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INFINITELY MANY SOLUTIONS FOR KIRCHHOFF TYPE EQUATIONS INVOLVING DEGENERATE OPERATOR

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Abstract. In this paper, we study the existence of infinitely many nontrivial solutions for a class of nonlinear Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)\Delta_\lambda u+V(x)u=f(x,u),\ \text{in }\mathbb{R}^N$$

where constants $a>0,\ b>0$, Δ_λ is a strongly degenerate elliptic operator, and f is a function with a more general superlinear conditions or sublinear conditions.

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1. Introduction

This paper is concerned with a class of nonlinear Kirchhoff type equations

$$(1.1)\qquad -\left(a+b\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)\Delta_\lambda u+V(x)u=f(x,u),\quad \text{in }\mathbb{R}^N$$

where constants $a,b>0$, $N\geqslant 1$, $V\in C(\mathbb{R}^N,\mathbb{R})$, $\nabla_\lambda=(\lambda_1\partial_{x_1}u,...,\lambda_N\partial_{x_N}u)$ and Δ_λ is a strongly degenerate elliptic operator of the following form

$$\Delta_\lambda:=\sum_{i=1}^N\partial_{x_i}(\lambda_i^2\partial_{x_i}),\quad \lambda=(\lambda_1,\cdots,\lambda_N):\mathbb{R}^N\to\mathbb{R}^N.$$

Kogoj and Lanconelli in [7] firstly introduced the strongly degenerate elliptic operator Δ_λ. After that, a growing attention has been devoted to Δ_λ-Laplacians. Kogoj and Lanconelli in [7] assume that the operator is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^N$. Kogoj and Sonner [8] showed that global well-posedness and long-time behavior of solutions of semilinear degenerate parabolic involving the Δ_λ-Laplacians, and this result was extended in [9], where hyperbolic problems were considered. Ahn and My [2] proved that Liouville-type theorems for elliptic inequalities involving the Δ_λ-Laplacians. Finally, Kogoj and Sonner remark that the Δ_λ-Laplacians belong to the more general class of $X-$

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elliptic operators. The Δ_λ operator contains many degenerate elliptic operators such as the Grushin-type operator

$$G_\alpha = \Delta_x + |x|^{2\alpha}\Delta_y, \quad \alpha > 0,$$

where (x, y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $N_1 + N_2 = N$, and the operator of the form

$$P_{\alpha,\beta,\gamma} = \Delta_x + |x|^{2\alpha}\Delta_y + |x|^{2\beta}|y|^{2\gamma}\Delta_z, \ (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}, \ N_1 + N_2 + N_3 = N,$$

where α, β and γ are real positive constants. We can refer the readers to [1] for some important properties of this operator.

In the last decades, Δ_λ elliptic equations

$$\begin{cases} -\Delta_\lambda u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, has been studied by many authors. See [3, 1, 7, 13, 14, 19] and the references therein. The nonlinear term f satisfies the Ambrosetti-Rabinowitz(AR) condition is studied in [7]. The (AR) condition guarantees the boundedness of the Palais-Smale(PS) sequence of the energy functional, which is essential for the application of the critical point theorem. When f does not satisfy the (AR) condition is studied in [3, 1, 14]. At present, some authors began to consider problem (1.2) on unbounded domain $\mathbb{R}^N$. The main difficulty in $\mathbb{R}^N$ is lack of compactness of Sobolev embedding. For this reason, some authors work on the subspace of Sobolev space to overcome this difficulty. Luyen and Tri [15] considered that $V(x)$ is a coercive potential, which ensures that the weighted Sobolev space embedding is compactness. They proved that Δ_λ equation possess infinity many solutions with the nonlinear term has (AR) condition.

Recently, a class of Kirchhoff-type elliptic equation

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

has received extensive attention and research by many authors. Cheng and Wu [4] proved the existence result of positive solutions to Kirchhoff-type problems with the variational method. Mao and Zhang [16] shows that in the absence of (PS) condition, the minimax methods and invariant sets of descent flow are used to study multiple solutions of Kirchhoff type problems. The problem (1.3) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla_x u|^2 dx\right)\Delta_x u = g(x, u) \tag{1.4}$$

which was proposed by Kirchhoff in 1883 as a generalization of the well-known d'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=g(x,u)$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Problem (1.4) models several physical systems, where u describes a process which depends on the average of itself. A parabolic version of equation (1.4) be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed dependent on the "energy" of the entire system with u being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which gives rise to equations of the type $u_t - a(\int_\Omega u dx)\Delta u = h$.

In this paper, we want to use the idea of [21] to study the existence of infinitely many nontrivial solutions for the Kirchhoff type problem with Δ_λ type operator. Now, we give the following assumptions on potential $V(x)$:

(V_1) $V \in C(\mathbb{R}^N,\mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

(V_2) There exists a constant $R > 0$ such that

$$\int_{|x|\geqslant R} V^{-1}dx < \infty.$$

For the nonlinearity f, we give the following assumptions:

(f_1) $f \in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ and there exist constants $C_1, C_2 > 0$ and $p \in (2, 2^*_\lambda)$ such that

$$|f(x,u)| \leqslant C_1|u| + C_2|u|^{p-1},\ \forall (x,u) \in \mathbb{R}^N\times\mathbb{R}.$$

where $2^*_\lambda = \frac{2Q}{Q-2}$ and Q denotes the homogeneous dimension of $\mathbb{R}^N$ with respect to a group of dilations(see Section 2 for more details).

(f_2) $f(x,u) = -f(x,-u),\ \forall(x,u) \in \mathbb{R}^N\times\mathbb{R}$.

(f_3) $\lim_{|u|\to\infty}\frac{|F(x,u)|}{|u|^4} = \infty$, uniformly in $x \in \mathbb{R}^N$, $Q < 4$, and there exists $r_0 \geqslant 0$, such that $F(x,u) \geqslant 0,\ \forall(x,u) \in \mathbb{R}^N\times\mathbb{R},\ |u| \geqslant r_0$, where $F(x,u) := \int_0^u f(x,t)dt$.

