ISSN 0002-306X.Изв. НАН РА и ГИУА. Сер. ТН. 2012. Т. LXV, № 4.

UDC 62-50 AUTOMATION AND CONTROL SYSTEMS

O.N. GASPARYAN, A.T. ULIKYAN

DESIGN OF MATRIX REGULATORS FOR CIRCULANT CONTROL SYSTEMS

The paper is devoted to the problem of designing matrix regulators for a special class of multivariable feedback control systems called circulant systems. The exposition is based on the characteristic transfer function method, which allows reducing the investigation of N-dimensional multivariable control system to investigation of N fictitious one-dimensional systems. An analytical formula for elements of circulant matrix regulators is derived.

Keywords: multivariable control system, circulant system, permutation matrix, characteristic transfer function, matrix regulator.

Introduction. The problem of multiple-input multiple-output (MIMO) control system design is one of the centrals in multivariable feedback control [1-3]. The paper is devoted to the issue of designing matrix regulators for a significant class of MIMO control systems described by circulant transfer matrices.Such systems are widespread in various technical applications, especially in process control and aerospace engineering [1-3]. The proposed approach is based on the characteristic transfer functions (CTF) method [4]. That method allows reducing the task of analysis and design of an *N*-dimensional square (i.e. having *N* inputs and *N* outputs) MIMO system to *N* one-dimensional tasks, which, in many cases, can be solved by conventional methods of classical control [5].

The general matrix block diagram of a MIMO system with the matrix regulator $K(s)$ is shown in Fig. 1, where: $\varphi(s)$ and $f(s)$ stand for Laplace transforms of the *N*-dimensional input and output vectors $\varphi(t)$ and $f(t)$; $W(s)$ denotes the transfer matrix of the plant with entries that are scalar rational functions in complex variable s. The destination of the matrix regulator $K(s)$ consists of providing the required performance indices of the closed-loopMIMO system [1, 2].

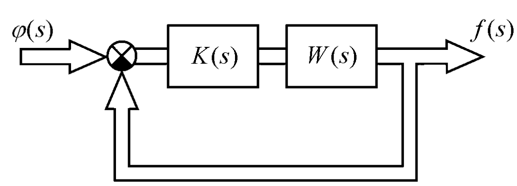


Fig. 1. Square MIMO control system with plant $W(s)$ and matrix regulator $K(s)$

Circulant control systems. The distinctive feature of *circulant* MIMO systems is that their transfer matrices are circulant [6, 7]. In a circulant matrix, each subsequent row is obtained from the preceding row by shifting all elements (except for the N th) by one position to the right; the N th element of the preceding row then becomes the first element of the following. For the circulant matrix $W(s)$ of the plant we have:

$$W(s) = \begin{pmatrix} w_0(s) & w_1(s) & w_2(s) & \dots\dots & w_{N-1}(s) \\ w_{N-1}(s) & w_0(s) & w_1(s) & \dots\dots & w_{N-2}(s) \\ \dots & \dots & \dots & \dots & \dots \\ w_1(s) & w_2(s) & w_3(s) & \dots\dots & w_0(s) \end{pmatrix}. \tag{1}$$

Each diagonal of a circulant matrix consists of the same elements, and the diagonals located at the same distance from the lower left corner and from the principal diagonal consist of identical elements. Physically, this means that in circulant systems, it is possible to single out some groups of subsystems with identical transfer functions of all cross-connections, i.e. having some internal symmetry. It is easy to see that any circulant matrix is completely defined by the first (or any other) row. Using the designations $w_0(s),\ w_i(s)\ (i = 1, 2, \dots, N-1)$ for the first row of the circulant matrix $W(s)$ (1), the latter can be represented in the matrix polynomial form [2, 3]:

$$W(s) = w_0(s)I + \sum_{k=1}^{N-1} w_k(s)U^k, \tag{2}$$

where

$$U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3}$$

is the orthogonal *permutation matrix* [6, 7].

