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ON A RIEMANN BOUNDARY VALUE PROBLEM IN THE SPACE OF p-SUMMABLE FUNCTIONS WITH INFINITE INDEX

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Abstract. The paper considers the Riemann boundary value problem in the half-plane in the space $L^p(\rho)$, where weight function $\rho(x)$ has infinite number of zeros. A necessary and sufficient condition is obtained for the normal solvability and Noetherianness of the considered problem. If the problem is solvable, solutions are represented in an explicit form.

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1. Introduction

Let $\Pi^{\pm}$ be the upper and lower half-planes of the complex plane C, and let A be the class of functions Φ analytic in $\Pi^{+} \cup \Pi^{-}$ satisfying the condition

$$|\Phi(z)| \leq C|z|^{n_0}, \quad |Imz| \geq y_0 > 0,$$

where n_0 is a natural number, $y_0 > 0$ is arbitrary and C is a constant, possibly depending on y_0. By $L^p(\rho), 1 < p < \infty$ we define the following space

$$L^p(\rho) := \Big\{ f : \|f\|_{L^p(\rho)} := \int_{-\infty}^{+\infty} |f(x)|^p \rho(x) dx < \infty \Big\},$$

where

$$\rho(x) = \prod_{k=1}^{\infty} \Big| \frac{x - x_k}{x + i} \Big|^{\alpha_k}, \tag{1.1}$$

at that

$$\sum_{k=1}^{\infty} \alpha_k < \infty, \quad \text{and} \quad 0 < \alpha_k < 1, \; k = 1, 2, ...$$

We investigate the Riemann boundary value problem in the half-plane in the space $L^p(\rho), 1 < p < \infty$ in the following setting:

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Problem R_p. *Let $f \in L^p(\rho), 1 < p < \infty$. Determine an analytic in $\Pi^+ \cup \Pi^-$ function $\Phi \in A$ to satisfy the boundary condition:*

$$\lim_{y\to+0} \|\Phi^+(x+iy) - a(x)\Phi^-(x-iy) - f(x)\|_{L^p(\rho)} = 0, \ (1<p<\infty), \tag{1.2}$$

where $\rho(x)$ is defined by (1.1), $a(x) \neq 0$ is an arbitrary function from the class $C^\delta(-\infty,+\infty), \delta > 0$ and $\Phi^\pm$ are the contractions of function Φ on $\Pi^\pm$ respectively.

The similar problem in $C(\rho)$ (the class of functions f continuous on the real axis with weight ρ) was investigated in the paper [19]. In that case it is shown that the homogeneous problem has one linearly independent solution. Note that a similar homogeneous problem in $L^1(\rho)$ has an infinite number of linearly independent solutions [20].

By T_p we denote

$$T_p = \{x_k : \ \alpha_k > \frac{1}{p}\}.$$

In this work, it is established that in the case $T_p = \emptyset$, the homogeneous problem R_p does not have a solution different from zero. When $T_p \neq \emptyset$ the homogeneous problem R_p has a finite number of linearly independent solutions.

2. Preliminary results

Let $\kappa = inda(t)$, $t \in (-\infty,+\infty)$,

$$\begin{aligned} S^+(z) &= exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z\in\Pi^+,\\ S^-(z) &= \Big(\frac{z+i}{z-i}\Big)^\kappa exp\Big\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln a_1(t)dt}{t-z}\Big\}, \qquad z\in\Pi^-, \end{aligned} \tag{2.1}$$

where

$$a_1(t) = \Big(\frac{t+i}{t-i}\Big)^\kappa a(t), \qquad inda_1(t) = 0.$$

In what follows, we assume that the sequence $\{x_k\}_1^\infty$ has a finite limit x_0.

Lemma 2.1. *Let the sequence $\{x_k\}_1^\infty$ satisfy the following conditions:*

$$\sum_{k=1}^\infty \alpha_k \ln|x_0 - x_k| > -\infty, \tag{2.2}$$

$$|x_k - x_j| > c|x_k - x_0|, \qquad j\neq k \tag{2.3}$$

for some fixed $c > 0$. Then

$$\inf \rho_m = \rho_0 > 0, \qquad m = 1,2,...,$$

where

$$\rho_m = \prod_{k\neq m}^\infty \Big|\frac{x_m - x_k}{x_m + i}\Big|^{\alpha_k}.$$

