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INTEGRAL INEQUALITIES FOR THE GROWTH AND HIGHER DERIVATIVE OF POLYNOMIALS

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Abstract. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| \leqslant 1$, it was proved by S. Gulzar [Anal Math 42, 339-352 (2016). https://doi.org/10.1007/s10476-016-0403-7] that

$$\left\| z^s P^{(s)}(z) + \beta \frac{n(n-1)...(n-s+1)}{2^s} P(z) \right\|_p \leqslant n(n-1)...(n-s+1) \left\| \left(1 + \frac{\beta}{2^s}\right) z + \frac{\beta}{2^s} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}$$

for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $1 \leqslant s \leqslant n$ and $0 \leqslant p < \infty$. In this paper we extend the above result to the growth of polynomials and also generalize the above and other related results in this direction.

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1. Introduction

Let $\mathcal{P}_n$ denote the space of all polynomials of degree at most n over the field of complex numbers. The subject of inequalities for polynomials and related classes of functions plays an important and crucial role in obtaining inverse theorems in Approximation Theory. The extremal problems of analytic functions and the results were some approaches to obtaining the classical inequalities are developed on using various methods of the geometric function theory are known for various norms and for many classes of functions such as polynomials with various constraints and in various regions of the complex plane. A classical result due to Bernstein [4] is that, for two polynomials $P(z)$ and $T(z)$ with degree of $P(z)$ not exceeding that of $T(z)$ and $T(z) \neq 0$ for $|z| > 1$, the inequality $|P(z)| \leqslant |T(z)|$ on the unit circle $|z| = 1$ implies the inequality of their derivatives $|P'(z)| \leqslant |T'(z)|$ on $|z| = 1$. In particular, for $T(z) = z^n \max_{|z|=1} |P(z)|$ gives a famous Bernstein inequality namely, if $P(z)$ is a polynomial of degree n then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1.1}$$

On the other hand, concerning the growth of polynomials we have for $P \in \mathcal{P}_n$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then $|Q(z)| = |P(z)|$ for $|z| = 1$. This implies $|Q(z)| \leqslant \max_{|z|=1} |P(z)|$ for $|z| = 1$. This further implies, by using maximum modulus theorem, that $|Q(z)| \leqslant \max_{|z|=1}|P(z)|$ for $|z| \leqslant 1$ or equivalently $|z^n\overline{P(\frac{1}{\bar{z}})}| \leqslant \max_{|z|=1}|P(z)|$. If we take $z = e^{i\theta}/R$ where $\theta \in [0, 2\pi)$ and $R \geqslant 1$, we get $|(e^{in\theta}/R^n)\overline{P(Re^{i\theta})}| \leqslant \max_{|z|=1}|P(z)|$. Hence, the growth estimate for $|P(z)|$ over a large cricle $|z| = R$ in comparsion with its maximum modulus over the unit circle $|z| = 1$ is given by

$$\max_{|z|=R}|P(z)| \leqslant R^n \max_{|z|=1}|P(z)|, \qquad R > 1. \tag{1.2}$$

These inequalities (1.1) and (1.2) are related with each other and have been the starting point of a considerable literature in polynomial approximations and these inequalities were generalized and extended in several directions, in different norms and for different classes of functions.

Define the standard Hardy space norm for $P \in \mathcal{P}_n$ by

$$\|P\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^p d\theta\right)^{1/p}, \quad 0 < p < \infty;$$

and the Mahler measure by

$$\|P\|_0 = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|P(e^{i\theta})|d\theta\right).$$

It is well known that $\lim_{p\to 0+}\|P\|_p = \|P\|_0$. We also note that the supremum norm of the space H^∞ satisfies $\|P\|_\infty := \lim_{p\to\infty}\|P\|_p = \max_{|z|=1}|P(z)|$.

