Известия НАН Армении, Математика, том 57, н. 5, 2022, стр. 25 – 32.

MULTIPLE FACTORIZATION OF SKEW-SYMMETRIC MATRICES

B. N. YENGIBARYAN, N. B. YENGIBARYAN

https://doi.org/10.54503/0002-3043-2022.57.5-25-32

Institute of Mathematics of Armenian National Academy of Sciences
E-mails: *b.yengibaryan@eif.am; yengib@instmath.sci.am*

Abstract. Multiple Factorization $A = \Pi D \Pi^T$ is proposed for real skew-symmetric matrix $A \neq 0$ of order $n \geq 3$. The block-diagonal factor D contains skew-symmetric invertible blocks of order 2 and the zero block of order $n - rank\ A$, if $rank\ A < n$. The matrix Π is an alternate product of Permutation matrices and Unit Lower triangular matrices with two columns. The applied approach is economical and contributes to the computational stability. The number of arithmetic operations $\sim \frac{1}{3}n^3$. The inverse matrix A^{-1} and the skeletal decomposition of A are presented in factorized form without additional calculations.

MSC2020 numbers: 15A23; 42B25.
Keywords: skew-symmetric matrix; multiple factorization; stability; inverse matrix.

1. Introduction

For a skew-symmetric (SS) matrix A the LDL^T factorization is well known(see [1]):

$$PAP^T = LDL^T. \tag{1.1}$$

Here P is a permutation matrix, L is unit lower triangular, D is block diagonal with SS blocks of order 2 or 1. L^T is the matrix transposed to L.

One modification of the LDL^T factorization is presented in this paper. It will be shown that a nonzero SS matrix $A \in R^{n\times n}$ of order $n \geq 3$ admits $\Pi D \Pi^T$ factorization:

$$A = \Pi D \Pi^T. \tag{1.2}$$

The block-diagonal factor D consists of invertible blocks of order 2 and one zero block of order $n - rank\ A$ (if A is non-invertible). The matrix Π is the alternate product of permutation matrices and unit lower triangular matrices with two columns.

The proposed method for constructing $\Pi D \Pi^T$ factorization is economical and contributes to the computational stability. The number of arithmetic operations $\sim \frac{1}{3}n^3$ (half the cost of LU factorization). The inverse matrix A^{-1} and the skeletal decomposition are presented in multiple factorized form without additional calculations.

Factorization $\Pi D \Pi^T$ has some similarity with multiple factorization of convolution https: //meet.google.com/syd-jmcc-ttg integral operators (see [2]).

Notations. We will consider real skew-symmetric (SS) matrices of fixed order $n \geq 3$:

$$A = (a_{km})_{k,m=1}^{n} \neq 0, \quad a_{mk} = -a_{km}, \quad n \geq 3, \tag{1.3}$$

and their same submatrices. Sizes of matrices will be noted as appropriate.

- The index k in W_k means that the given square matrix has an order k.
- E_s is the unit matrix of order s. The notation E is used for E_n.
- Zero matrix of any size is denoted by 0.
- $\Lambda_2 \{W\}$ is the leading principal submatrix of order 2 of W.

2. Preparatory factorization

Consider the following SS matrix $C(s)$ of order $n - 2s + 2 \geq 3$:

$$C(s) = (c_{km}(s))_{k,m=2s-1}^{n} \neq 0, \quad c_{mk}(s) = -c_{km}(s), \tag{2.1}$$

where:

$$2 \leq 2s \leq n-1, \quad J(s) := \Lambda_2 \{C(s)\} \neq 0. \tag{2.2}$$

We have:

$$J(s) = \begin{pmatrix} 0 & \lambda(s) \\ -\lambda(s) & 0 \end{pmatrix}, \quad \lambda(s) := c_{2s-1,2s}(s) \neq 0, \tag{2.3}$$

$$\exists\ J^{-1}(s) = \begin{pmatrix} 0 & -[\lambda(s)]^{-1} \\ [\lambda(s)]^{-1} & 0 \end{pmatrix}.$$

Let's represent $C(s)$ in the form

$$C(s) = \begin{pmatrix} J(s) & U^T(s) \\ -U(s) & G(s) \end{pmatrix}. \tag{2.4}$$

Here $U(s)$ is a $(n-2s) \times 2$ matrix.