Proof. From condition (2.3) we have

$$\Big|\frac{x_j - x_k}{x_j + i}\Big|^{\alpha_k} > c^{\alpha_k}\Big|\frac{x_0 - x_k}{x_j + i}\Big|^{\alpha_k}$$

and

$$\prod_{k\neq j}^{\infty}\Big|\frac{x_j - x_k}{x_j + i}\Big|^{\alpha_k} > \prod_{k=1}^{\infty} c^{\alpha_k}\prod_{k\neq j}^{\infty}\Big|\frac{x_0 - x_k}{x_j + i}\Big|^{\alpha_k}.$$

According to the condition (2.2) there exists $\delta > 0$ such that $\inf \rho_m = \delta > 0$, $m = 1, 2, ...$ □

Let us denote

$$\delta_k(x) = \prod_{j\neq k}^{\infty}\Big|\frac{x - x_j}{x + i}\Big|^{\alpha_j} \tag{2.4}$$

and

$$\delta(x) = \delta_{k+1}(x) - \delta_k(x), \qquad x \in [x_k, x_{k+1}).$$

Here we state Lemmas 2.2 and 2.3, which were proved in [19].

Lemma 2.2. *There exist $x'_k \in [x_k, x_{k+1}),\ k = 1, 2, ...$ such that $\delta(x'_k) = 0$.*

Let $X_1 = (-\infty, x'_1)$ and $X_k = [x'_{k-1}, x'_k),\ k = 2, 3, ...$. It is clear that $X_k \cap X_{k+1} = \emptyset,\ k = 1, 2, 3, ...$.

Lemma 2.3. *Let the sequence of points $\{x_k\}_1^\infty$ satisfy either conditions (2.2) and (2.3). Then there exists $\delta > 0$ such that for any $k = 1, 2, ...$:*

$$\inf_{x\in X_k} \delta_k(x) > \delta > 0.$$

Denote $\widetilde{\delta}(x) = \{\delta_k(x), x \in X_k\}$, $k = 1, 2, ...$ From Lemmas 2.2 and 2.3 it follows that function $\widetilde{\delta}(x)$ is continuous, and $\inf \widetilde{\delta}(x) > 0$, $x \in (-\infty, \infty)$.

Here we consider two cases:

1. We assume that $T_p = \emptyset$. Let $f(z) \in L^p(\rho)$. Define the function $\Phi(z)$ as follows

$$\Phi(z) = \frac{S(z)}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(t)dt}{S^+(t)(t-z)}, \qquad z \in \Pi^{\pm}. \tag{2.5}$$

Then $\Phi(z) \in H^p(\rho)$ (see [4], [5]).

2. Consider $T_p \neq \emptyset$. Define the function $\Phi_k(z)$ as follows

$$\Phi_k(z) = \frac{S(z)}{2\pi i(x_k - z)}\int_{X_k}\frac{f(t)(x_k - t)dt}{S^+(t)(t-z)},\ \ k = 1, 2, ... \qquad z \in \Pi^{\pm}. \tag{2.6}$$

Theorem 2.1. *The estimate*

$$\|\Phi_k^+(x + iy) - a(x)\Phi_k^-(x - iy)\|_{L^p(\rho)} \le C\|f\|_{L^p(\rho)},$$

where the constant C is independent of y and k, is true. The limit relation

$$\lim_{y\to+0}\|\Phi_k^+(x + iy) - a(x)\Phi_k^-(x - iy) - f(x)\|_{L^p(\rho)} = 0$$

also holds.

Proof: Consider

$$\Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy) =$$

$$= \frac{S(x+iy)}{2\pi i(x_k - x - iy)} \int_{X_k} \frac{f(t)}{S^+(t)} \frac{dt}{t-x-iy} -$$

$$- \frac{a(x)S(x-iy)}{2\pi i(x_k - x + iy)} \int_{X_k} \frac{f(t)}{S^+(t)} \frac{dt}{t-x+iy} =$$

$$= I_1(f,x,y) + I_2(f,x,y),$$

where

$$I_1(f,x,y) = \frac{y}{\pi} \frac{S(x+iy)}{x_k - x - iy} \int_{X_k} \frac{f(t)(x_k-t)dt}{S^+(t)((t-x)^2+y^2)},$$

$$I_2(f,x,y) = \frac{yT(x;y)}{2\pi i} \int_{X_k} \frac{f(t)(x_k-t)dt}{S^+(t)(t-x+iy)},$$

where

$$T(x;y) = \frac{S(x+iy)}{x_k - x - iy} - \frac{a(x)S(x-iy)}{x_k - i + iy}.$$

As

$$\int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{|x+i|^{\alpha_k}} \frac{y|dx|}{|x_k - x - iy|\,((t-x)^2+y^2)} \le const,$$

then

$$\|I_1(f,x,y)\|_{L^1(\rho)} =$$

$$= \widetilde{C_1} \int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{(1+|x|)^{\alpha}|x+i|^{\alpha_k}} \frac{y|dx|}{(x_k - x - iy)} \int_{X_k} \frac{|f(t)||x_k - t||dt|}{((t-x)^2+y^2)}$$

$$\le C_1 \|f\|_{L^1(\rho)} \int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{(1+|x|)^{\alpha}|x+i|^{\alpha_k}} \frac{y|dx|}{(x_k - x - iy)} \le M_1' \|f\|_{L^1(\rho)},$$

where M_1' is a constant does not depend on y and k.