The eigenvalues β_i of the permutation matrix U are the roots of the equation

$$\det[\beta I - U] = \beta^N - 1 = 0, \tag{4}$$

and, for any N, are expressed in the analytical form:

$$\beta_i = \exp\left\{ j\frac{2\pi(i-1)}{N} \right\} \qquad (i = 1, 2, \dots, N). \tag{5}$$

The CTFs $q_i(s)$ of the circulant matrix $W(s)$ (1), (2) can be represented, for any number N of separate channels, as

$$q_i(s) = W_0(s) + \sum_{k=1}^{N-1} W_k(s) \exp\left\{ j\frac{2\pi(i-1)}{N}k \right\} \quad (i = 1, 2, \ldots, N). \tag{6}$$

Besides, the canonical basis of the circulant matrix $W(s)$ and the modal matrix C are inherited from the permutation matrix U (3) [2, 3].

Design of circulant systems with matrix regulators. Let both the plant $W(s)$ and the regulator $K(s)$ in Fig. 1 are circulant, i.e. are described by circulant transfer matrices having the following canonical representations [2, 3]:

$$W(s) = C\,diag\{q_i(s)\}C^{-1}, K(s) = C\,diag\{p_i(s)\}C^{-1}, \tag{7}$$

where the orthogonal modal matrix C is composed of the normalized eigenvectors c_i of the permutation matrix U (3). Then, the transfer matrix $G(s)$ of the open-loop corrected system is equal to

$$G(s) = W(s)K(s) = \begin{pmatrix} G_0(s) & G_1(s) & G_2(s) & \ldots. & G_{N-1}(s) \\ G_{N-1}(s) & G_0(s) & G_1(s) & \ldots & G_{N-2}(s) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ G_1(s) & G_2(s) & G_3(s) & \ldots. & G_0(s) \end{pmatrix}, \tag{8}$$

or, taking into account (7),to

$$G(s) = C\,diag\{q_i(s)\}C^{-1}C\,diag\{p_i(s)\}C^{-1} = C\,diag\{q_i(s)p_i(s)\}C^{-1} = C\,diag\{g_i(s)\}C^{-1}, \tag{9}$$

where

$$g_i(s) = q_i(s)p_i(s) \quad (i = 1, 2, \ldots, N) \tag{10}$$

are the CTFs of the corrected circulant system with the circulantmatrix regulator.

Let us find the relationships between the transfer matrices $G(s)$, $W(s)$ and $K(s)$ for circulant systems with matrix regulators. Giving the *desired* transfer matrix of the open-loop system $G(s)$, we can immediately write down, based upon equations (8)-(10),

$$K(s) = W^{-1}(s)G(s) = Cdiag\{p_i(s)\}C^{-1}, \tag{11}$$

where

$$p_i(s) = \frac{g_i(s)}{q_i(s)} \quad (i = 1, 2, \ldots, N) \tag{12}$$

are the CTFs of the required matrix regulator $K(s)$ in (7).

Formally, the N equations in (12) are quite similar to the corresponding equations for single-input single-output (one-dimensional) control systems. On the other hand, here, we have a *set* of N equations, which depend on $2N$ transfer functions $w_0(s)$, $w_1(s)$, ..., $w_{N-1}(s)$ and $G_0(s), G_1(s), ..., G_{N-1}(s)$ in (1) and (8).

The issue is to find the rational transfer functions $k_0(s), k_1(s), ..., k_{N-1}(s)$ so that, the given transfer matrices $W(s)$ (1) and $G(s)$ (8), the N scalar equations (10) [or the matrix equation (8)] be satisfied. The analytical solution of that task cannot be generally found. Therefore, we shall try to bring the equations in question to a form, which will simplify the numerical solution of the task by means of modern computer aids.

