$. Assume that any one of the following conditions $(H_1), (H_1^1)$ or (H_1^2) hold true:*

$$(H_1): \begin{cases} (i). & \min(\alpha,\beta,\gamma) \geq 1, \alpha\gamma \geq 2, \\ (ii). & [\Gamma(\beta)]^\gamma(e-1) < [\Gamma(\alpha+\beta)]^\gamma \\ (iii). & \frac{e[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma} + \frac{4(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma} < 1, \end{cases}$$

$$(H_1^1):\begin{cases}(i). & \min(\alpha,\gamma)\geq 1, \alpha+\beta\geq \max(4^{\frac{1}{\gamma}},2),\\ (ii). & \text{The function } L^{\gamma}_{\alpha,\beta}: z\mapsto (z+1)^2-z(\alpha z+\beta)^{\gamma} \text{ is decreasing on } [1,\infty),\\ (iii). & \theta^{(\gamma)}_{\alpha,\beta}<\frac{1}{e},\end{cases}$$

$$(H_1^2):\ [\Gamma(\alpha+\beta)]^{\gamma}\geq 4[\Gamma(\beta)]^{\gamma},$$

Then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ *is starlike in* $\mathcal{D}$.

Proof. First we assume that the hypothesis (H_1) holds. By using the triangle inequality and using the fact that the sequence (x_n) is decreasing (see the proof of Lemma 2.1), then for all $z\in\mathcal{D}$ we get

$$\begin{aligned}\left|\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}\right| &\geq 1-\sum_{k=1}^{\infty}A_{k+1}|z|^k\geq 1-[\Gamma(\beta)]^{\gamma}\sum_{k=1}^{\infty}\frac{x_k}{k!}\\ &> 1-[\Gamma(\beta)]^{\gamma}\sum_{k=1}^{\infty}\frac{x_1}{k!}=1-\theta^{(\gamma)}_{\alpha,\beta}(e-1)>0,\end{aligned}\tag{3.6}$$

where $\theta^{(\gamma)}_{\alpha,\beta}$ and $(x_k)_k$ are defined in (2.2) and (2.3) respectively. On the other hand, we have

$$(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'-\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}=\sum_{k=1}^{\infty}\frac{B_k z^k}{k!},\ z\in\mathcal{D},\tag{3.7}$$

where $(B_k)_k$ is defined by

$$B_k=\frac{k\Gamma(k+1)[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}},\ k\geq 1.$$

The sequence $(B_k)_{k\geq 2}$ is monotonically decreases for all $\alpha\geq 1,\beta\geq 2$ and $\gamma\geq 1$ such that $\alpha\gamma\geq 2$. Indeed, for this we consider the function $g^{\gamma}_{\alpha,\beta}(z)$ defined by

$$f^{(\gamma)}_{\alpha,\beta}(\xi)=\frac{\xi\Gamma(\xi+1)}{[\Gamma(\alpha\xi+\beta)]^{\gamma}},\ \xi>0.$$

Then

$$(f^{(\gamma)}_{\alpha,\beta}(\xi))'=f^{(\gamma)}_{\alpha,\beta}(\xi)\left[\frac{1}{\xi}+\psi(\xi+1)-\alpha\gamma\psi(\alpha\xi+\beta)\right].\tag{3.8}$$

Since the digamma function $\psi(\xi)$ is increasing on $(0,\infty)$, then for $\alpha\geq 1,\beta\geq 2$ and $\gamma\geq 1$ we have

$$\psi(\alpha\xi+\beta)\geq\psi(\xi+2),\ \xi\geq 1.$$

With the aid the functional relation

$$\psi(\xi+1)=\psi(\xi)+\frac{1}{\xi},\ \xi>0$$

combining with the above inequality and (3.8) we thus get

$$(f_{\alpha,\beta}^{(\gamma)}(\xi))' \leq f_{\alpha,\beta}^{(\gamma)}(\xi)\left[(1-\alpha\gamma)\psi(\xi)+\frac{2-\alpha\gamma}{\xi}-\frac{\alpha\gamma}{\xi+1}\right]\leq 0, \tag{3.9}$$