So

$$\|I_1(f,x,y)\|_{L^1(\rho)} \le M_1' \|f\|_{L^1(\rho)}.$$

Similarly we get

$$\|I_1(f,x,y)\|_{L^\infty(\rho)} \le M_1'' \|f\|_{L^\infty(\rho)},$$

where $f \in L^\infty(\rho)$. By applying Riesz-Thorin interpolation theorem [3], we obtain

$$\|I_1(f,x,y)\|_{L^p(\rho)} \le M_1 \|f\|_{L^p(\rho)}, \quad 1 < p < \infty. \tag{2.7}$$

As S^+ is bounded, then using the fact (Lemma 3 in [16]) that for sufficiently large R at $|x| > R$ the following estimate we have

$$|S^+(x+iy) - a(x)S^-(x-iy)| \le C_2 |S^+(x+iy)| \frac{y}{|x+i|}, \tag{2.8}$$

where $C_2 > C_1 > 0$ some constant independent of y, we get

$$|T(x)| \leq \frac{Cy}{(x_k - x)^2 + y^2},$$

where $C > \max_k \{x_k\}$ is a constant.

Since

$$\int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{|x+i|^{\alpha_k}} \frac{y|dx|}{((x_k - x)^2 + y^2)} \leq const,$$

then

$$\|I_2(f,x,y)\|_{L^1(\rho)}$$
$$\leq \widetilde{C_2} \int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{|x+i|^{\alpha_k}} \frac{y|dx|}{((x_k - x)^2 + y^2)} \int_{X_k} \frac{|f(t)||x_k - t||dt|}{|(t - x + iy)|}$$
$$\leq C_2 \|f\|_{L^1(\rho)} \int_{-\infty}^{+\infty} \frac{|x_k - x|^{\alpha_k}}{|x+i|^{\alpha_k}} \frac{y|dx|}{((x_k - x)^2 + y^2)} \leq M_2^{'} \|f\|_{L^1(\rho)},$$

where $M_2^{'}$ is a constant does not depend on y and k.

So

$$\|I_2(f,x,y)\|_{L^1(\rho)} \leq M_2^{'} \|f\|_{L^1(\rho)}.$$

Similarly we get

$$\|I_2(f,x,y)\|_{L^\infty(\rho)} \leq M_2^{''} \|f\|_{L^\infty(\rho)},$$

where $f \in L^\infty(\rho)$. By applying Riesz-Thorin interpolation theorem, we get

$$\|I_2(f,x,y)\|_{L^p(\rho)} \leq M_2 \|f\|_{L^p(\rho)}, \ \ 1 < p < \infty. \tag{2.9}$$

Hence, from (2.7) and (2.9), we obtain

$$\|\Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy)\|_{L^p(\rho)} \leq M\|f\|_{L^p(\rho)}$$

where $1 < p < \infty$ and $M = \max\{M_1, M_2\}$ is a constant independent of y and k. The estimate of the theorem is proved.

Now let's prove the second statement of the theorem. Let $f_n(x) \in C^\alpha$ be a sequence of finite functions such that

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_{L^p(\rho)} = 0, 1 < p < \infty. \tag{2.10}$$

For any n we set

$$\widetilde{\Phi}_n(z) = \frac{S(z)}{2\pi i(x_k - z)} \int_{-\infty}^{+\infty} \frac{f_n(t)(x_k - t)dt}{S^+(t)(t - z)}, \qquad z \in \Pi^\pm.$$

and from Sokhotski-Plemelj formula (see [2]), we get

$$\lim_{y\to+0} \left\| \widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - f_n(x) \right\|_{L^1(\rho)} = 0. \tag{2.11}$$

Using the estimate of this theorem, we obtain

$$\lim_{y\to+0} \left\| \Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy) - f(x) \right\|_{L^p(\rho)}$$

$$\leq \left\| \widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - f_n(x) \right\|_{L^p(\rho)} + \|f_n(x) - f(x)\|_{L^p(\rho)}$$
$$+ \left\| \left(\widetilde{\Phi}_n^+(x+iy) - \Phi_k^+(x+iy) \right) - a(x) \left(\widetilde{\Phi}_n^-(x-iy) - \Phi_k^-(x-iy) \right) \right\|_{L^p(\rho)}$$
$$\leq \left\| \widetilde{\Phi}_n^+(x+iy) - a(x)\widetilde{\Phi}_n^-(x-iy) - f_n(x) \right\|_{L^p(\rho)} + 2 \|f_n(x) - f(x)\|_{L^p(\rho)} .$$