If $P \in \mathcal{P}_n$, then

$$\left\|P'(z)\right\|_p \leqslant n\left\|P(z)\right\|_p, \qquad p \geqslant 1, \tag{1.3}$$

and for $R \geqslant 1$

$$\left\|P(Rz)\right\|_p \leqslant R^n\left\|P(z)\right\|_p, \qquad p > 0. \tag{1.4}$$

The inequality (1.3) is due to Zygmund [16], whereas the inequality (1.4) is a simple consequences of a result due to Hardy [8]. Arestov [2] verified that (1.3) remains true for $0 \leqslant p < 1$ as well. Also inequalities (1.3) and (1.4) are further generalized by Aziz and Rather [3] as

$$\left\|zP'(z) + \beta\frac{n}{2}P(z)\right\|_p \leqslant n\left|1 + \frac{\beta}{2}\right|\left\|P(z)\right\|_p, \qquad p > 0, \tag{1.5}$$

and

$$\left\|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right\|_p \leqslant \left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|\left\|P(z)\right\|_p, \qquad p > 0, \tag{1.6}$$

respectively for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $R \geqslant 1$. For $p = \infty$, inequalities (1.5) and (1.6) are due to Jain [10].

The inequalities (1.3) and (1.4) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z|<1$. In fact if $P(z)\neq 0$ for $|z|<1$, the inequality (1.3) can be replaced by

$$\|P'(z)\|_p \leqslant n\frac{\|P(z)\|_p}{\|1+z\|_p}, \qquad 0\leqslant p\leqslant\infty, \tag{1.7}$$

whereas the inequality (1.4) can be replaced by

$$\|P(Rz)\|_p \leqslant n\frac{\|1+R^nz\|_p}{\|1+z\|_p}\|P(z)\|_p\,, \qquad 0\leqslant p\leqslant\infty. \tag{1.8}$$

For $p\geqslant 1$, inequality (1.7) is due to de Brujin [6] and inequality (1.8) is due to Boas and Rahman. Rahman and Schmeisser [14] extended both for $0\leqslant p<1$. For $p=\infty$, inequality (1.7) was conjectured by Erdös and later verified by Lax [12] and inequality (1.8) by Ankeny and Rivlin [1]. Inequalities (1.7) and (1.8) are further generalized by Aziz and Rather [[3] corollary 5, 6] as

$$\left\|zP'(z)+\beta\frac{n}{2}P(z)\right\|_p \leqslant n\left\|\left(1+\frac{\beta}{2}\right)z+\frac{\beta}{2}\right\|_p\frac{\|P(z)\|_p}{\|1+z\|_p}, \qquad p>0, \tag{1.9}$$

and

$$\begin{aligned}&\left\|P(Rz)+\beta\left(\frac{R+1}{2}\right)^nP(z)\right\|_p\\ &\leqslant\left\|\left(R^n+\beta\left(\frac{R+1}{2}\right)^n\right)z+1+\beta\left(\frac{R+1}{2}\right)^n\right\|_p\frac{\|P(z)\|_p}{\|1+z\|_p}, \qquad p>0,\end{aligned} \tag{1.10}$$

respectively for every $\beta\in\mathbb{C}$ with $|\beta|\leqslant 1$ and $R\geqslant 1$. For $p=\infty$, inequalities (1.9) and (1.10) are due to Jain [10] which were further generalized by Hans and Lal [9] for s^{th} derivative of polynomials. Recently S. Gulzar [7] obtained an L_p version of Hans and Lal [9] results and proved following theorems:

Theorem A. *If $P\in\mathcal{P}_n$, then for $\beta\in\mathbb{C}$ with $|\beta|\leqslant 1$, $1\leqslant s\leqslant n$, and $0\leqslant p<\infty$*

$$\left\|z^sP^{(s)}(z)+\beta\frac{n_s}{2^s}P(z)\right\|_p \leqslant n_s\left|1+\frac{\beta}{2^s}\right|\|P(z)\|_p\,, \tag{1.11}$$

where $n_s=n(n-1)(n-2)\dots(n-s+1)$.