for all $\xi \geq 2$. Consequently, the sequence $(B_k)_{k\geq 2}$ is decreasing. It follows that

$$\begin{aligned}\left|(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))' - \frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| &\leq \sum_{k=1}^{\infty}\frac{B_k}{k!}\\ &\leq B_1 + B_2\sum_{k=2}^{\infty}\frac{1}{k!} = B_1 + B_2(e-2).\end{aligned} \tag{3.10}$$

Keeping (3.6) and (3.10) in mind, we get

$$\left|\frac{z(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))'}{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}-1\right| \leq \frac{\theta_{\alpha,\beta}^{(\gamma)}+B_2(e-2)}{1-\theta_{\alpha,\beta}^{(\gamma)}(e-1)} < 1, \tag{3.11}$$

for all $z \in \mathcal{D}$, under the given hypothesis. This implies that

$$\Re\left(\frac{z(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))'}{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}\right) > 0,$$

for all $z \in \mathcal{D}$ which implies that the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is starlike in $\mathcal{D}$ under the conditions (H_1). Now, we assume that (H_1^1) is valid. Since $\alpha \geq 1$ such that $\alpha+\beta \geq x^*$ we obtain

$$[\Gamma(\alpha k+\alpha+\beta)]^{\gamma} \geq [\Gamma(\alpha k+1+\beta)]^{\gamma}. \tag{3.12}$$

Thus, we get

$$\frac{B_{k+1}}{B_k} \leq \frac{(k+1)^2}{k(\alpha k+\beta)^{\gamma}}.$$

Moreover, since the function $z \mapsto L_{\alpha,\beta}^{\gamma}(z)$ is decreasing on $[1,\infty)$ such that $L_{\alpha,\beta}^{\gamma}(1) \leq 0$ we conclude that the sequence $(B_k)_{k\geq 1}$ is decreasing. Therefore, we have

$$\left|\frac{z(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))'}{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}-1\right| \leq \frac{B_1}{1-\theta_{\alpha,\beta}^{(\gamma)}(e-1)} < 1, \tag{3.13}$$

for all $z \in \mathcal{D}$. Finally, we suppose that the hypothesis (H_1^2) is valid. In view of Lemma 2.7, we have to show that both $\{kA_k\}$ and $\{kA_k-(k+1)A_{k+1}\}$ are nonincreasing sequences for all $n \geq 1$. In Theorem 3.2 we have already proved that $\{kA_k\}$ is nonincreasing sequence for all $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^{\gamma} \geq 2[\Gamma(\beta)]^{\gamma}$. Now it remains to show that $\{kA_k-(k+1)A_{k+1}\}$ is nonincreasing or $\{kA_k\}$ is convex sequence (see Definition 1.1). That is $kA_k-2(k+1)A_{k+1}+(k+2)A_{k+2} \geq 0$ (for all $k \geq 1$).

Neglecting the third term and taking difference of first two term, i.e.

$$kA_k - 2(k+1)A_{k+1} = \frac{k[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha k+\beta)]^\gamma}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - \frac{2(k+1)}{k}\right].$$

As the function $z \mapsto \frac{\Gamma(z+a)}{\Gamma(z)}$, $a>0$ is increasing and hence the sequence

$$\left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma\right\}_{k\geq 1} \tag{3.14}$$

is increasing provided that $\alpha > 0$ and $\gamma > 0$, on the other hand the sequence $\left\{\frac{2(k+1)}{k}\right\}_{k\geq 1}$ is decreasing sequence. This implies that the sequence

$$\{u_k\}_{k\geq 1} := \left\{\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - \frac{2(k+1)}{k}\right\}_{k\geq 1} \tag{3.15}$$