Towards that end, let us derive the analytical equations for CTFs $g_i(s)$ for $N=3$. Based on (6)-(12), we have

$$\begin{gathered} q_1(s) = w_0(s) + w_1(s) + w_2(s), \\ q_2(s) = w_0(s) + w_1(s)\exp\left\{j\frac{2\pi}{3}\right\} + w_2(s)\exp\left\{j\frac{4\pi}{3}\right\}, \\ q_3(s) = w_0(s) + w_1(s)\exp\left\{j\frac{4\pi}{3}\right\} + w_2(s)\exp\left\{j\frac{8\pi}{3}\right\}, \end{gathered} \tag{13}$$

$$\begin{gathered} p_1(s) = k_0(s) + k_1(s) + k_2(s), \\ p_2(s) = k_0(s) + k_1(s)\exp\left\{j\frac{2\pi}{3}\right\} + k_2(s)\exp\left\{j\frac{4\pi}{3}\right\}, \\ p_3(s) = k_0(s) + k_1(s)\exp\left\{j\frac{4\pi}{3}\right\} + k_2(s)\exp\left\{j\frac{8\pi}{3}\right\}. \end{gathered} \tag{14}$$

The substitution of equations (13) and (14) into (10) and examination of the obtained equations shows that the relationship between the elements of the transfer matrices $G(s)$, $W(s)$ and $K(s)$ can be written in the following compact form:

$$\begin{gathered} \begin{pmatrix} G_0(s) \\ G_1(s) \\ G_2(s) \end{pmatrix} = \begin{pmatrix} k_0(s)w_0(s) + k_1(s)w_2(s) + k_2(s)w_1(s) \\ k_0(s)w_1(s) + k_1(s)w_0(s) + k_2(s)w_2(s) \\ k_0(s)w_2(s) + k_1(s)w_1(s) + k_2(s)w_0(s) \end{pmatrix} = \\ = \begin{pmatrix} w_0(s) & w_2(s) & w_1(s) \\ w_1(s) & w_0(s) & w_2(s) \\ w_2(s) & w_1(s) & w_0(s) \end{pmatrix} \begin{pmatrix} k_0(s) \\ k_1(s) \\ k_2(s) \end{pmatrix}. \end{gathered} \tag{15}$$

Note that the matrix

$$\tilde{W}(s) = \begin{pmatrix} w_0(s) & w_2(s) & w_1(s) \\ w_1(s) & w_0(s) & w_2(s) \\ w_2(s) & w_1(s) & w_0(s) \end{pmatrix} \tag{16}$$

in the equation (15) is transposed with respect to the transfer matrix of the plant $W(s)$, i.e. $\tilde{W}(s) = W^T(s)$.

It can be shown that analogous [to (15)] relationships hold true for $N = 4$, $N = 5$ and $N = 6$. Therefore, by induction, the relationship between the N-dimensional column vectors $\tilde{G}(s) = [G_0(s)\ G_1(s)\ ....\ G_{N-1}(s)]^T$ and $\tilde{K}(s)$ has the form

$$\tilde{G}(s) = W^T(s)\tilde{K}(s). \tag{17}$$

From (17), we get the formula

$$\tilde{K}(s) = [W^T(s)]^{-1}\tilde{G}(s) \tag{18}$$

relating the vector $\tilde{K}(s)$, composed of the elements of the first row of the circulant regulator $K(s)$, with the transfer matrix of the plant $W(s)$ and the vector $\tilde{G}(s)$, composed of the elements of the first row of the desired open-loop transfer matrix $G(s)$.

Since the matrix $\tilde{W}(s) = W^T(s)$ is circulant, it has a standard canonical representation

$$\tilde{W}(s) = C diag\{\tilde{q}_i(s)\}C^{-1}, \tag{19}$$

where the CTFs $\tilde{q}_i(s)$ are given by the following expressions:

$$\tilde{q}_i(s) = w_0(s) + \sum_{k=1}^{N-1} w_{N-k}(s)\exp\left\{ j\frac{2\pi(i-1)}{N}k \right\}. \tag{20}$$

Taking into account (19) we have, instead of (18), the final expression

$$\tilde{K}(s) = C diag\left\{\frac{1}{\tilde{q}_i(s)}\right\}C^{-1}\tilde{G}(s) = C diag\left\{\frac{1}{\tilde{q}_i(s)}\right\}C^*\tilde{G}(s), \tag{21}$$

which relates two vectors $\tilde{G}(s)$ and $\tilde{K}(s)$ and is well-suited for numerical computations.