Theorem B. *If $P\in\mathcal{P}_n$ and $P(z)$ does not vanish in $|z|\leqslant 1$, then for $\beta\in\mathbb{C}$ with $|\beta|\leqslant 1$, $1\leqslant s\leqslant n$, and $0\leqslant p<\infty$*

$$\left\|z^sP^{(s)}(z)+\beta\frac{n_s}{2^s}P(z)\right\|_p \leqslant n_s\left\|\left(1+\frac{\beta}{2^s}\right)z+\frac{\beta}{2^s}\right\|_p\frac{\|P(z)\|_p}{\|1+z\|_p}, \tag{1.12}$$

where $n_s=n(n-1)(n-2)\dots(n-s+1)$.

2. Main results

In this paper, we first present the following interesting result which is compact generalization of inequalities (1.3) – (1.6) and (1.11).

Theorem 2.1. *If $P \in \mathcal{P}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) + \beta (R+1)^{n-s} \frac{s! C(n,s)}{2^n} P(z) \right\|_p$$
$$\leqslant s! C(n,s) \left| R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| \|P(z)\|_p \,. \tag{2.1}$$

The result is best possible and equality in (2.1) *holds for* $P(z) = cz^n$, $c \neq 0$.

For taking $R = 1$ in (2.1) we obtain (1.11). The following result is obtained by letting $p \to \infty$ in (2.1).

Corollary 2.1. *If $P \in \mathcal{P}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) + \beta (R+1)^{n-s} \frac{s! C(n,s)}{2^n} P(z) \right\|_\infty$$
$$\leqslant s! C(n,s) \left| R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| \|P(z)\|_\infty \,. \tag{2.2}$$

The result is best possible and equality holds for $P(z) = cz^n$, $c \neq 0$.

Taking $\beta = 0$ in (2.1), we get the following compact generalization of inequalities of (1.3) and (1.4).

Corollary 2.2. *If $P \in \mathcal{P}_n$, then for $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) \right\|_p \leqslant s! C(n,s) R^{n-s} \|P(z)\|_p \,. \tag{2.3}$$

For taking both $s = 1$ and $R = 1$ in (2.3), we get inequality (1.3) and for taking $s = 0$, inequality (2.3) reduces to (1.4).

Remark 1. Inequality (1.5) can be obtained by putting $s = 1$ and $R = 1$ in (2.1) and for $s = 0$, inequality (2.1) reduces to (1.6).

Next, we present the following compact generalization of the inequalities (1.7), (1.8), (1.9), (1.10) and (1.12).

Theorem 2.2. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| \leqslant 1$, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

(2.4)
$$\left\| z^s P^{(s)}(Rz) + (R+1)^{n-s} \beta \frac{s! C(n,s)}{2^n} P(z) \right\|_p$$
$$\leqslant s! C(n,s) \left\| \left(R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right) z + \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\beta}{2^n} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}.$$

The result is best possible and equality in (2.4) *holds for* $P(z) = az^n + b$, $|a| = |b| = 1$.

Remark 2. By letting $p \to \infty$ in (2.4), we obtain a result due to Jain [[11], Theorem 3]. Inequality (1.12) can be obtained by putting $R = 1$ in (2.4).
The following is compact generalization of inequalities (1.7) and (1.8) is obtained by putting $\beta = 0$ in (2.4).

Corollary 2.3. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| \leqslant 1$, then for $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\left\| z^s P^{(s)}(Rz) \right\|_p \leqslant s! C(n,s) \left\| R^{n-s} z + \frac{d^s(1)}{dz^s} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}. \tag{2.5}$$

For $s = 1$ and $R = 1$, inequality (2.5) reduces to (1.7) and inequality (1.8) is obtained by putting $s = 0$ in (2.5). Also for $s = 1$ and $R = 1$ in (2.4), we obtain (1.9) and inequality (1.10) can be obtained by putting $s = 0$ in (2.4).

Finally, we establish the following result for self-inversive polynomials.