is increasing and consequently

$$u_k \geq u_1 = \frac{[\Gamma(\alpha+\beta)]^\lambda - 4[\Gamma(\beta)]^\lambda}{[\Gamma(\beta)]^\lambda}$$

which is non-negative under the given hypotheses. Hence

$$kA_k - 2(k+1)A_{k+1} + (k+2)A_{k+2} \geq 0$$

for all $k \geq 1$. This completes the proof of Theorem 3.3. $\square$

Corollary 3.3. *If $\beta > \sqrt{e} \approx 1.6487212707$, then the function $\mathbb{F}^{(2)}_{1,\beta}(z)$ is starlike on $\mathcal{D}$.*

Proof. Upon setting $\alpha = 1$ and $\gamma = 2$ in the hypotheses (H_1^1) of Theorem 3.3. Then, the condition "$(H_1^1) : (i)$" and "$(H_1) : (ii)$" hold true for all $\beta > 1$. In addition, the condition "$(H_1^) : (iii)$" holds true if and only if $\beta^2 > e$. $\square$

Corollary 3.4. *If $\beta > 1.29$, then the function $\mathbb{F}_{2,\beta}(z)$ is starlike on $\mathcal{D}$.*

Proof. Specifying $\alpha = 2$ and $\gamma = 1$ in the conditions (H_1) of Theorem 3.3. Then the conditions "$(H_1) : (i)$" and "$(H_1) : (ii)$" are valid for all $\beta \geq 1$. Using mathematical software, we can verify that the condition "$(H_1) : (iii)$" holds true for all $\beta > 1.29$. $\square$

Remark 3.2. In [1, Example 2.1], the authors proved that the function $\mathbb{F}_{2,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq (-1+\sqrt{17})/2 \approx 1.5615...$ Further, according to [1, Theorem 2.2], $\mathbb{F}_{2,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq (3+\sqrt{17})/2 \approx 3.56155$. Moreover, [17, Theorem 6] indicates that $\mathbb{F}_{2,\beta}(z)$ is starlike in $\mathcal{D}$ if $\beta \geq 3.214319744$. Hence, Corollary 3.4 provides results for $\mathbb{F}_{2,\beta}(z)$, better than the results available in [1, Theorem 2.1, Theorem 2.2] and [17, Theorem 6].

Theorem 3.4. *Let $\alpha, \beta > 0$ and γ be positive real numbers, and also let the following conditions (H_3) or (H_3^1) be satisfied:*

$$(H_2) : \begin{cases} (i) & \alpha \geq 1, \beta \geq 3, \alpha\gamma \geq 2, \\ (ii) & 2(e-1)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma, \\ (iii) & \frac{2e[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma} + \frac{12(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma} < 1. \end{cases}$$

$$(H_2^1) : [\Gamma(\alpha+\beta)]^\gamma \geq 8[\Gamma(\beta)]^\gamma,$$

then the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex function in $\mathcal{D}$.

Proof. It is well known that $f(z)$ is convex if and only if $zf'(z)$ is starlike. So in order to prove $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ is convex it is sufficient to prove that the function

$$\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z) := z(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))'$$

is starlike. We have

$$(\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z))' - \mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)/z = \sum_{k=1}^{\infty} \frac{C_k z^k}{k!}, \tag{3.16}$$

where $(C_k)_{k\geq 1}$ is defined by

$$C_k = \frac{[\Gamma(\beta)]^\gamma k\Gamma(k+2)}{[\Gamma(\alpha k+\beta)]^\gamma},\ k \geq 1.$$

Next, we define the function $g_{\alpha,\beta}^{(\gamma)}$ by

$$g_{\alpha,\beta}^{(\gamma)}(\xi) = \frac{\xi\Gamma(\xi+2)}{[\Gamma(\alpha\xi+\beta)]^\gamma},\ \xi \geq 1.$$