The following factorization holds (see [1]):

$$C(s) = \begin{pmatrix} E_2 & 0 \\ \tilde{V}(s) & E_{n-2s} \end{pmatrix} \begin{pmatrix} J(s) & 0 \\ 0 & W(s) \end{pmatrix} \begin{pmatrix} E_2 & \tilde{V}^T(s) \\ 0 & E_{n-2s} \end{pmatrix}. \tag{2.5}$$

Here

$$\tilde{V}(s) = U(s) J^{-1}(s), \tag{2.6}$$

the SS matrix $W(s)$ of order $n - 2s$ is the Schur complement of the block $J(s)$:

$$W(s) = G(s) + U(s) J^{-1}(s) U^T(s). \tag{2.7}$$

Permutation of rows and columns. Let $B(s)$ be nonzero SS matrix of order $n-2s+2 \geq 3$:

$$B(s) = (b_{km}(s))_{k,m=2s-1}^{n} \neq 0, \quad b_{mk}(s) = -b_{km}(s), \quad 2 \leq 2s \leq n-1. \tag{2.8}$$

Unlike $C(s)$, invertibility of block $\Lambda_2\{B(s)\}$ is not assumed.

Consider the application of such a permutation $\tilde{P}(s)$, that the SS matrix

$$C(s) = \tilde{P}(s)\, B(s)\, \tilde{P}^T(s) \tag{2.9}$$

satisfies condition (2.2) and promotes computational stability (when constructing (2.5)).

We will assume that the rows and columns of the permutation matrices are numbered in the same way as $B(s)$: from $2s-1$ to n.

Let $b_{ij}(s)$ be any nonzero element of the matrix $B(s)$. By virtue of skew-symmetry of $B(s)$, one can take $i > j$:

$$b_{ij}(s) \neq 0, \quad i > j. \tag{2.10}$$

Let's translate the element $b_{ij}(s)$ into the position $(2s+2, 2s+1)$ using the following two elementary permutation matrices $Q(s)$ and $F(s)$. The matrix $Q(s)$ permutes rows or columns $2s-1$ and i, the matrix $F(s)$ permutes positions $2s$ and j. Consider transform (2.9), where

$$\tilde{P}(s) = F(s)\, Q(s), \quad \tilde{P}^T(s) = Q(s)\, F(s). \tag{2.11}$$

The matrix $C(s)$ determined by (2.9) is skew-symmetric, hence $c_{kk}(s) = 0$, $2s-1 \leq k \leq n$.

The Block $J(s) = \Lambda_2\{C(s)\}$ has the form (2.3), where

$$\lambda(s) = b_{ij}(s) \neq 0. \tag{2.12}$$

Hence the decomposition (2.5) takes place.

We have:

$$B(s) = \tilde{P}^T(s)\, C(s)\, \tilde{P}(s). \tag{2.13}$$

The following lemma holds.

Lemma 2.1. *Skew-symmetric matrix $B(s)$ of the form (2.8) admits factorization*

$$B(s) = \tilde{P}^T(s) \begin{pmatrix} E_2 & 0 \\ \tilde{V}(s) & E_{n-2s} \end{pmatrix} \begin{pmatrix} J(s) & 0 \\ 0 & W(s) \end{pmatrix} \begin{pmatrix} E_2 & \tilde{V}^T(s) \\ 0 & E_{n-2s} \end{pmatrix} \tilde{P}(s), \tag{2.14}$$

The permutation matrix $\tilde{P}(s)$ is given by (2.11). Matrices $V(s)$ and $W(s)$ are determined by (2.6) and (2.7) respectively.

Selected element $b_{ij}(s)$ will be called the Pivot element of factorization (2.14).

Choice of a pivot element. To facilitate the computational stability of constructing factorization (2.14), one can take as a pivot element $b_{ij}(s)$ (any) largest modulus element of the matrix $B(s)$.

In the framework of pivoting strategy of Bunch (see [1]), the pivot element is the largest modulus element of the first column (if this column is nonzero). Then $\tilde{P}(s) = Q(s)$.

The following version of the Bunch strategy can be used. Let the first column (and first row) of the matrix $B(s)$ be zero or consist of elements small in modulus, and sometime any element $b_{i,2s}(s)$ of the second column is ''sufficiently large'' in modulus. Then $J(s)$ can be formed by translating $b_{i,2s}(s)$ into the position $(2s-1, 2s)$, and element $b_{2s,i}(s) = -b_{i,2s}(s)$ into the position $(2s, 2s-1)$. Then only one elementary permutation matrix is used, as in the case of the Bunch strategy. However, the degree of computational stability may be significantly higher.