Theorem 2.3. *If $P \in \mathcal{P}_n$ and $P(z)$ is a self-inversive polynomial, then for $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, and $0 \leqslant p < \infty$*

$$\begin{aligned} &\left\| z^s P^{(s)}(Rz) + (R+1)^{n-s} \beta \frac{s! C(n,s)}{2^n} P(z) \right\|_p \\ &\leqslant s! C(n,s) \left\| \left(R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right) z + \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\beta}{2^n} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}. \end{aligned} \tag{2.6}$$

If we let $p \to \infty$ in (2.6) , we obtain the following result:

Corollary 2.4. *If $P \in \mathcal{P}_n$ and $P(z)$ is a self-inversive polynomial, then for $R \geqslant 1$, and $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$*

$$\begin{aligned} &\left\| z^s P^{(s)}(Rz) + (R+1)^{n-s} \beta \frac{s! C(n,s)}{2^n} P(z) \right\|_\infty \\ &\leqslant \frac{s! C(n,s)}{2} \left\{ \left| R^{n-s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| + \left| \frac{d^s(1)}{dz^s} + (R+1)^{n-s} \frac{\beta}{2^n} \right| \right\} \|P(z)\|_\infty . \end{aligned} \tag{2.7}$$

3. Lemmas

For the proof of these theorems, we need the following lemmas. The first lemma is the following well known-result ([[13] Theorem 14.1.2 and its proof , corollary 12.1.3] and [[6] Theorem 1 and its proof]).

Lemma 3.1. *Let $F \in \mathcal{P}_n$ and let P be a polynomial of degree at most n, such that $|P(z)| \leqslant |F(z)|$ for $|z| = 1$. If $F(z) \neq 0$ for $|z| < 1$ (resp. $|z| > 1$) and for every $z \in \mathbb{C}$ and every $\alpha, P(z) \neq e^{i\alpha} F(z)$, then*
(i) $|P(z)| \leqslant |F(z)|$ for $|z| < 1$ (resp. $|z| > 1$),

(ii) $F(z)+\beta P(z)\neq 0$ *for* $|z|<1$ *(resp.* $|z|>1$*) and* $\beta\in\mathbb{C}$ *with* $|\beta|\leqslant 1$ *and*
(iii)$P(z)+\lambda F(z)\neq 0$ *for* $|z|<1$ *(resp.* $|z|>1$*) and* $\lambda\in\mathbb{C}$ *with* $|\lambda|\geqslant 1$.

Lemma 3.2. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for every* $R>1$*, and* $|z|=1$*,*

$$|P(Rz)|\geqslant\left(\frac{R+1}{2}\right)^n|P(z)|.$$

Proof. Since all the zeros of $P(z)$ lie in $|z|\leqslant 1$, we write

$$P(z)=c\prod_{j=1}^{n}\left(z-r_je^{i\theta_j}\right),$$

where $r_j\leqslant 1$. Now for $0\leqslant\theta<2\pi$, $R>1$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta}-r_je^{i\theta_j}}{e^{i\theta}-r_je^{i\theta_j}}\right|&=\left\{\frac{R^2+r_j^2-2Rr_j\cos(\theta-\theta_j)}{1+r_j^2-2r_j\cos(\theta-\theta_j)}\right\}^{1/2}\\&\geqslant\left\{\frac{R+r_j}{1+r_j}\right\}\geqslant\left\{\frac{R+1}{2}\right\},\quad\text{for } j=1,2,\cdots,n.\end{aligned}$$

Hence

$$\left|\frac{P(Re^{i\theta})}{P(e^{i\theta})}\right|=\prod_{j=1}^{n}\left|\frac{Re^{i\theta}-r_je^{i\theta_j}}{e^{i\theta}-r_je^{i\theta_j}}\right|\geqslant\prod_{j=1}^{n}\left(\frac{R+1}{2}\right)=\left(\frac{R+1}{2}\right)^n$$

for $0\leqslant\theta<2\pi$. This implies for $|z|=1$ and $R>1$,

$$|P(Rz)|\geqslant\left(\frac{R+1}{2}\right)^n|P(z)|,$$

which completes the proof of Lemma 3.2. □

By applying lemma 3.2 to the polynomial $P^s(z)$, $(1\leqslant s\leqslant n)$, we obtain