Thus we get

$$(g_{\alpha,\beta}^{(\gamma)}(\xi))' = g_{\alpha,\beta}^{(\gamma)}(\xi)\left[\frac{1}{\xi} + \psi(\xi+2) - \alpha\gamma\psi(\alpha\xi+\beta)\right].$$

Again, by using the fact that the digamma function is increasing on $(0, \infty)$ we have

$$\psi(\alpha\xi+\beta) \geq \psi(\xi+3)$$

for all $\xi \geq 1, \alpha \geq 1$ and $\beta \geq 3$. Keeping in mind the above relations we obtain

$$\begin{aligned}(g_{\alpha,\beta}^{(\gamma)}(\xi))' &\leq g_{\alpha,\beta}^{(\gamma)}(\xi)\left[\frac{2-\alpha\gamma}{\xi} + \frac{1-\alpha\gamma}{\xi+1} - \frac{\alpha\gamma}{\xi+2} + (1-\alpha\gamma)\psi(\xi)\right] \\ &\leq 0,\end{aligned}$$

for all $\xi \geq 2$ and $\alpha\gamma \geq 2$. This implies that the sequences $(C_k)_{k\geq 2}$ is decreasing. Then, by (3.16) we get

$$\left|(\mathbb{G}_{\alpha,\beta}^{(\gamma)}(z))' - \mathbb{G}_{\alpha,\beta}^{(\gamma)}(z)/z\right| \leq C_1 + \sum_{k=2}^{\infty} \frac{C_2}{k!} = C_1 + C_2(e-2). \tag{3.17}$$

We see that the sequence $(y_k)_{k\geq 1}$ defined in (3.3) is also decreasing under the conditions (H_2). Therefore, by (3.2) we get

$$|\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)/z| = |(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'| \geq 1 - \sum_{k=1}^{\infty} \frac{y_k z^k}{k!} \geq 1 - y_1(e-1). \tag{3.18}$$

Having (3.18) and (3.17) in mind we obtain

$$\left| \frac{z(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z))'}{(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)} - 1 \right| \leq \frac{C_1 + C_2(e-2)}{1 - y_1(e-1)}. \tag{3.19}$$

The above inequality needs to be less than 1, this gives the conditions (H_3) : (iii). Thus we get

$$\Re\left(\frac{z(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z))'}{(\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)}\right) > 0$$

for all $z \in \mathcal{D}$. This implies that the function $\mathbb{G}^{(\gamma)}_{\alpha,\beta}(z)$ is starlike on $\mathcal{D}$ and consequently the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is convex on $\mathcal{D}$ under the conditions (H_2). Now, assume that the condition (H_2^1) is valid.

$$\mathbb{G}^{\gamma}_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} k\left[\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}\right]^{\gamma} z^k =: z + \sum_{k=2}^{\infty} \tilde{B}_k z^k. \tag{3.20}$$

In view of Lemma 2.7, we have to show that the sequence $\{k\tilde{B}_k\}$ is both decreasing and convex for all $k \geq 1$.

$$k\tilde{B}_k - (k+1)\tilde{B}_{k+1} = \frac{k^2[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^{\gamma} - \frac{(k+1)^2}{k^2}\right]. \tag{3.21}$$

Now using the same argument as in the proof of Theorem 3.3 under the conditions (H_1^1), we have $k\tilde{B}_k - (k+1)\tilde{B}_{k+1} \geq 0$ for all $k \geq 1$ under the condition $[\Gamma(\alpha+\beta)]^{\gamma} \geq 4[\Gamma(\beta)]^{\gamma}$, which is true under the hypothesis of Theorem 3.4. Now it remains to show that $\{k\tilde{B}_k\}$ is convex sequence. That is $k\tilde{B}_k - 2(k+1)\tilde{B}_{k+1} + (k+2)\tilde{B}_{k+2} \geq 0$, for all $k \geq 1$. Neglecting the third term and taking difference of first two term i.e.