(f_4) There exist $\beta \geqslant 0$ such that $F(x,u) \leqslant \frac{1}{4}f(x,u)u + \beta u^2$, $\forall(x,u) \in \mathbb{R}^N\times\mathbb{R}$.

(f_5) $F(x,u) \geqslant 0, \forall(x,u) \in \mathbb{R}^N\times\mathbb{R}$ and $G(x,h) \leqslant G(x,l)$ whenever $(h,l) \in \mathbb{R}^+\times\mathbb{R}^+$ and $h \leqslant l$, where $G(x,u) := \frac{1}{4}f(x,u)u - F(x,u)$.

In the following theorem, we give the multiplicity result of the solution of problem (1.1) when f satisfies the superlinear condition.

Theorem 1.1. *Assume that the potential $V(x)$ satisfies $(V_1),(V_2)$ and nonlinearity $f(x,u)$ satisfies $(f_1)-(f_4)$. Then the problem(1.1) has possesses infinitely many nontrivial solutions $\{u_k\}$ such that*

$$\lim_{k\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\left(a|\nabla_\lambda u_k|^2+V(x)u_k^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_k|^2dx\right)^2-\int_{\mathbb{R}^N}F(x,u_k)dx=+\infty.$$

Theorem 1.2. *Assume that the potential $V(x)$ satisfies $(V_1),(V_2)$ and nonlinearity $f(x,u)$ satisfies $(f_1)-(f_3)$ and (f_5). Then the problem(1.1) has possesses infinitely many nontrivial solutions $\{u_k\}$ such that*

$$\lim_{k\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\left(a|\nabla_\lambda u_k|^2+V(x)u_k^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_k|^2dx\right)^2-\int_{\mathbb{R}^N}F(x,u_k)dx=+\infty.$$

Next, in addition to discussing the above results, we also consider the multiplicity result that can still obtain a solution of problem (1.1) when f satisfies the sublinear.

(f_6) $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, there exist constant $1<q_1<q_2<2$, such that

$$|f(x,t)|\leqslant q_1|t|^{q_1-1}+q_2|t|^{q_2-1}.$$

(f_7) There exist a bounded open set $\tilde{B}\subset\mathbb{R}^N$ and constants $\delta,\xi>0,\ q_3\in(1,2)$ such that

$$F(x,u)\geqslant\xi|u|^{q_3},\ \forall(x,u)\in\tilde{B}\times[-\delta,\delta].$$

Now, we give the second result:

Theorem 1.3. *Assume that the potential $V(x)$ satisfies $(V_1),(V_2)$ and nonlinearity $f(x,u)$ satisfies $(f_2),(f_6),(f_7)$. Then the problem (1.1) has possesses infinitely many nontrivial solutions $\{u_k\}$.*

Remark 1.1. Compared with problem (1.1), we extend the equation to operator Δ_λ, because operator Δ_λ is more complicated with the addition of function λ. As can be seen from [7], when the function λ is smooth, then Δ_λ is the general operator class studied by Hömander in [5], and is hypoelliptic. The typical example is the Grushin-type operator, which means that Δ_λ is a generalization of Grushin-type operator. Later, Δ_λ belongs to the more general $X-$*elliptic operators* introduced in [10], and has some of the same important homogeneity as the classical Laplacian. Therefore, it is meaningful for us to extend the problem (1.3) to a more general Kirchhoff-type equation, and it is applicable to more environments.

Now, we give an example that satisfies all the assumptions of Theorem 1.1, as follows $f(x,u)=u|\sin x|+|u|^3u|\cos x|$, obviously, $F(x,u)=\frac{1}{2}u^2|\sin x|+\frac{1}{5}|u|^5|\cos x|$.

Of course, there is also f that satisfies the sublinear condition, such as $f(x,u) = \frac{4}{3}|u|^{-\frac{2}{3}}u\sin^2 x + \frac{5}{4}|u|^{-\frac{3}{4}}\cos^2 x$, and $F(x,u) = |u|^{\frac{4}{3}}\sin^2 x + |u|^{\frac{5}{4}}\cos^2 x$. Through simple calculations, it can be verified that the assumptions of each theorem are satisfied.

The main structure of this article is as follows. In the second section, we give some preliminary knowledge and main theorems. In the third section, we use the symmetric mountain pass theorem to prove Theorems 1.1 and 1.2. In the fourth section, we apply the theorem in [18], to get the multiplicity result of the solution.

2. Preliminaries

We recall the functional setting in [7, 3]. We consider the operator of the form

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2 \partial_{x_i}),$$

where $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, N$. Here the function $\lambda_i : \mathbb{R}^N \to \mathbb{R}$ are continuous, strictly positive and of C^1 outside the coordinate hyperplane, i.e. $\lambda_i > 0, i = 1, \ldots, N$ in $\mathbb{R}^N \setminus \prod$, where $\prod = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0\}$. As in [7], we assume that λ_i satisfy the following properties:

(1) $\lambda_1(x) \equiv 1,\ \lambda_i(x) = \lambda_i(x_1,\ \ldots, x_{i-1}),\ i = 2, \ldots N$;

(2) For every $x \in \mathbb{R}^N$, $\lambda_i(x) = \lambda_i(x^*),\ i = 1, \ldots, N$, where $x^* = (|x_1|, \ldots, |x_N|)$ if $x = (x_1, \ldots, x_N)$;

(3) There exists a constant $\rho \geqslant 0$ such that

$$0 \leqslant x_k \partial_{x_k} \lambda_i(x) \leqslant \rho \lambda_i(x), \quad \forall k \in \{1, \ldots, i-1\}, i = 2, \ldots, N,$$

and for every $x \in \mathbb{R}_+^N := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_i \geqslant 0,\ \forall i = 1, \ldots, N\}$;