Lemma 3.3. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for* $1\leqslant s\leqslant n$

$$\left|P^{(s)}(Rz)\right|\geqslant\left(\frac{R+1}{2}\right)^{n-s}\left|P^{(s)}(z)\right|,\qquad R\geqslant 1\quad\textit{and}\quad |z|=1.$$

Lemma 3.4. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for* $0\leqslant s\leqslant n$*,*

$$|z^sP^{(s)}(z)|\geqslant\frac{s!C(n,s)}{2^s}|P(z)|,\qquad R\geqslant 1\quad\textit{and}\quad |z|=1.$$

The above lemma is simply consequences of repeated application of Turán theorem [15].

Lemma 3.4 along with lemma (3.3) leads to following lemma:

Lemma 3.5. *If* $P\in\mathcal{P}_n$ *and* $P(z)$ *have all its zeros in* $|z|\leqslant 1$*, then for* $0\leqslant s\leqslant n$*,*

$$|z^sP^{(s)}(Rz)|\geqslant(R+1)^{n-s}\frac{s!C(n,s)}{2^n}|P(z)|,\qquad R\geqslant 1\quad\textit{and}\quad |z|=1,$$

and for every $\beta\in\mathbb{C}$ *with* $|\beta|\leqslant 1$*, the zeros of polynomial*

$$z^sP^{(s)}(Rz)+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(z)$$

lies in $|z| \leqslant 1$.

The second part of above lemma is the consequences of lemma 3.1.
The next lemma is due to Jain [11].

Lemma 3.6. *Let* $F(z)$ *be a polynomial of degree* n *having all its zeros in* $|z| \leqslant 1$ *and* $P(z)$ *be a polynomial of degree not exceeding that of* $F(z)$ *such that*

$$|P(z)| \leqslant |F(z)|, \qquad |z| = 1,$$

then for $R \geqslant 1$, $0 \leqslant s \leqslant n$, *and* $|\beta| \leqslant 1$

$$\left|z^s P^{(s)}(Rz) + \beta(R+1)^{n-s} \tfrac{s!C(n,s)}{2^n} P(z)\right| \leqslant$$
$$\left|z^s F^{(s)}(Rz) + \beta(R+1)^{n-s} \tfrac{s!C(n,s)}{2^n} F(z)\right| \quad for \quad |z| \geqslant 1.$$

The next lemma follows immediately from lemma 3.6 by taking $F(z) = Q(z)$ where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 3.7. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, *and* $R \geqslant 1$

$$\begin{aligned} &\left|z^s P^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} P(z)\right| \\ &\leqslant \left|z^s Q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} Q(z)\right| \quad for |z| \geqslant 1, \end{aligned} \tag{3.1}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 3.8. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$ *and* $Q(z) = z^n \overline{P(1/\overline{z})}$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, *and* α *real*

$$\left(z^s P^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} P(z)\right) e^{i\alpha} + z^n \overline{M(1/\overline{z})} \neq 0 \qquad for \quad |z| < 1,$$

where $M(z) = z^s Q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} Q(z)$.

Proof. Since $P(z) = \sum_{j=0}^{n} a_j z^j$ does not vanish in $|z| < 1$, therefore by lemma 3.7 for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $|z| = 1$, we have

$$\begin{aligned} \left|z^s P^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} P(z)\right| &\leqslant \left|z^s Q^{(s)}(Rz) + \beta(R+1)^{n-s} \frac{s!C(n,s)}{2^n} Q(z)\right| \\ &= |M(z)| = |z^n \overline{M(1/\overline{z})}|. \end{aligned}$$

Since $P(0) \neq 0$ implies $\deg Q(z) = n$. Moreover $Q(z) \neq 0$ for $|z| > 1$ and then lemma 3.5 implies that $M(z) \neq 0$ for $|z| > 1$. Therefore $z^n \overline{M(1/\overline{z})} \neq 0$ for $|z| < 1$. Then by lemma 3.1 for $|z| < 1$

$$\left(z^s P^{(s)}(Rz) + \beta(R+1)^{n-s} \tfrac{s!C(n,s)}{2^n} P(z)\right) e^{i\alpha} + z^n \overline{M(1/\overline{z})} \neq 0. \qquad \square$$

Next we describe a result of Arestov [2].
For $\delta = (\delta_0, \delta_1, \cdots, \delta_n) \in \mathbb{C}^{n+1}$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible if it preserves one of the following properties:
(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leqslant 1\}$,
(ii) $P(z)$ has all its zeros in$\{z \in \mathbb{C} : |z| \geqslant 1\}$.
The result of Arestov [2] may now be stated as follows.