$$k\tilde{B}_k - 2(k+1)\tilde{B}_{k+1} = \frac{k^2[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha k+\beta)]^{\gamma}}\left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^{\gamma} - \frac{2(k+1)^2}{k^2}\right].$$

which is non-negative under the hypothesis that $[\Gamma(\alpha+\beta)]^{\gamma} \geq 8\Gamma(\beta)]^{\gamma}$. □

If we set $(\alpha = 1, \gamma = 2)$ and $(\alpha = 2, \gamma = 1)$ respectively in the second hypotheses of Theorem 3.4, we get the following results as follows:

Corollary 3.5. *The following assertions hold true:*
(a). If $\beta \geq 2\sqrt{2}$, *then the function* $\mathbb{F}^{(2)}_{1,\beta}(z)$ *is convex in* $\mathcal{D}$.
(b). If $\beta \geq \frac{-1+\sqrt{33}}{2} \approx 2.3722...$, *then* $\mathbb{F}_{2,\beta}(z)$ *is convex in* $\mathcal{D}$.

Remark 3.3. Recently, the authors [17, Theorem 7] proved that $\mathbb{F}_{\alpha,\beta}(z)$ is convex in $\mathcal{D}$ if $\alpha \geq 1$ and $\beta \geq 3.56155281$. Therefore, the second assertions of Corollary 3.5 improve the results in [17] for $\alpha = 2$.

Theorem 3.5. *Let $\alpha \geq 1, \beta \geq 1, \gamma \geq 1$ such that $\alpha\gamma \geq 2$. Also, suppose that the following conditions*

$$[\Gamma(\beta)]^\gamma(e-1) < [\Gamma(\alpha+\beta)]^\gamma_; \text{ and } \frac{(1+M(e-1))[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma} + \frac{4(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma} < M,$$

are valid, where M is a solution of the equation $\cos(M) = M$. *Then*

$$\Re\left(\left[\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right]'\right) > 0.$$

Proof. The proof of this result go along the lines introduced in the proof of Theorem 3.3, when we used Lemma 2.5 such that the function

$$\frac{\theta^{(\gamma)}_{\alpha,\beta} + B_2(e-2)}{1-\theta^{(\gamma)}_{\alpha,\beta}(e-1)} < M,$$

where M is a solution of the equation $\cos(M) = M$, we omit the details. □

Theorem 3.6. *Let $\alpha \geq 1, \beta \geq 1, \gamma \geq 1$ such that $\alpha\gamma \geq 2$. Also, suppose that the following conditions*

$$[\Gamma(\beta)]^\gamma(e-1) < [\Gamma(\alpha+\beta)]^\gamma_; \text{ and } \frac{(e+1))[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma} + \frac{8(e-2)[\Gamma(\beta)]^\gamma}{[\Gamma(2\alpha+\beta)]^\gamma} < 1,$$

are valid. Then

$$\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) \in S_p(\mathcal{D}).$$

Proof. The proof of this result is very similar to the proof of Theorem 3.3 when we used the part (2) of Lemma 2.6, such that

$$\frac{\theta^{(\gamma)}_{\alpha,\beta} + B_2(e-2)}{1-\theta^{(\gamma)}_{\alpha,\beta}(e-1)} < 1/2,$$

thus, we omit the details in this case also. □

Theorem 3.7. *Let $\alpha \geq 1, \beta \geq 1$ and $\gamma \geq 1$ such that $\alpha+\beta \geq 3$. In addition, assume that the following conditions hold true:*

$$(6e-2)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma.$$

Then the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ is uniformly convex in $\mathcal{D}$.