Taking into account (2.10) and (2.11), we conclude

$$\lim_{y \to +0} \left\| \Phi_k^+(x+iy) - a(x)\Phi_k^-(x-iy) - f(x) \right\|_{L^p(\rho)} = 0.$$

Theorem is proved. □

3. The main result

3.1. The problem R_p for $T_p = \emptyset$.

Theorem 3.1. *Let $T_p = \emptyset$. Then the homogeneous problem R_p ($f \equiv 0$) does not have solution different from zero.*

Theorem 3.2. *Let $f \in L^p(\rho)$ and $T_p = \emptyset$. Also let the sequence of points $\{x_k\}_1^\infty$ satisfy the conditions (2.2), (2.3). Then the following assertions hold:*
a) If $\kappa \geq 0$, then the general solution of the inhomogeneous Problem R_p may be represented as

$$\Phi(z) = \frac{S(z)}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)dt}{S^+(t)(t-z)}, \quad z \in \Pi^+ \cup \Pi^-. \tag{3.1}$$

b) If $\kappa < 0$, then the inhomogeneous Problem R_p is solvable if and only if the function f satisfies the conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)} \frac{dt}{(t+i)^j} = 0, \qquad j = 1, 2, ..., -\kappa - 1.$$

The general solution can be represented by (3.1).

Proof. The proof of the point a) follows from Lemma 2.3 and Theorem 3.1.

b) Let $\kappa < 0$. Denote

$$\Phi^+(x+iy) - a(x)\Phi^-(x-iy) = f_y(x). \tag{3.2}$$

Taking into account that $a(x) = \dfrac{S^+(x)}{S^-(x)}$, we get

$$\frac{\Phi^+(x+iy)}{S^+(x)} - \frac{\Phi^-(x-iy)}{S^-(x)} = \frac{f_y(x)}{S^+(x)}.$$

Denoting

$$\Phi_y^+(z) = \frac{\Phi^+(z+iy)}{S^+(z)}, \ z \in \Pi^+, \qquad \Phi_y^-(z) = \frac{\Phi^-(z-iy)}{S^-(z)}, \ z \in \Pi^-,$$

we get

$$\Phi_y^+(x) - \Phi_y^-(x) = \frac{f_y(x)}{S^+(x)}.$$

In the case $\kappa < 0$, the function $\Phi_y^-(z)$ has zero of order $|\kappa - 1|$ at the point of $z = -i$. Consequently, $f(x)$ satisfies the conditions

$$\int_{-\infty}^{+\infty} \frac{f(t)}{S^+(t)} \frac{dt}{(t+i)^j} = 0, \qquad j = 1, 2, ..., -\kappa - 1.$$

Theorem 3.2 is proved. □

3.2. **The problem R_p for $T_p \neq \emptyset$.**

Theorem 3.3. *Let $T_p \neq \emptyset$. Then the general solution of the homogeneous problem R_p $(f \equiv 0)$, can be represented as:*

$$\Phi_0(z) = S(z) \sum_{x_k \in T_p} \frac{A_k}{x_k - z},$$

where $\{A_k\} \in l^p$.

Proof. It is clear that the number of points $x_k \in T_p$ is finite and by n_p we denote those points. It is sufficient to establish that the function $r_k(z) = S(z)(x_k - z)^{-1}$ does not satisfy condition (1.2) if $x_k \notin T_p$. Indeed

$$|r_k(x+iy) - r_k(x-iy)| = |R_1(x,y) + R_2(x,y)|,$$

where

$$R_1(x,y) = \frac{(x_k - x)\left(S^+(x+iy) - a(x)S^-(x-iy)\right)}{(x_k - x)^2 + y^2},$$

$$R_2(x,y) = \frac{iy\left(S^+(x+iy) + a(x)S^-(x-iy)\right)}{(x_k - x)^2 + y^2}.$$

Using inequality (2.8), we get

$$\|R_1(x,y)\|_{L^p(\rho)}$$

$$\leq C_1 y^p \left(\int_{-\infty}^{+\infty} \frac{|x_k - x|^{p(1+\alpha_k)} dx}{|x+i|^p \, |1+|x||^{p\alpha_0} \left((x_k - x)^2 + y^2\right)^p} \right)^{\frac{1}{p}} < C.$$