Lemma 3.9. [2, Theorem 4] *Let $\phi(x) = \psi(logx)$ where ψ is a convex non decreasing function on $\mathbb{R}$. Then for all $P \in \mathcal{P}_n$ and each admissible operator Λ_δ,*

$$\int_0^{2\pi} \phi(|\Lambda_\delta P(e^{i\theta})|)d\theta \leqslant \int_0^{2\pi} \phi(A(\delta, n)|P(e^{i\theta})|)d\theta,$$

where $A(\delta, n) = max(|\delta_0|, |\delta_n|)$.

In particular, Lemma 3.9 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$. Therefore, we have

$$\left\{\int_0^{2\pi} (|\Lambda_\delta P(e^{i\theta})|^p)d\theta\right\}^{1/p} \leqslant A(\delta, n)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{3.2}$$

From lemma 3.9, we deduce the following result:

Lemma 3.10. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $0 \leqslant s \leqslant n$, $R \geqslant 1$, α real, and $p > 0$,*

$$\begin{aligned}\int_0^{2\pi} &\left|\left(e^{is\theta}P^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(e^{i\theta})\right)e^{i\alpha} + e^{in\theta}\overline{M(e^{i\theta})}\right|^p d\theta \\ &\leqslant (s!C(n,s))^p\left|\left(R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha} + \frac{d^s(1)}{dz^s} + (R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p \int\limits_0^{2\pi} |P(e^{i\theta})|^p d\theta,\end{aligned} \tag{3.3}$$

where $M(z) = z^sQ^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}Q(z)$.

Proof. Since $P(z) = \sum\limits_{j=o}^{n} a_j z^j$ does not vanish in $|z| < 1$. Therefore by lemma 3.8, the polynomial

$$\begin{aligned}\Lambda_\delta P(z) &= \left(z^sP^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(z)\right)e^{i\alpha} + z^n\overline{M(1/\overline{z})} \\ &= s!C(n,s)\left\{\left(R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha} + \frac{d^s(1)}{dz^s} + (R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right\}a_nz^n + \\ &\quad ... + s!C(n,s)\left\{\left(\frac{d^s(1)}{dz^s} + (R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha} + R^{n-s} + (R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right\}a_0\end{aligned}$$

does not vanish in $|z| < 1$ for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and α real. Therefore Λ_δ is an admissible operator. Applying (3.2) we get desired result for $p > 0$. This completes the proof of lemma 3.10. $\square$

4. Proofs of the theorems

Proof of Theorem 2.1. By hypothesis $P \in \mathcal{P}_n$, we can write

$$P(z) = P_1(z)P_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(z - z_j), \quad k \geqslant 1,$$

where the zeros $z_1, z_2, \ldots, z_k$ of $P_1(z)$ lie in $|z| \leqslant 1$ and the zeros $z_{k+1}, z_{k+2}, \ldots, z_n$ of $P_2(z)$ lie in $|z| > 1$. Since all the zeros of $P_2(z)$ lie in $|z| > 1$, the polynomial $Q_2(z) = z^{n-k}\overline{P_2(1/\overline{z})}$ has all its zeroes in $|z| < 1$ and $|Q_2(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$T(z) = P_1(z)Q_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(1 - z\overline{z_j}),$$