Proof. Simple computation gives

$$(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'' = \sum_{k=0}^{\infty} \frac{D_k z^k}{k!}, \tag{3.22}$$

where the sequence $(D_k)_{k\geq 0}$ is defined by

$$D_k = \frac{[\Gamma(\beta)]^\gamma \Gamma(k+3)}{[\Gamma(\alpha k+\beta+\alpha)]^\gamma},\ k \geq 0.$$

We define the function $h^{(\gamma)}_{\alpha,\beta}$ defined by

$$h^{(\gamma)}_{\alpha,\beta}(\xi) = \frac{\Gamma(\xi+3)}{[\Gamma(\alpha\xi+\beta+\alpha)]^\gamma},\ \xi > 0.$$

Therefore

$$(h^{(\gamma)}_{\alpha,\beta}(\xi))' = h^{(\gamma)}_{\alpha,\beta}(\xi)\left[\psi(\xi+3) - \alpha\gamma\psi(\alpha\xi+\beta+\alpha)\right], \xi \geq 0.$$

Again, by using the fact that the digamma function is increasing, we deduce that the function $h^{(\gamma)}_{\alpha,\beta}(\xi)$ is decreasing on $[0,\infty)$ for all $\alpha \geq 1, \beta \geq 1$ and $\gamma \geq 1$ such that $\alpha+\beta \geq 3$. This implies that the sequence $(D_k)_{k\geq 0}$ is decreasing. Then

$$\left|(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))''\right| \leq \frac{2e[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma},\ z \in \mathcal{D}. \tag{3.23}$$

We observe that the sequence $(y_k)_{k\geq 1} = (D_k/(k+2))_{k\geq 1}$ is also decreasing under the conditions of this Theorem. Then implies that the inequality (3.18) holds true. Now, bearing in mind the inequalities (3.18) and (3.23) we conclude

$$\left|\frac{z(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))''}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'}\right| \leq \frac{2e\theta^{(\gamma)}_{\alpha,\beta}}{1-2(e-1)\theta^{(\gamma)}_{\alpha,\beta}},\ z \in \mathcal{D},$$

where $\theta^{(\gamma)}_{\alpha,\beta}$ is defined in (2.2). So, for the uniformly convex of the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ the above bound needs to be less than $\frac{1}{2}$, by the means of part (1) of Lemma 2.6. This gives the condition

$$(6e-2)\theta^{(\gamma)}_{\alpha,\beta} < 1,$$

or equivalently

$$(6e-2)[\Gamma(\beta)]^\gamma < [\Gamma(\alpha+\beta)]^\gamma.$$

With this, the proof of Theorem 3.7 is complete. □

Specifying $\alpha = 2$ and $\gamma = 1$ in Theorem 3.7, we conclude the following result as follows:

Corollary 3.6. *If* $\beta > \frac{-1+\sqrt{24e-7}}{2} \approx 3.3157163$, *then the function* $\mathbb{F}_{2,\beta}(z)$ *is uniformly convex in* $\mathcal{D}$.

Remark 3.4. In [16, Theorem 2.6], Noreen et al. proved that the function $\mathbb{F}_{2,\beta}(z)$ is uniformly convex in $\mathcal{D}$ if $\beta \geq 9.11125$. Hence, Corollary 3.6 improves Theorem 2.6 in [16].

Theorem 3.8. *For $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$ we have*

$$\Re\left(\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{z}\right) > \frac{1}{2}, \quad z \in \mathcal{D}. \tag{3.24}$$

Proof. In view of Lemma 2.8, it is sufficient to prove that the sequence $\{A_k\}_{k\geq 1}$, where A_k is defined by (1.3), is decreasing and convex.

$$A_k - A_{k+1} = \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\right)^\gamma \left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - 1\right] \tag{3.25}$$

and

$$A_k - 2A_{k+1} + A_{n+2} = \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\right)^\gamma \left[\left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)}\right)^\gamma - 2 + \left(\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k+1)+\beta)}\right)^\gamma\right]. \tag{3.26}$$

Now using the same argument as in Theorem 3.2, $A_k - A_{k+1} \geq 0$ for all $n \geq 1$ under the condition $[\Gamma(\alpha+\beta)]^\gamma \geq [\Gamma(\beta)]^\gamma$, which is true under the hypothesis of Theorem 3.8. Similarly $A_k - 2A_{k+1} \geq 0$ for all $k \geq 1$ (neglecting the third term) under the hypothesis that $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$. $\square$

Corollary 3.7. *For $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$, the sequence*

$$\left\{\left(\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}\right)^\gamma\right\}_{n=1}^\infty$$

is a subordinating factor sequence for the class $\mathcal{K}$.