(4) Exists a group of dilations $\{\delta_t\}_{t>0}$

$$\delta_t : \mathbb{R}^N \to \mathbb{R}, \quad \delta_t(x) = \delta_t(x_1, \ldots, x_N) = (t^{\epsilon_1}x_1, \ldots, t^{\epsilon_N}x_N),$$

where $1 \leqslant \epsilon_1 \leqslant \epsilon_2 \leqslant \ldots \leqslant \epsilon_N$, such that λ_i is $\delta_t -$ homogeneous of degree $\epsilon_i - 1$, i.e.

$$\lambda_i(\delta_t(x)) = t^{\epsilon_i - 1}\lambda_i(x), \quad \forall x \in \mathbb{R}^N,\ t > 0,\ i = 1, \ldots, N.$$

This implies that the operation Δ_λ is $\delta_t -$ homogeneous of degree two, i.e.

$$\Delta_\lambda(u(\delta_t(x))) = t^2(\Delta_\lambda u)(\delta_t(x)), \quad \forall u \in C^\infty(\mathbb{R}^N).$$

We denote by Q the homogeneous dimension of $\mathbb{R}^N$ with respect to group of dilations $\{\delta_t\}_{t>0}$, i.e.

$$Q := \epsilon_1 + \cdots + \epsilon_N.$$

The homogeneous Q plays a crucial role, both in the geometry and the functional associated to the operator Δ_λ.

Now, we denote by $W_\lambda^{1,2}(\mathbb{R}^N)$ the closure of $C_0^1(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{W_\lambda^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\nabla_\lambda u|^2 + u^2)dx\right)^{\frac{1}{2}},$$

is Hilbert space with the inner product

$$(u, v) = \int_{\mathbb{R}^N} (\nabla_\lambda u \nabla_\lambda v + uv)dx.$$

Under the hypotheses (V_1), we define space

$$E = \left\{u \in W_\lambda^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2dx < +\infty\right\},$$

with the inner product

$$(u, v) = \int_{\mathbb{R}^N} (\nabla_\lambda u \nabla_\lambda v + V(x)uv)\, dx,$$

and the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla_\lambda u|^2 + V(x)u^2)dx.$$

Here, we denote $\|\cdot\|_p$ as the norm of Lebesgue space $L^p(\mathbb{R}^N)$.

Proposition 2.1. *Under the assumptions (V_1) and (V_2), the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for every $p \in [1, 2_\lambda^*)$.*

Proof. In [15], we know that under the assumption of (V_1), the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $p \in [2, 2_\lambda^*]$, and $E \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ is compact for $p \in [1, 2_\lambda^*)$. Then there are constant C_p such that

$$\|u\|_p \leqslant C_p\|u\|,\ \forall u \in E. \tag{2.1}$$

When we want to embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in [1, 2_\lambda^*)$ under the assumption of (V_1) and (V_2), it suffices to prove the result for $p = 1$. Assume $u_n \rightharpoonup u$ in E. For any $R > 0$, write

$$\int_{\mathbb{R}^N} |u_n - u|dx = \int_{|x|\leqslant R} |u_n - u|dx + \int_{|x|>R} |u_n - u|dx. \tag{2.2}$$

By the Hölder inequality to obtain that

$$\int_{|x|>R} |u_n - u|dx \leqslant \left(\int_{|x|>R} V|u_n - u|^2dx\right)^{\frac{1}{2}} \left(\int_{|x|>R} V^{-1}dx\right)^{\frac{1}{2}} = o_R(1), \tag{2.3}$$

where $o_R(1)$ is a quantity that converges to 0 as $R \to \infty$ uniformly for n. Then $u_n \to u$ strongly in $L^1(\mathbb{R}^N)$ since $u_n \to u$ in $L^1_{loc}(\mathbb{R}^N)$. □

Remark 2.1. Several important problems arising in many research fields such as physics and differential geometry lead to consider semilinear variational elliptic equations defined on unbounded domains of the Euclidean space and a great deal of work has been devoted to their study. From the mathematical point of view, probably the main interest relies on the fact that often the tools of nonlinear functional analysis, based on compactness arguments, can not be used, at least in a straight forward way, and some new techniques have to be developed. The seminal paper [11] by Lions has inspired a (nowadays usual) way to overcome the lack of compactness by exploiting symmetry. This approach is fruitful in the study of variational elliptic problems in presence of a suitable continuous action of a topological group on the Sobolev space where the solutions are being sought.

Here, we use another skill following the idea of Rabinowitz [17] to get the Sobolev embedding is compact by the potential V. Luyen and Tri [15] use the idea of Rabinowitz to get the Sobolev compact embedding, but they only obtained the embedding map from E into $L^p(\mathbb{R}^N)$ is compact for $2 \leqslant p < 2^*_\lambda$. We want to study the sublinear case, so we give a wider interval for the Sobolev embedding. Moreover, Assumption (V_2) makes V look like a well-shaped potential.

Now, we define the following energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (a|\nabla_\lambda u|^2 + V(x)u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)dx,\ u \in E.$$

Obviously, given constant $a > 0$, $\int_{\mathbb{R}^N}(|\nabla_\lambda u|^2 + V(x)u^2)dx$ is equivalent to $\int_{\mathbb{R}^N}(a|\nabla_\lambda u|^2 + V(x)u^2)dx$. Hence, the norm of u in E denoted by

$$\|u\| = \left(\int_{\mathbb{R}^N} (a|\nabla_\lambda u|^2 + V(x)u^2)dx\right)^{\frac{1}{2}}$$

that is,

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)dx. \tag{2.4}$$

Definition 2.1. A sequence $\{u_n\} \subset E$ is said to be a $(C)_c-$*sequence* if $J(u_n) \to c$ and $J'(u_n)(1 + \|u_n\|) \to 0$. J is said to satisfy the $(C)_c -$ *condition* if any $(C)_c -$ *sequence* has a convergent subsequence.

Definition 2.2. A sequence $\{u_n\} \subset E$ is said to be a $(PS)-$*sequence* if $J(u_n) \leqslant c$ and $J'(u_n) \to 0,\ n \to \infty$. J is said to satisfy $(PS)-$*condition* if any $(PS)-$*sequence* has a convergent subsequence.