then all the zeros of $T(z)$ lie in $|z| \leqslant 1$, and for $|z| = 1$,

$$|T(z)| = |P_1(z)|\,|Q_2(z)| = |P_1(z)|\,|P_2(z)| = |P(z)|\,.$$

Now on applying lemma 3.6 we get for $R \geqslant 1$, $0 \leqslant s \leqslant n$, and $|\beta| \leqslant 1$

$$\begin{aligned}&\left|z^s P^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(z)\right| \\ &\leqslant \left|z^s T^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(z)\right| \quad \text{for} \quad |z| \geqslant 1,\end{aligned}$$

which in particular gives for $p > 0$,

$$\begin{aligned}&\int_0^{2\pi}\left|e^{is\theta}P^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(e^{i\theta})\right|^p d\theta \\ &\leqslant \int_0^{2\pi}\left|e^{is\theta}T^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(e^{i\theta})\right|^p d\theta.\end{aligned} \tag{4.1}$$

Since all the zeros of $T(z)$ lies in $|z| \leqslant 1$, by lemma 3.5 the polynomial

$$z^s T^{(s)}(Rz) + \beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}T(z),$$

has also all its zeros in $|z| \leqslant 1$ for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$. Therefore if $T(z) = c_n z^n + c_{n-1}z^{n-1} + ... + c_1 z + c_0$, then the operator Λ_δ defined by

$$\begin{aligned}\Lambda_\delta T(z) &= z^s T^{(s)}(Rz) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(z)\\ &= s!C(n,s)\left(R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right)c_n z^n + ...\\ &+ s!C(n,s)\left(\frac{d^s(1)}{dz^s} + (R+1)^{n-s}\frac{\beta}{2^n}\right)c_0,\end{aligned}$$

is admissible. Hence by (3.2) of lemma 3.9 for each $p > 0$, we have

(4.2)
$$\int_0^{2\pi}\left|e^{is\theta}T^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(e^{i\theta})\right|^p d\theta \leqslant (c(\delta))^p \int_0^{2\pi}|T(e^{i\theta})|^p d\theta,$$

where $c(\delta) = max\left(s!C(n,s)\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|, \quad s!C(n,s)\left|\frac{d^s(1)}{dz^s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|\right)$. For every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $R \geqslant 1$, it can be easily verified that $c(\delta) = s!C(n,s)\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|$. Thus from (4.2), we have

$$\begin{aligned}&\int_0^{2\pi}\left|e^{is\theta}T^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}T(e^{i\theta})\right|^p d\theta\\ &\leqslant (s!C(n,s))^p\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|^p\int_0^{2\pi}|T(e^{i\theta})|^p d\theta.\end{aligned} \tag{4.3}$$

Combining inequalities (4.1) and (4.3) and noting that $|T(e^{i\theta})| = |P(e^{i\theta})|$, we obtain

$$\begin{aligned}&\int_0^{2\pi}\left|e^{is\theta}P^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(e^{i\theta})\right|^p d\theta\\ &\leqslant (s!C(n,s))^p\left|R^{n-s} + (R+1)^{n-s}\frac{\beta}{2^n}\right|^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.\end{aligned}$$

This proves theorem (2.1) for $p > 0$. To obtain this result for $p = 0$, we simply make $p \to 0+$. $\square$

Proof of Theorem 2.2. By hypothesis $P(z)$ does not vanish in $z < 1$, therefore by lemma 3.6 for every $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $0 \leqslant \theta \leqslant 2\pi$

$$\begin{aligned}&\left|e^{is\theta}P^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}P(e^{i\theta})\right|\\ &\leqslant \left|e^{is\theta}Q^{(s)}(Re^{i\theta}) + \beta(R+1)^{n-s}\frac{s!C(n,s)}{2^n}Q(e^{i\theta})\right|,\end{aligned} \tag{4.4}$$

where $Q(z)=z^n\overline{P(1/\overline{z})}$. Also by lemma 3.10,

(4.5)

$$\int\limits_0^{2\pi}\left|e^{i\alpha}F(\theta)+e^{in\theta}\overline{M(e^{i\theta})}\right|^p d\theta$$

$$\leqslant (s!C(n,s))^p\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p\int\limits_0^{2\pi}|P(e^{i\theta})|^p d\theta,$$

where

$$F(\theta)=e^{is\theta}P^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(e^{i\theta})$$

and

$$M(e^{i\theta})=e^{is\theta}Q^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}Q(e^{i\theta}).$$