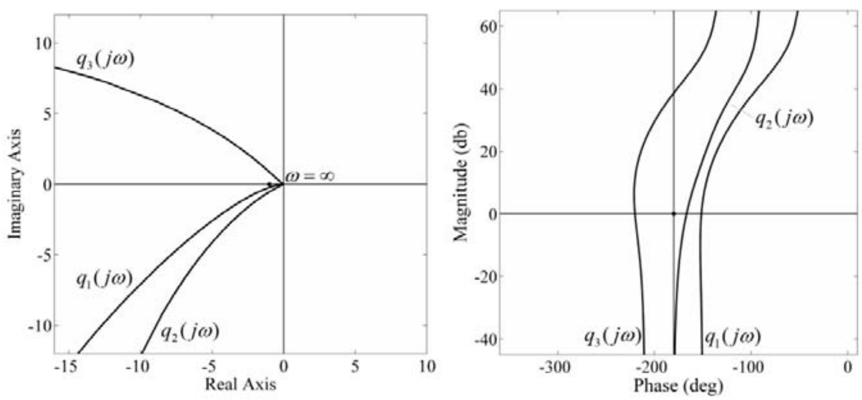
Example. Assume we have a three-dimensional circulant plant with the following elements of the first row:

$$w_0(s) = \frac{100}{0.8s^2 + s}, \qquad w_1(s) = \frac{50}{0.8s^2 + s}, \qquad w_2(s) = \frac{-60}{0.16s^3 + s^2 + s}. \quad (22)$$

The CTFs $q_i(s)$ of that matrix can be represented analytically in the form:

$$q_1(s) = \frac{10(3s+9)}{0.16s^3 + s^2 + s}, \qquad q_{2,3}(s) = \frac{10\left[(1.5 - 0.866j)s + 10.5 \mp 9.526j\right]}{0.16s^3 + s^2 + s}. \quad (23)$$

The Nyquist and Nichols plots of the CTFs (23) are shown in Fig. 2. The inspection of the graphs in Fig. 2 indicates that the initial circulant system is unstable.



(a) (b)

Fig. 2. Frequency characteristics of the three-dimensional circulant plant $W(s)$ (22).
(a) Nyquist plots; (b) Nichols plots

Let us find a circulant matrix regulator $K(s)$, which will provide stability ofthe given system and, in addition, provide that the value of the oscillation index M with respect to output signals be equal to unity, i.e. $M = 1$. Towards that end, we need to find such a circulant transfer matrix of the corrected system $G(s)$, which possesses the required performance characteristics. The analysis shows that the goal can be achieved by the 3×3 matrix $G(s)$ with the following elements of the first row:

$$G_0(s) = \frac{0.1907s^3 + 1.477s^2 + 2.691s + 1}{0.004s^5 + 0.081s^4 + 0.535s^3 + 1.35s^2 + s},$$

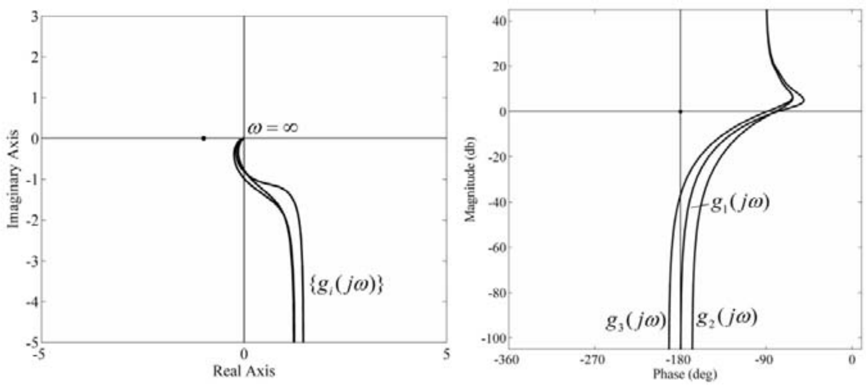
$$G_1(s) = \frac{0.05638s^3 + 0.141s^2 + 0.05638s}{0.004s^5 + 0.081s^4 + 0.535s^3 + 1.35s^2 + s}, \quad (24)$$