Definition 2.3. Let X be a Banach space, $J \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. Set

$$\Sigma = \{A \subset X - \{0\} : A \text{ is closed in } X \text{ and symmetric with respect to } 0\},$$
$$K_c = \{u \in X : J(u) = c, J'(u) = 0\},\ J^c = \{u \in X : J(u) \leqslant c\},$$

for $A \in \Sigma$, we say genus of A is n denoted by $\gamma(A) = n$ if there is an odd map $\phi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.

Theorem 2.1. ([18]) *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $J \in C^1(X,\mathbb{R})$ satisfies $(C)_c$ − condition for all $c > 0$, and*

- (J_1) *$J(0) = 0, J(-u) = J(u)$ for all $u \in X$;*
- (J_2) *there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap Z} > \alpha$;*
- (J_3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $J(u) \leqslant 0$ on $\tilde{X} \setminus B_R$.*

Then J possesses an unbounded sequence of critical values.

Theorem 2.2. ([18]) *Let X be a Banach space, J be an even C^1 functional on X and satisfy the (PS) − condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geqslant n\},\ c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} J(u).$$

- (i) *If $\Sigma_n \neq 0$ and $c_n \in \mathbb{R}$, then c_n is critical value of J;*
- (ii) *If there exists $k \in \mathbb{N}$ such that $c_n = c_{n+1} = \cdots = c_{n+k} = c \in \mathbb{R}$, and $c \neq J(0)$, then $\gamma(K_c) \geqslant k + 1$.*

3. The superlinear case

Lemma 3.1. *Assume (V_1), (V_2) and (f_1) are satisfied. Then $J(u)$ is well-defined and of class $C^1(E,\mathbb{R})$ and*

(3.1)
$$\langle J'(u), v\rangle = (u, v) + b\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)\int_{\mathbb{R}^N} \nabla_\lambda u \nabla_\lambda v dx - \int_{\mathbb{R}^N} f(x,u)v dx,\ u, v \in E.$$

And, the critical points of $J(u)$ in E are also solutions of problem (1.1).

Proof. We can get from (f_1), one has

$$|F(x,u)| \leqslant \frac{C_1}{2}|u|^2 + \frac{C_2}{p}|u|^p, \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.2}$$

for $2 \leqslant p < 2^*_\lambda$, where $F(x,u) = \int_0^u f(x,t)dt$. It can be known from Proposition 2.1 and the above formula, $J(u)$ defined by (2.4) is well-defined on E.

Let $H(u)=\int_{\mathbb{R}^N}F(x,u)dx$. For all $u,v\in E$ and $0<|t|<1$, by the Mean Value Theorem and (f_1), there exist $\theta\in(0,1)$ such that

$$\begin{aligned}&\frac{|F(x,u(x)+tv(x))-F(x,u(x))|}{|t|}=|f(x,u(x)+\theta tv(x))v(x)|\\ \leqslant&C_1|u(x)||v(x)|+C_1|v(x)|^2+C_2|u(x)+\theta tv(x)|^{p-1}|v(x)|\\ \leqslant&C_1|u(x)||v(x)|+C_1|v(x)|^2+2^{p-1}C_2(|u(x)|^{p-1}|v(x)|+|v(x)|^p).\end{aligned}$$

The Hölder inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^N}|u(x)||v(x)|dx\leqslant&\|u(x)\|_p\|v(x)\|_{\frac{p}{p-1}},\\ \int_{\mathbb{R}^N}|v(x)|^2dx\leqslant&\|v(x)\|_p\|v(x)\|_{\frac{p}{p-1}},\\ \int_{\mathbb{R}^N}|u(x)|^{p-1}|v(x)|dx\leqslant&\|u(x)\|_p^{p-1}\|v(x)\|_p,\\ \int_{\mathbb{R}^N}|v(x)|^pdx\leqslant&\|v(x)\|_{\frac{p^2}{2}}^{p/2}\|v(x)\|_{\frac{p^2}{2(p-1)}}^{p/2(p-1)}.\end{aligned}$$

Hence,

$$\nu(x):=C_1|u(x)||v(x)|+C_1|v(x)|^2+2^{p-1}C_2(|u(x)|^{p-1}|v(x)|+|v(x)|^p)\in L^1(\mathbb{R}^N).$$

which implies $H(u)\in C^1(E,\mathbb{R})$. By Lebesgue's Dominated Convergence Theorem and Mean Value Theorem, we obtain

$$\begin{aligned}\langle H'(u),v\rangle=&\lim_{t\to0^+}\frac{H(u+tv)-H(u)}{t}=\lim_{t\to0^+}\int_{\mathbb{R}^N}\frac{F(x,u+tv)-F(x,u)}{t}dx.\\ =&\lim_{t\to0^+}\int_{\mathbb{R}^N}f(x+t\theta v)vdx=\int_{\mathbb{R}^N}f(x,u)vdx.\end{aligned}$$

Next, we prove the continuity of H'. Let $u_n\to u$ in E, then $u_n\to u$ in $L^p(\mathbb{R}^N)$ by Proposition 2.1 for $p\in[1,2_\lambda^*)$. Note that

$$\begin{aligned}&\|H'(u_n)-H'(u)\|=\sup_{\|v\|\leqslant1}|\langle H'(u_n)-H'(u),v\rangle|\\ =&\sup_{\|v\|\leqslant1}\left|\int_{\mathbb{R}^N}[f(x,u_n)-f(x,u)]vdx\right|\leqslant\sup_{\|v\|\leqslant1}\int_{\mathbb{R}^N}\big|f(x,u_n)-f(x,u)\big||v|dx.\end{aligned}$$

By the Hölder inequality

$$\begin{aligned}&\sup_{\|v\|\leqslant1}\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)||v|dx\\ \leqslant&\sup_{\|v\|\leqslant1}\left(\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}v^pdx\right)^{\frac{1}{p}}\to0,\end{aligned}$$

as $n\to\infty$. Hence, H' is continuous. This shows that (3.1) holds. Moreover, by a standard argument, it is easy to show that the critical points of J in E are solutions of problem (1.1). □

Lemma 3.2. *Assume that* $(V_1), (V_2), (f_1), (f_3), (f_4)$ *are satisfied. Then any* $(C)_c-$ *sequence* $\{u_n\}$ *of* J *is bounded in* E.