3. One step of multiple factorization

Consider the following SS matrices $A(s)$ of order n:

$$A(1) = B(1), \quad A(s) = \begin{pmatrix} D(s) & 0 \\ 0 & B(s) \end{pmatrix}, s \geq 2, \ 2s \leq n-1. \tag{3.1}$$

Here $B(s) \neq 0$ is a matrix of the form (2.9); $D(s)$ is SS matrix of order $2s-2$.

Factorization (2.14) of $B(s)$ generates the following factorization for $A(s)$, $s \geq 2$:

(3.2)

$$A(s) = \begin{pmatrix} E_{2s} & 0 \\ 0 & \tilde{P}(s) \end{pmatrix} \begin{pmatrix} E_{2s} & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & \tilde{V}(s) & E_{n-2s-2} \end{pmatrix} \times$$

$$\times \begin{pmatrix} D(s) & 0 & 0 \\ 0 & J(s) & 0 \\ 0 & 0 & W(s) \end{pmatrix} \begin{pmatrix} E_{2s} & 0 & 0 \\ 0 & E_2 & \tilde{V}^T(s) \\ 0 & 0 & E_{n-2s-2} \end{pmatrix} \begin{pmatrix} E_{2s} & 0 \\ 0 & \tilde{P}^T(s) \end{pmatrix}, s \geq 2.$$

Let $V(s)$ and $P(s)$ are the following continuations of $\tilde{V}(s)$ and $\tilde{P}(s)$, up to n-order matrices:

$$V(s) = \begin{pmatrix} 0 & 0 \\ \tilde{V}(s) & 0 \end{pmatrix}, P(s) = \begin{pmatrix} E_{2s-2} & 0 \\ 0 & \tilde{P}(s) \end{pmatrix}. \tag{3.3}$$

Denote also

$$D(s+1) = \begin{pmatrix} D(s) & 0 \\ 0 & J(s) \end{pmatrix}, \quad B(s+1) = W(s); \tag{3.4}$$

$$A(s+1) = \begin{pmatrix} D(s+1) & 0 \\ 0 & B(s+1) \end{pmatrix}.$$

In the new notation equality (3.3) takes the following recursive form at $s \geq 2$, $2s \leq n-1$:

$$A(s) = P(s)(E + V(s))A(s+1)\left(E + V^T(s)\right)P^T(s). \tag{3.5}$$

Equality (3.6) for $s = 1$ follows from (2.14) in which

$$D(2) = J(1) = \Lambda_2\{C(1)\} \neq 0. \tag{3.6}$$

The following lemma holds.

Lemma 3.1. *The matrix $A(s)$, $s \geq 1$, given by* (3.1)*, admits factorization* (3.5)*, where $V(s)$, $P(s)$, $A(s+1)$ are determined by relations* (3.3),(3.4).

Thus, defining matrices $\tilde{V}(s)$ and $W(s)$ by (2.6) and (2.7), decomposition (2.5) is constructed, which leads to (3.5).

The matrices $V(s)$ and $V^T(s)$ that appear in (3.5) are nilpotent with index 2. Therefore

$$(E + V(s))^{-1} = E - V(s), \quad \left(E + V^T(s)\right)^{-1} = E - V^T. \tag{3.7}$$

4. Multiple Factorization

Let the matrix $A = A(1)$ have the form (1.3). Consider the question of recursive constructing of the following multiple factorizations

$$A = P(1)(E + V(1))\cdots P(k)(E + V(k))\cdot \cdot A(k+1)\left(E + V^T(k)\right)P^T(k)\cdots\left(E + V^T(1)\right)P^T(1), \tag{4.1}$$

with suitable values of $k \geq 1$, using recurrent relations (3.6).

The existence of (4.1) for $k = 1$ is contained in Lemma 3.1. Let (4.1) be constructed for some $k \geq 1$. Thus, all intermediate matrices $J(s)$ have been constructed. It follows from (3.5) that the matrix $D(k+1)$ of order $2k$ has the form

$$D(k+1) = \begin{pmatrix} J(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J(k) \end{pmatrix}, \; 2 \leq 2k \leq n-1, \tag{4.2}$$

with invertible SS blocks $J(1), \cdots, J(k)$ of order 2.