Proof. The result can be easily proved using Theorem 3.8 and Lemma 2.9, so we omit details here.

Theorem 3.9. *For $\alpha, \gamma > 0$ and $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$*

$$\Re\left\{(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))'\right\} > \frac{1}{2}, \quad z \in \mathcal{D}. \tag{3.27}$$

Proof. From (1.3), we get

$$(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))' = 1 + \sum_{k=2}^\infty \tilde{B}_k z^{k-1}, \tag{3.28}$$

where $(\tilde{B}_k)_k$ is defined in (3.20), and proceeding similarly as in Theorem 3.8, we achieve the desired result by the means of Lemma 2.8. $\square$

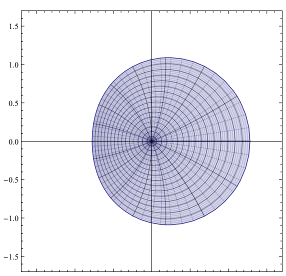
Corollary 3.8. *For* $\alpha, \gamma > 0$ *and* $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma$*, the sequence*

$$\left\{ \frac{(n+1)(\Gamma(\beta))^\gamma}{(\Gamma(\alpha n+\beta))^\gamma} \right\}_{n=1}^{\infty}$$

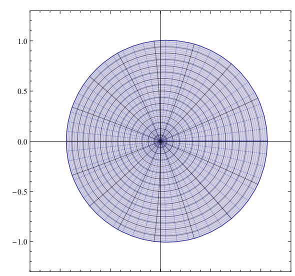
is a subordinating factor sequence for the class $\mathcal{K}$*.*

Proof. The claim follows by the means of Theorem 3.9 and Lemma 2.9, hence we omit details here.

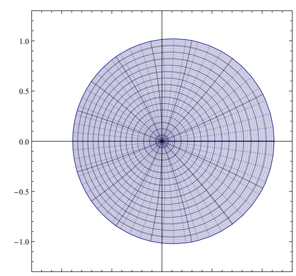
Remark 3.5. The following are graphs of the functions $\mathbb{F}^{(2)}_{1,2}(z), \mathbb{F}_{2,\frac{7}{2}}(z)$ and $\mathbb{F}_{2,\frac{5}{2}}(z)$ over $\mathcal{D}$. These figures depict the validity of our results.



Mapping of $\mathbb{F}^{(2)}_{1,2}(z)$ over $\mathcal{D}$



Mapping of $\mathbb{F}_{2,\frac{7}{2}}(z)$ over $\mathcal{D}$



Mapping of $\mathbb{F}_{2,\frac{5}{2}}(z)$ over $\mathcal{D}$

Список литературы

[1] D. Bansal, J. K. Prajapat, "Certain geometric properties of the Mittag-Leffler functions", Complex Variables and Elliptic Equations, **61** (3), 338 – 350 (2016).

[2] A. Baricz, Generalized Bessel Functions of the First Kind, Lecture Notes in Mathematics, **1994**, Springer-Verlag, Berlin (2010).

[3] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York (1983).

[4] L. Féjer, "Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge", Acta Literarum Sci., **8**, 89 – 115 (1936).

[5] R. Garra, F. Polito, "On some operators involving Hadamard derivatives", Int. Trans. Spec. Func., **24** (10), 773 – 782 (2013).

[6] S. Gerhold, "Asymptotics for a variant of the Mittag-Leffler function", Int. Trans. Spec. Func., **23** (6), 397 – 403 (2012).