Proof. We will use the contradiction method to prove the boundness of $\{u_n\}$, assume that $\|u_n\| \to \infty,$ as $n \to \infty$. Let $\{u_n\} \subset E$ be $(C)_c - $ *sequence* such that

$$J(u_n) \to c, \ (1 + \|u_n\|)J'(u_n) \to 0, \tag{3.3}$$

then we have

$$c + 1 \geqslant J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle. \tag{3.4}$$

Setting $v_n := \frac{u_n}{\|u_n\|}$, then $\|v_n\|$=1. And assume that

$$\begin{aligned}
&v_n \rightharpoonup v \text{ in } E,\\
&v_n \to v \text{ in } L^p(\mathbb{R}^N), \text{ for } 1 \leqslant p < 2^*_\lambda,\\
&v_n(x) \to v(x) \text{ a.e. } x \in \mathbb{R}^N.
\end{aligned}$$

If $v = 0$, then $v_n \to 0$ in $L^p(\mathbb{R}^N),\ \forall p \in [1, 2^*_\lambda)$, and $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$. By (f_4) and (3.4), we have

$$\begin{aligned}
\frac{c+1}{\|u_n\|^2} &\geqslant \frac{1}{\|u_n\|^2}\left(J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle\right)\\
&= \frac{1}{\|u_n\|^2}\left(\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^N} \frac{1}{4} f(x, u_n)u_n - F(x, u_n)dx\right)\\
&\geqslant \frac{1}{4} - \beta\int_{\mathbb{R}^N} \frac{u_n^2}{\|u_n\|^2}dx = \frac{1}{4} - \beta\int_{\mathbb{R}^N} v_n^2 dx,
\end{aligned} \tag{3.5}$$

as $n \to \infty$, which implies $\frac{1}{4} \leqslant 0$. Thus, it is a contradiction.

If $v \neq 0$. For $0 \leqslant \delta_0 < \delta_1$, let $A_n(\delta_0, \delta_1) = \{x \in \mathbb{R}^N : \delta_0 \leqslant |u_n| < \delta_1\}$. Setting $B := \{x \in \mathbb{R}^N : v(x) \neq 0\}$. Thus, $meas(B) > 0$. For almost every $x \in B$, we have $\lim_{n\to\infty} |v_n(x)| = \infty$. Hence, $B \subset A_n(r_0, \infty)$ for large $n \in \mathbb{N}$, where r_0 is given in (f_3). By (f_3), we have

$$\lim_{n\to\infty} \frac{|F(x, u_n)|}{\|u_n\|^4} = \lim_{n\to\infty} \frac{|F(x, u_n)|}{|u_n|^4}|v_n|^4 = \infty.$$

From Fatou's Lemma, (3.2) and (3.3) we can get

$$
\begin{aligned}
0 = & \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^4} = \lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|^4} \\
= & \lim_{n\to\infty} \frac{1}{\|u_n\|^4}\left(\frac{1}{2}\|u_n\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u_n|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u_n)dx\right) \\
= & \frac{b}{4}\lim_{n\to\infty} \frac{(\int_{\mathbb{R}^N} |\nabla_\lambda u_n|^2 dx)^2}{\|u_n\|^4} + \lim_{n\to\infty} \frac{1}{\|u_n\|^4}\left(\frac{1}{2}\|u_n\|^2\right. \\
& \left. - \int_{A_n(0,r_0)} F(x,u_n)dx - \int_{A_n(r_0,+\infty)} F(x,u_n)dx\right) \\
\leqslant & \frac{b}{4} + \lim_{n\to\infty}\left(\frac{1}{2\|u_n\|^2} - \int_{A_n(0,r_0)} \frac{F(x,u_n)}{|u_n|^2}\frac{|v_n|^2}{|u_n|^2}|v_n|^2 dx - \int_{A_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^4}dx\right) \\
\leqslant & \frac{b}{4} + \limsup_{n\to\infty}\left[\frac{1}{2\|u_n\|^2} + \left(\frac{C_1}{2} + \frac{C_2}{p}r_0^{p-2}\right)\frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} |v_n|^2 dx - \int_{A_n} \frac{F(x,u_n)}{|u_n|^4}|v_n|^4 dx\right] \\
\leqslant & \frac{b}{4} + C_3 - \liminf_{n\to\infty} \int_{A_n} \frac{F(x,u_n)}{|u_n|^4}|v_n|^4 dx = -\infty,
\end{aligned}
\tag{3.6}
$$

which is a contradiction. Thus, $\{u_n\}$ is bounded in E. □

Lemma 3.3. *Assume that $(V_1),(V_2),(f_1)-(f_3)$ and (f_5) are satisfied. Then any $(C)_c-$ sequence $\{u_n\}$ of J is bounded in E.*

Proof. The proof method is similar to Lemma 3.2, also assuming that $\|u_n\| \to \infty$, as $n \to \infty$. We may assume that $v_n \rightharpoonup v$ in E, by Proposition 2.1, $v_n \to v$ in $L^p(\mathbb{R}^N)$ for $1 \leqslant p < 2^*_\lambda$, and $v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$.