To continue the decomposition (4.1), the matrix $A(k+1)$ must satisfy conditions of Lemma 3.1, that is:

$$B(k+1) \neq 0 \quad \text{and} \quad 2k \leq n-3. \tag{4.3}$$

These conditions will be violated for some value of k if one of the following two situations a) and b) arises:

a) **The case of invertible matrix** A**.** Let the number n is even: $n = 2m$; $k = m - 1$ and the matrix $B(s+1)$ of order 2 is invertible:

$$2k = n - 2, B(k+1) \neq 0. \tag{4.4}$$

Then the matrix $B(k+1) = J(k+1)$ joins to the matrix $D(s+1)$ as the last diagonal block.

We obtain the factorization (1.2) where

$$D = \begin{pmatrix} J(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J(k+1) \end{pmatrix} \tag{4.5}$$

$$\Pi = P(1)(E + V(1)) \cdots P(k)(E + V(k)). \tag{4.6}$$

b) The case of non-invertible matrix A. Let

$$2k \leq n - 1 \quad \text{and} \quad B(k+1) = 0. \tag{4.7}$$

Then we obtain the factorization (1.2) where Π is given by (4.6) and

$$D = \begin{pmatrix} D(k) & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.8}$$

zero diagonal block of which has the order $n - 2k + 2$.

Because of invertibility of blocks $J(1), \cdots, J(k)$ we have $rank\ D = rank\ D(k) = 2k - 2$. It follows from invertibility of matrices $E + V(s)$ (see (3.7)) that

$$r := rank\ A = 2k - 2. \tag{4.9}$$

Hence the previously unknown rank r of the matrix A is also determined. The following Theorem holds.

Theorem 4.1. *Any real SS matrix $A \neq 0$ of order $n \geq 3$ admits complete factorization* (1.2)*, in which Π is the product of the form* (4.6)*. The equality* (4.9) *holds. In the case* (4.4) *the matrix D has the form* (4.5) *and $r = n$. In the case* (4.7) *the matrix D has the form* (4.8)*.*

5. Construction of the inverse matrix A^{-1} and skeletal decomposition

Construction of the inverse matrix. In the case of invertible matrix A, taking into account (4.1), we obtain the following expression for A^{-1}:

(5.1)
$$A^{-1} = P(1)(E - V(1)) \cdots P(k)(E - V(k)) \begin{pmatrix} J^{-1}(1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J^{-1}(k+1) \end{pmatrix} \times$$
$$\times \left(E - V^T(k)\right) P(k) \cdots \left(E - V^T(1)\right) P(1)$$

Numerical construction of the inverse matrix in the form (5.1) does not imply additional calculations compared with the direct decomposition (4.1), except for calculating $J^{-1}(k+1)$. All other matrices involved in (5.1) were determined when constructing a direct decomposition (4.1).

Skeletal decomposition. Let $r = rank\ A < n$. Then the factorization (4.2) leads to the following skeletal decomposition of the matrix A:

$$A = \left[\Pi \begin{pmatrix} D(k+1) \\ 0 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} E_r & 0 \end{pmatrix} \Pi^T\right], \quad r = 2k - 2. \tag{5.2}$$

Note, that the Skeletal decomposition applies to the construction of the pseudoinverse matrix A^+.

The number of arithmetic operations. The numerical construction of the decomposition (3.5) is reduced to the calculation of the elements of the matrices $\tilde{V}(s)$ and $W(s)$ in accordance with the formulas (2.6), (2.7). In the case of an invertible matrix, the total number of arithmetic operations is about $\frac{1}{3}n^3$ (half the cost of LU factorization). In the case $rank\ A < n$, the number of arithmetic operations may be significantly less.

After constructing the inverse matrix A^{-1} in the factorized form (5.2), the following question arises: Is it necessary to expand the parentheses in the product $P(1)(E - V_1) \cdots P(k)(E - V_m)$ and obtain a three-factor decomposition, while performing a large amount of additional calculations? We only note that in the question of solving the equations $Ax = b$ there is no need for it.

Low-rank approximation. Suppose that in the recurrent construction of the decompositions (3.5). For some all elements of the matrix $B(k+1)$ are small enough (according to the chosen criterion of smallness). Then the decomposition can be stopped, the matrix $B(k+1)$ is replaced by a zero matrix.

Список литературы

[1] Nicholas J. Higham, Accuracy and Stability of Numerical Algorithms, University of Manchester, Manchester, England (2002).

[2] B. N. Engibarian (Yengibaryan), “Multiple factorization of convolution-type integral operators”, Comput. Math. Math. Phys., **37**: 4, 435 – 446 (1997).

Поступила 30 марта 2022

После доработки 01 апреля 2022

Принята к публикации 20 апреля 2022