$$G_2(s) = \frac{0.08624s^3 + 0.2156s^2 + 0.08624s}{0.004s^5 + 0.081s^4 + 0.535s^3 + 1.35s^2 + s}.$$

The CTFs $g_1(s)$, $g_2(s)$ and $g_3(s)$ of the corresponding system have the form: $g_1(s) = \dfrac{0.3332s^3 + 1.8336s^2 + 2.8336s + 1}{0.004s^5 + 0.081s^4 + 0.535s^3 + 1.35s^2 + s}$,

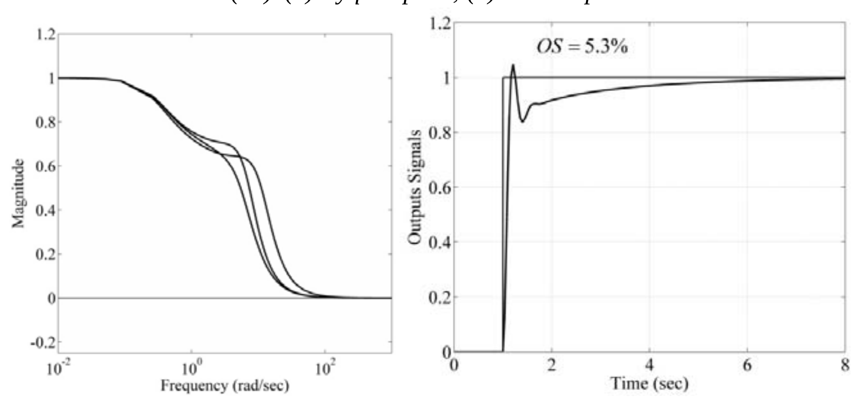
$$g_{2,3}(s) = \frac{(0.1939 \mp j0.02586)s^3 + (1.2987 \mp j0.064605)s^2 + (2.6197 \mp j0.02586)s + 1}{0.004s^5 + 0.081s^4 + 0.535s^3 + 1.35s^2 + s}.$$

The Nyquist and Nichols frequency characteristics of these CTFs are shown in Fig. 3. The frequency characteristics of the closed-loop CTFs are presented in Fig. 4(a).



(a) (b)

Fig. 3. Frequency characteristics of the three-dimensional corrected circulant system $G(s)$ (24). (a) Nyquist plots; (b) Nichols plots



(a) (b)

Fig. 4. Frequency (a) and transient (b) responses of the closed-loop circulant system

The transient responses of the system under the unit steps applied simultaneously to all inputs at time $t = 1.0\ s$ are given in Fig. 4(b). The same overshoot of all channels is $OS = 5.3\,\%$. Note that the transient responses of all channels of the system are the same, which is explained by the fact that, in the circulant system, only the first characteristic system is activated under the applied input unit steps.

The solution of the equation (21) gives the following elements of the first row of the circulant matrix regulator $K(s)$:

$$k_0(s) = \frac{0.4312s^2 + 1.078s + 0.4312}{s^2 + 14s + 40},$$

$$k_1(s) = \frac{-0.04643s^2 - 0.1161s - 0.04643}{s^2 + 14s + 40},$$

$$k_2(s) = \frac{0.2819s^2 + 0.705s + 0.2819}{s^2 + 14s + 40}.$$

Conclusion. An analytical formula relating the elements of the first row of the matrix regulator and the given transfer matrices of the corrected circulant system and the plant is derived in the paper. That formula exploits the canonical representation of circulant control systems on the bases of the CTFs method. It should be emphasized, that for circulant systems, the CTFs and the modal matrices can be written in analytical form for any number of channels N. That allows one to develop effective program codes (e.g., in the MATLAB language [8]) for computer-aided design of circulant control systems of an arbitrary dimension. It can be shown that the derived formula applies also to anticirculant systems [2, 3], i.e. to MIMO control systems with anticirculant transfer matrices.