If $v = 0$, we define

$$J(t_n u_n) = \max_{t\in[0,1]} J(tu_n).$$

For any $K > 0$, set $\overline{v}_n = \sqrt{4K}\frac{u_n}{\|u_n\|} = \sqrt{4K}v_n$, then $\|\overline{v}_n\|^2 = 4K$. By (3.2) and Proposition 2.1, we have

$$\left|\int_{\mathbb{R}^N} F(x,\overline{v}_n)dx\right| \leqslant \frac{C_1}{2}\int_{\mathbb{R}^N} |\overline{v}_n|^2 dx + \frac{C_2}{p}\int_{\mathbb{R}^N} |\overline{v}_n|^p dx \to 0,\ n\to\infty.$$

Therefore, for a sufficiently large n such that

$$J(t_n u_n) \geqslant J(\overline{v}_n) = \frac{1}{2}\|\overline{v}_n\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda \overline{v}_n|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,\overline{v}_n)dx \geqslant K. \tag{3.7}$$

Hence, by$(f_3), (f_5)$, we obtain

$$\begin{aligned} J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle =& \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right) dx \\ \geqslant& \frac{1}{4}\|t_n u_n\|^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4}f(x,t_n u_n)t_n u_n - F(x,t_n u_n)\right) dx \\ =& J(t_n u_n) - \frac{1}{4}\langle J'(t_n u_n), t_n u_n\rangle . \end{aligned}$$

According to (3.7), which implies $\lim_{n\to\infty} J(t_n u_n) = \infty$, and due to the choice of t_n we know $\langle J'(t_n u_n), t_n u_n\rangle = 0$. That is, $J(u_n) - \frac{1}{4}\langle J'(u_n), u_n\rangle \geqslant \infty$, which contradicts with (3.4).

If $v \neq 0$, contradictions can be obtained by similar argument as (3.6). The proof is complete. □

Lemma 3.4. ([20]) *Assume that $p_1, p_2 > 1, r, q \geqslant 1$ and $\Omega \subseteq \mathbb{R}$. Let $g(x,t)$ be a Carathéodory function on $\mathbb{R}^N \times \mathbb{R}$ and satisfy*

$$|g(x,t)| \leqslant a_1|t|^{(p_1-1/r)} + a_2|t|^{(p_2-1/r)},\ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.8}$$

where $a_1, a_2 \geqslant 0$. If $u_n \to u$ in $L^{p_1}(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N)$, $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$, then for any $v \in L^{p_1 q}(\mathbb{R}^N) \cap L^{p_2 q}(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|^r |v|^q dx = 0. \tag{3.9}$$

Lemma 3.5. ([20]) *Assume that $p_1, p_2 > 1, r \geqslant 1$ and $\Omega \subseteq \mathbb{R}$. Let $g(x,t)$ be a Carathéodory function on $\mathbb{R}^N \times \mathbb{R}$ and satisfy (3.8). If $u_n \to u$ in $L^{p_1}(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N)$, $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|^r |u_n - u| dx = 0. \tag{3.10}$$

Lemma 3.6. *Assume that $(V_1), (V_2), (f_1), (f_3)$ and (f_4) or (f_5) are satisfied. Then any $(C)_c -$ sequence $\{u_n\}$ has a convergent subsequence in E.*

Proof. By the previous lemma, we know that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can suppose that $u_n \rightharpoonup u$ in E. By Proposition 2.1, $u_n \to u$ in $L^p(\mathbb{R}^N)$ for $1 \leqslant p < 2^*_\lambda$, and together with by Lemma 3.5, one has

$$\int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||u_n - u| dx \to 0, n \to \infty. \tag{3.11}$$

Observe that,

$$
\begin{aligned}
\langle J'(u_n)-J'(u),u_n-u\rangle =&\|u_n-u\|^2+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u_n\nabla_\lambda(u_n-u)dx\\
&-b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx\\
&-\int_{\mathbb{R}^N}\big[f(x,u_n)-f(x,u)\big](u_n-u)dx\\
=&\|u_n-u\|^2+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda|u_n-u|^2dx\\
&-b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx\\
&-\int_{\mathbb{R}^N}\big[f(x,u_n)-f(x,u)\big](u_n-u)dx\\
\geqslant&\|u_n-u\|^2-\int_{\mathbb{R}^N}\big[f(x,u_n)-f(x,u)\big](u_n-u)dx\\
&-b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx.
\end{aligned}
\tag{3.12}
$$

It is clear that,

$$
\begin{aligned}
\|u_n-u\|^2\leqslant&\langle J'(u_n)-J'(u),u_n-u\rangle+\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx\\
&+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx.
\end{aligned}
\tag{3.13}
$$

By the definition of weak convergence, we have

$$
\langle J'(u_n)-J'(u),u_n-u\rangle\to 0,\ n\to\infty. \tag{3.14}
$$

Set $\overline{E}=\{u\in L^2(\mathbb{R}^N):\nabla_\lambda u\in L^2(\mathbb{R}^N)\}$ with the norm $\|u\|_{\overline{E}}=\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2dx\right)^{\frac{1}{2}}$. Then the embedding $E\hookrightarrow\overline{E}$ is continuous. Hence, $u_n\rightharpoonup u$ in $\overline{E}$. According to the boundedness of $\{u_n\}$ in E, one has

$$
b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx\to 0,\ \text{as } n\to\infty. \tag{3.15}
$$

From the (3.11)-(3.15) we can get $u_n\to u$ in $E,$ as $n\to\infty$. □

Let $\{e_j\}$ is an orthonormal basis of E and define $X_j=\mathbb{R}e_j$,

$$
Y_k=\oplus_{j=1}^k X_j,\ Z_k=\overline{\oplus_{j=k+1}^\infty X_j},\ k\in\mathbb{Z}.
$$

Lemma 3.7. *Assume that* (V_1) *and* (V_2) *are satisfied. Then*

$$
\beta_k:=\sup_{u\in Z_k,\|u\|=1}\|u\|_p\to 0,\ k\to\infty,\ p\in[1,2^*_\lambda).
$$

Proof. It is clear that $0<\beta_{k+1}\leqslant\beta_k$, so that $\beta_k\to\beta\geqslant 0,\ k\to\infty$. For every $k\in\mathbb{N}$, there exists $u_k\in Z_k$ such that $\|u_k\|_2>\frac{\beta_k}{2}$ and $\|u_k\|=1$. We denote