Integrating both sides of (4.5) with respect to α from 0 to 2π, we get for each $p>0$,

$$\int\limits_0^{2\pi}\int\limits_0^{2\pi}\left|e^{i\alpha}F(\theta)+e^{in\theta}\overline{M(e^{i\theta})}\right|^p d\theta d\alpha$$

(4.6)

$$\leqslant (s!C(n,s))^p\int\limits_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p d\alpha\int\limits_0^{2\pi}|P(e^{i\theta})|^p d\theta.$$

Now for every real α, $t\geqslant 1$ and $p>0$, we have

$$\int\limits_0^{2\pi}|t+e^{i\alpha}|^p d\alpha\geqslant\int\limits_0^{2\pi}|1+e^{i\alpha}|^p d\alpha.$$

If $F(\theta)\neq 0$, we take $t=|M(e^{i\theta})/F(\theta)|$, then by (4.4), $t\geqslant 1$ and we get

$$\int\limits_0^{2\pi}\left|e^{i\alpha}F(\theta)+e^{in\theta}\overline{M(e^{i\theta})}\right|^p d\alpha=|F(\theta)|^p\int\limits_0^{2\pi}\left|e^{i\alpha}+\frac{e^{in\theta}\overline{M(e^{i\theta})}}{F(\theta)}\right|^p d\alpha$$

$$=|F(\theta)|^p\int\limits_0^{2\pi}\left|e^{i\alpha}+\left|\frac{M(e^{i\theta})}{F(\theta)}\right|\right|^p d\alpha\geqslant|F(\theta)|^p\int\limits_0^{2\pi}|1+e^{i\alpha}|^p d\alpha.$$

For $F(\theta)=0$, this inequality is trivially true. Using this in (4.6) , we conclude that for every $\beta\in\mathbb{C}$ with $|\beta|\leqslant 1$

(4.7)

$$\int\limits_0^{2\pi}|F(\theta)|^p d\theta\int\limits_0^{2\pi}|1+e^{i\alpha}|^p d\alpha$$

$$\leqslant (s!C(n,s))^p\int\limits_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p d\alpha\int\limits_0^{2\pi}|P(e^{i\theta})|^p d\theta.$$

Since

$$\int_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\overline{\beta}}{2^n}\right|^p d\alpha$$

$$=\int_0^{2\pi}\left|\left|R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right|e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\left|\frac{\overline{\beta}}{2^n}\right|\right|^p d\alpha$$

$$=\int_0^{2\pi}\left|\left(R^{n-s}+(R+1)^{n-s}\frac{\beta}{2^n}\right)e^{i\alpha}+\frac{d^s(1)}{dz^s}+(R+1)^{n-s}\frac{\beta}{2^n}\right|^p d\alpha, \tag{4.8}$$

the desired result follows immediately by combining (4.7) and (4.8). This proves Theorem 2.2 for $p>0$. To establish this result for $p=0$, we simply make $p\to 0$. □

Proof of Theorem 2.3. Since $P(z)$ is a self-inversive polynomial, we have $P(z)=uQ(z)$ for all $z\in\mathbb{C}$ where $|u|=1$ and $Q(z)=z^n\overline{P(1/\overline{z})}$. Therefore for every $\beta\in\mathbb{C}$ with $|\beta|\leqslant 1$,

$$\left|e^{is\theta}P^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}P(e^{i\theta})\right|=$$
$$\left|e^{is\theta}Q^{(s)}(Re^{i\theta})+\beta(R+1)^{n-s}\tfrac{s!C(n,s)}{2^n}Q(e^{i\theta})\right|,$$

for all $z\in\mathbb{C}$. Using (4.4) and proceeding similarly as in the proof of Theorem 2.2 we get the desired result. This completes the proof of Theorem 2.3. □

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