On the other hand

$$\|R_2(x,y)\|_{L^p(\rho)} \geq C_0 \Big(2 \int_{|x - x_k| < \delta} \frac{y^p |x_k - x|^{p\alpha_k} dx}{\left((x_k - x)^2 + y^2\right)^p} \Big)^{\frac{1}{p}} > C_1 > 0.$$

where C_1 does not depend on δ. So $\|r_k(x+iy) - r_k(x-iy)\|_{L^p(\rho)} \geq M > 0$. □

Theorem 3.4. *Let $f \in L^p(\rho)$ and $T_p \neq \emptyset$. Also let the sequence of points $\{x_k\}_1^\infty$ satisfy conditions* (2.2), (2.3) *and $\kappa \geq 0$. Then the general solution of the inhomogeneous Problem R_p may be represented as $\Phi(z) = \Phi_0(z) + \Phi_1(z)$ where Φ_0 is the general solution of the homogeneous problem and*

$$\Phi_1(z) = \sum_{k=1}^{\infty} \Phi_k(z), \tag{3.3}$$

where

$$\Phi_k(z) = \frac{S(z)}{2\pi i(x_k - z)} \int_{X_k} \frac{f(t)(x_k - t)dt}{(t - z)}, k = 1, 2, ...$$

Proof. Since $\kappa \geq 0$, then from Theorem 2.1 we have

$$\|\Phi_1^+(x + iy) - a(x)\Phi_1^-(x - iy)\|_{L^1(\rho)} \leq C\|f\|_{L^p(\rho)},$$

where the constant C is independent on y and k. Therefore, similar to the proof of the second part of Theorem 2.1, we obtain

$$\lim_{y\to+0} \|\Phi_1^+(x + iy) - a(x)\Phi_1^-(x - iy) - f(x)\|_{L^p(\rho)} = 0.$$

Taking into account Theorem 3.3 we get the proof of the theorem. □

Theorem 3.5. *Let $f \in L^p(\rho)$ and $T_p \neq \emptyset$. Also let the sequence of points $\{x_k\}_1^\infty$ satisfy conditions* (2.2), (2.3) *and $\kappa < 0$. Then the general solution of the in-homogeneous problem R_p may be represented as $\Phi(z) = \Phi_0(z) + \Phi_1(z)$, where $\Phi_1(z)$ is defined by* (3.3) *and*

$$\Phi_0(z) = S(z) \sum_{x_k \in T_p} \frac{A_k}{x_k - z},$$

here $\{A_k\}_{k=1}^\infty \in l^1$, $A_{-\kappa+1}, A_{-\kappa+2}, ...$ are arbitrary complex numbers, and the numbers $A_1, A_2, ..., A_\kappa$ are uniquely defined by the system of linear equations

$$\begin{cases} \sum_{k=1}^\infty \frac{A_k}{(x_k+i)} = -\sum_{j=1}^\infty I_{1\,1} \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^2} = -\sum_{j=1}^\infty (I_{2\,1} + I_{1\,2}) \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^3} = -\sum_{j=1}^\infty (I_{3\,1} + 2I_{2\,2} + I_{1\,3}) \\ \cdots \\ \sum_{k=1}^\infty \frac{A_k}{(x_k+i)^{-\kappa}} = -\sum_{j=1}^\infty \sum_{m=1}^{-\kappa} C_m^{-\kappa} I_{m\,-\kappa-m} \end{cases}, \tag{3.4}$$

where C_m^n are the binomial coefficients and

$$I_{m\,n} = \frac{1}{2\pi i(x_k + i)^m} \int_{X_k} \frac{f(t)(x_k - t)}{S^+(t)(t + i)^n} dt, \quad m, n = 1, 2, ..., -\kappa\ .$$

Proof. In the case $\kappa < 0$, $S^-(z)$ has a pole of order $-\kappa(\kappa < 0)$ at the point $z = -i$. Hence in order $\Phi(z)$ to be solution of the in-homogeneous problem R_p, for $A_1, A_2, ..., A_{-\kappa}$ it must be hold (3.4). Note that the determinant of the linear system (3.4) is a Vandermonde determinant and is determined by the following formula:

$$det = \prod_{1\leq k<j\leq -\kappa} \left(\frac{1}{x_j + i} - \frac{1}{x_k + i} \right).$$

Since $\frac{1}{x_k+i}$, $k = 1, 2, ..., -\kappa$ are distinct, the determinant is non-zero. Hence the numbers $A_1, A_2, ..., A_{-\kappa}$ may be uniquely defined by the system of linear equations (3.4). □

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