REFERENCES

1. **Skogestad S. and Postlethwaite I.** Multivariable Feedback Control. Analysis and Design.- John Wiley and Sons Ltd., Chichester, Sussex, UK, 2005.-595 p.
2. **Gasparyan O.N.** Linear and Nonlinear Multivariable Feedback Control: A Classical Approach. - John Wiley & Sons, UK, 2008.-356 p.
3. **Гаспарян О.Н.** Теория многосвязных систем автоматического регулирования.- Ереван: Изд-во “Асогик”, 2010.- 380 с.
4. **MacFarlane A.G.J. and Belletrutti J.J.** The characteristic locus design method // Automatica.-1970.-Vol. 9, № 5. - P. 575-588.
5. **Бессекерский В.А., Попов Е.П.** Теория систем автоматического регулирования.- М.: Наука, 2003.- 575 с.
6. **Bellman R.**Introduction to Matrix Analysis.-McGraw-Hill, NY, 1970. - 367 p.
7. **Davis P.J.**Circulant Matrices. - Wiley, NY, 1979. - 284 p.
8. Using MATLAB. The Math Works, Inc., 2012. - 874 p.

SEUA. The material is received 12.09.2012.

Օ.Ն. ԳԱՍՊԱՐՅԱՆ, Ա.Թ. ՈՒԼԻԿՅԱՆ

ՑԻՐԿՈՒԼՅԱՆՏ ԿԱՌԱՎԱՐՄԱՆ ՀԱՄԱԿԱՐԳԵՐԻ ՄԱՏՐԻՑԱՅԻՆ ԿԱՐԳԱՎՈՐԻՉՆԵՐԻ ՆԱԽԱԳԾՈՒՄԸ

Դիտարկվում է հետադարձ կապով բազմաչափ կառավարման համակարգերի հատուկ դասի, այսպես կոչված, ցիրկուլյանտ համակարգերի մատրիցայի նկարգավորիչների նախագծման խնդիրը: Ներկայացվածը կատարված է բնութագրիչ փոխանցման ֆունկցիաների մեթոդով, որը թույլ է տալիս N−չափանի փոխադարձ կապերով կառավարման համակարգերի հետազոտումը հանգեցնել մեկ մուտքով և ելքով N հատեր նակայական համակարգերի հետազոտման: Դուրս է բերված անալիտիկ բանաձև՝ ցիրկուլյանտ մատրիցայի նկարգավորիչի տարրերի որոշման համար:

Առանցքային բառեր. Բազմաչափ կառավարման համակարգ, ցիրկուլյանտ համակարգ, տեղափոխությունների մատրից, բնութագրիչ փոխանցման ֆունկցիա, մատրիցայի նկարգավորիչ:

О.Н. ГАСПАРЯН, А.Т. УЛИКЯН

ПРОЕКТИРОВАНИЕ МАТРИЧНЫХ РЕГУЛЯТОРОВ ДЛЯ ЦИРКУЛЯНТНЫХ СИСТЕМ УПРАВЛЕНИЯ

Рассматривается задача проектирования матричных регуляторов для специального класса многомерных систем управления с обратной связью, называемых циркулянтными системами. Изложение основано на методе характеристических передаточных функций, который позволяет свести исследование N-мерной взаимосвязанной системы управления к исследованию N фиктивных систем с одним входом и выходом. Выведена аналитическая формула для определения элементов циркулянтного матричного регулятора.

Ключевые слова: многомерная система управления, циркулянтная система, матрица перестановок, характеристическая передаточная функция, матричный регулятор.