$v = \sum_{j=1}^{\infty} c_j e_j$, for any $v \in E$ by the Cauchy-Schwarz inequality, one has

$$\begin{aligned}|(u_k, v)| = &\left|\left(u_k, \sum_{j=1}^{\infty} c_j e_j\right)\right| = \left|\left(u_k, \sum_{j=k}^{\infty} c_j e_j\right)\right| \leqslant \|u_k\| \left\|\sum_{j=k}^{\infty} c_j e_j\right\| \\ &\leqslant \left(\sum_{j=k}^{\infty} c_j^2\right)^{\frac{1}{2}} \left(\sum_{j=k}^{\infty} e_j^2\right)^{\frac{1}{2}} = \left(\sum_{j=k}^{\infty} c_j^2\right)^{\frac{1}{2}} \to 0, \text{ as } k \to \infty,\end{aligned}$$

which implies that $u_k \rightharpoonup 0$ in E. By Proposition 2.1, we have $u_k \to 0$ in $L^p(\mathbb{R}^N)$. Hence, letting $k \to \infty$, we get $\beta = 0$. □

Lemma 3.8. *Assume that* (V_1), (V_2) *and* (f_1) *are satisfied, there exist constants* $\rho,\ \alpha \geqslant 0$ *such that* $J_\lambda|_{\partial B_\rho \cap Z_m} \geqslant \alpha$.

Proof. By Lemma 3.7, we can choose an integer $m \geqslant 1$ such that

$$\|u\|_2^2 \leqslant \frac{1}{2C_1}\|u\|^2,\ \|u\|_p^p \leqslant \frac{p}{4C_2}\|u\|^p,\ \forall u \in Z_m. \tag{3.16}$$

According to (2.4) (3.2) and (3.16), for $u \in Z_m$, we have

$$\begin{aligned}J(u) =&\frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ \geqslant&\frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \geqslant \frac{1}{2}\|u\|^2 - \frac{C_1}{2}\|u\|_2^2 - \frac{C_2}{p}\|u\|_p^p \\ \geqslant&\frac{1}{4}(\|u\|^2 - \|u\|^p) = \frac{2^{p-2}-1}{2^{p+2}} := \alpha \geqslant 0,\end{aligned}$$

choosing $\rho = \|u\| = \frac{1}{2}$. □

Lemma 3.9. *Assume that* (V_1), (V_2), (f_1) *and* (f_3) *are satisfied. Then for any finite dimensional subspace* $\overline{E} \subset E$, *there is* $R = R(\overline{E}) > 0$ *such that*

$$J(u) \leqslant 0,\ \forall u \in \overline{E} \setminus B_R.$$

Proof. For any $\overline{E} \subset E$, there is a positive integral number m such that $\overline{E} \subset E_m$. Since all norms are equivalent in finite dimensional space, there is a constant $\eta > 0$ such that

$$\|u\|_4 \geqslant \eta\|u\|,\ \forall u \in E_m. \tag{3.17}$$

By (f_1) and (f_3), one has

$$F(x,u) \geqslant \delta|u|^4 - C_\delta|u|^2,\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.18}$$

for any $\delta > \frac{b}{4C_4^4}$ and constant $C_\delta > 0$. Hence, by (3.17) and (3.18), we have

$$J(u) \leqslant \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \delta\|u\|_4^4 + C_\delta\|u\|_2^2 \leqslant \left(\frac{1}{2} + C_\delta C_2^2\right)\|u\|^2 - \left(\delta C_4^4 - \frac{b}{4}\right)\|u\|^4, \forall u \in E_m.$$

Hence, there is a large $R = R(\overline{E}) > 0$ such that $J(u) \leqslant 0$ for all $u \in \overline{E} \setminus B_R$. □

Proof of Theorem 1.1. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. Obviously, $J(0) = 0$ and (f_2) implies J is even. By Lemmas 3.2, 3.6, 3.8 and 3.9, all conditions of Theorem 2.1 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial sequence solutions $\{u_k\}$ such that $J(u_k) \to \infty$ as $k \to \infty$. □

Proof of Theorem 1.2. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. Obviously, $J(0) = 0$ and (f_2) implies J is even. By Lemmas 3.3, 3.6, 3.8 and 3.9, all conditions of Theorem 2.1 are satisfied. Thus, problem (1.1) possesses infinitely many nontrivial sequence solutions $\{u_k\}$ such that $J(u_k) \to \infty$ as $k \to \infty$. □

4. The sublinear case

Lemma 4.1. *Assume that (V_1),(V_2),(f_2), (f_6),(f_7) are satisfied. Then the J satisfies the (PS)-condition.*

Proof. Obviously, from (V_1), (f_6), we know the functional $J \in C^1$ and also have the derivative functional (3.1). According to the (f_6), one has

$$|F(x,u)| \leqslant |u|^{q_1} + |u|^{q_2},\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{4.1}$$

From the above formula, for $1 < q_1 < q_2 < 2$, we can get

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \geqslant \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left(|u|^{q_1} + |u|^{q_2}\right)dx \\ &\geqslant \frac{1}{2}\|u\|^2 - C_1'\left(\|u\|^{q_1} + \|u\|^{q_2}\right) \to \infty. \end{aligned} \tag{4.2}$$

as $\|u\| \to \infty$. Hence J is bounded from below. Next we show that J satisfies (PS)-condition. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset E$ is (PS)-sequence. Therefore, according to (2.1), there exist a constant $\eta > 0$, such that

$$\|u\|_2 \leqslant C_2\|u\| < \eta. \tag{4.3}$$

By Proposition 2.1 let a subsequence still denoted by $\{u_n\}$, such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } E, \\ &u_n \to u \text{ in } L^p(\mathbb{R}^N),\ \text{for } 1 \leqslant p < 2^*_\lambda. \end{aligned}$$