[7] A. W. Goodman, Univalent functions, Vol. I, Mariner Publishing Co., Inc., Tampa, FL, (1983).

[8] A. W. Goodman, Univalent functions, Vol. II, Mariner Publishing Co., Inc., Tampa, FL (1983).

[9] A. W. Goodman, "On uniformly starlike functions", J. Math. Anal. Appl., **155** (2), 364 – 370 (1991).

[10] A. W. Goodman, "On uniformly convex functions", Ann. Polon. Math., **56** (1), 87 – 92 (1991).

[11] T. H. MacGregor, "The radius of univalence of certain analytic functions II", Proc Amer. Math. Soc., **14**, 521 – 524 (1963).

[12] T. H. MacGregor, "A class of univalent functions", Proc. Amer. Math. Soc., **15**, 311 – 317 (1964).

[13] K. Mehrez, "Some geometric properties of a class of functions related to the Fox-Wright functions", Banach J. Math. Anal., **14**, 1222 – 1240 (2020).

[14] K. Mehrez, S. Das, A. Kumar, Geometric Properties of the Products of Modified Bessel Functions of the First Kind, Bull. Malays. Math. Sci. Soc. (2021).

[15] D. S. Mitrinović, P. M. Vasić, Analytic Inequalities, Springer-Verlag, New York (1970).
[16] S. Noreen, M. Raza, M. U. Din, S. Hussain, "On Certain Geometric Properties of Normalized Mittag-Leffler Functions", U. P. B. Sci. Bull. Series A, **81** (4), 167 – 174 (2019).
[17] S. Noreen, M. Raza, J. -L. Liu, M. Arif, Geometric Properties of Normalized Mittag-Leffler Functions, Symmetry, **11**, 45 (2019).
[18] E. Le Roy, "Valeurs asymptotiques de certaines séries procédant suivant les puissances entièreset positives d'une variable réelle", Bull des Sci Math., **24** (2), 245 – 268 (1900).
[19] S. Ozaki, "On the theory of multivalent functions. II", Sci. Rep. Tokyo Bunrika Daigaku, Sect. A, **4**, 45 – 87 (1941).
[20] F. Rö nning, "Uniformly convex functions and a corresponding class of starlike functions", Proc. Amer. Math. Soc., **118** (1), 189 – 196 (1993).
[21] S. Ponnusamy, "Close-to-convexity properties of Gaussian hypergeometric functions", J. Comput. Appl. Maths., **88**, 327 – 337 (1997).
[22] S. Ponnusamy, M. Vuorinen, "Univalence and convexity properties for gaussian hypergeometric functions", Rocky Mountain J. Math., **31** (1), 327 – 353 (2001).
[23] H. M. Srivastava, S. M. El-Deeb, "Fuzzy differential subordinations based upon the Mittag–Leffler type Borel distribution", Symmetry, 13, 1023 (2021).
[24] X. I. Xanthopoulos, "Subclasses of starlike functions with $\operatorname{Re} f'(z) > 0$", Studia Univ. Babeş-Bolyai Math., **38** (1), 39 – 47 (1993).
[25] V. Ravichandran, "On uniformly convex functions", Ganita **53**, no. 2, 117 – 124 (2002).
[26] V. Singh, "Univalent functions with bounded derivative in the unit disc", Indian J. Pure Appl. Math., **8**, 1370 – 1377 (1977).
[27] J. K. Prajapat, "Certain geometric properties of the Wright function", Integral Transforms Spec. Funct., **26** (3), 203 – 212 (2015).
[28] J. K. Prajapat, S. Maharana, D. Bansal, "Radius of starlikeness and Hardy space of Mittag-Leffler functions", Filomat, **32** (18), 6475 – 6486 (2018).
[29] H. S. Wilf, "Subordinating factor sequences for convex maps of the unit circle", Proc. Amer. Math. Soc., **12**, 689 – 693 (1961).

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