It follows from (f_6) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \big|f(x,u_n)-f(x,u)\big|\big|u_n-u\big|dx \\
\leqslant&\int_{\mathbb{R}^N} \Big|q_1(|u_n|^{q_1-1}-|u|^{q_1-1})+q_2(|u_n|^{q_2-1}-|u|^{q_2-1})\Big|\Big|u_n-u\Big|dx \\
\leqslant& q_1\left(\int_{\mathbb{R}^N}|u_n-u|^{q_1}dx\right)^{\frac{1}{q_1}}\left[\left(\int_{\mathbb{R}^N}|u_n|^{q_1}dx\right)^{\frac{q_1-1}{q_1}}-\left(\int_{\mathbb{R}^N}|u|^{q_1}dx\right)^{\frac{q_1-1}{q_1}}\right] \\
&+q_2\left(\int_{\mathbb{R}^N}|u_n-u|^{q_2}dx\right)^{\frac{1}{q_2}}\left[\left(\int_{\mathbb{R}^N}|u_n|^{q_2}dx\right)^{\frac{q_2-1}{q_2}}-\left(\int_{\mathbb{R}^N}|u|^{q_2}dx\right)^{\frac{q_2-1}{q_2}}\right]\to 0,
\end{aligned}
\tag{4.4}
$$

as $n\to\infty$. According to (3.12), we know

$$
\begin{aligned}
\|u_n-u\|^2\leqslant&\langle J'(u_n)-J'(u),u_n-u\rangle+\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx \\
&+b\left(\int_{\mathbb{R}^N}|\nabla_\lambda u|^2-\int_{\mathbb{R}^N}|\nabla_\lambda u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla_\lambda u\nabla_\lambda(u_n-u)dx.
\end{aligned}
\tag{4.5}
$$

It follow from (3.14), (3.15), (4.4) and (4.5), we have $\|u_n-u\|\to 0$, as $n\to\infty$. $\square$

Proof of Theorem 1.3. We take n disjoint open sets $\tilde{B}_i$ for any $n\in\mathbb{N}$, such that $\bigcup_{i=1}^n\tilde{B}_i\subset\tilde{B}$. Let $u_i\in\left(W_0^{1,2}(\tilde{B}_i)\bigcap E\right)\setminus\{0\}$ and $\|u_i\|_E=1$, $i=1,2,...,n$, and

$$
\Lambda_1=span\{u_1,u_2,\cdots,u_n\},\ \ \Lambda_2=\{u\in\Lambda_1:\|u\|_E=1\}.
$$

For any $u\in\Lambda_1$, there exist $\tau_i\in\mathbb{R},\ i=1,2,...,n$ such that

$$
u(x)=\sum_{i=1}^n\tau_iu_i(x),\ \ x\in\mathbb{R}^N. \tag{4.6}
$$

Hence,

$$
\begin{aligned}
\|u\|_{q_3}=\left(\int_{\mathbb{R}^N}|u|^{q_3}dx\right)^{\frac{1}{q_3}}&=\left(\int_{\mathbb{R}^N}\left|\sum_{i=1}^n\tau_iu_i(x)\right|^{q_3}dx\right)^{\frac{1}{q_3}} \\
&=\left(\sum_{i=1}^n|\tau_i|^{q_3}\int_{\tilde{B}_i}|u_i(x)|^{q_3}dx\right)^{\frac{1}{q_3}},
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
\|u\|^2&=\int_{\mathbb{R}^N}(a|\nabla_\lambda u|^2+V(x)u^2)dx=\sum_{i=1}^n\tau_i^2\int_{\tilde{B}_i}(a|\nabla_\lambda u_i|^2+V(x)u_i^2)dx \\
&=\sum_{i=1}^n\tau_i^2\|u_i\|^2=\sum_{i=1}^n\tau_i^2,
\end{aligned}
\tag{4.8}
$$

which together with (4.7) implies there exists a constant $\kappa>0$ such that

$$
\kappa\|u\|\leqslant\|u\|_{q_3},\ \ u\in\Lambda_1. \tag{4.9}
$$

It follows from (4.6) − (4.9) and (f_7), we have

$$\begin{aligned} J(tu) &= t^2\|u\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N} |\nabla_\lambda u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x, tu)dx \\ &\leqslant t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \sum_{i=1}^{n}\int_{\tilde{B}_i} F(x, t\tau_i u_i)dx \\ &\leqslant t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \xi t^{q_3}\sum_{i=1}^{n}|\tau_i|^{q_3}\int_{\tilde{B}_i} |u_i|^{q_3}dx \\ &= t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \xi t^{q_3}\|u\|_{q_3}^{q_3} \\ &\leqslant t^2\|u\|^2 + \frac{bt^4}{4}\|u\|^4 - \xi(t\kappa)^{q_3}\|u\|^{q_3} \\ &= t^2 + \frac{bt^4}{4} - \xi(t\kappa)^{q_3} := -\sigma,\ u \in \Lambda_2. \end{aligned}$$

Hence, there exist $0 < t < 1$ and $\sigma > 0$ such that $J(tu) < -\sigma$, $u \in \Lambda_2$. Let

$$\Lambda_2' = \{tu : u \in \Lambda_2\},\ \tilde{B} = \left\{(\tau_1, \tau_2, \cdots, \tau_n) \in \mathbb{R}^n : \sum_{i=1}^{n}\tau_i^2 < t^2\right\}.$$

Therefore $J(u) < -\sigma,\ u \in \Lambda_2'$. And by ($f_2$), we know J is even and $J(0) = 0$, can deduce $\Lambda_2' \subset J^{-\sigma} \in \Sigma$. Also, in view of $(4.6), (4.8)$, there exist an odd mapping $\varphi \in C(\Lambda_2', \partial\tilde{B})$. By properties of the genus, we obtain that

$$\gamma(J^{-\sigma}) \geqslant \gamma(\Lambda_2') = n. \tag{4.10}$$

Hence, we get for any $n \in \mathbb{N}$, there exists $\sigma > 0$ such that $\gamma(J^{-\sigma}) \geqslant n$. Now let

$$c_n = \inf_{A\in\Sigma_n}\sup_{u\in A} J(u).$$

In view of J is bounded below on E and (4.10), one has

$$-\infty < c_n < -\sigma < 0. \tag{4.11}$$

In other words, for any $n \in \mathbb{N},\ c_n$ is negative real number. Thus, we can apply the Theorem 2.2 to get that problem (1.1) has infinitely many solutions.

□

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