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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING THEIR DERIVATIVES AND SHIFTS WITH PARTIALLY SHARED VALUES

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Abstract. The uniqueness problems of the j-th derivative of a meromorphic function $f(z)$ and the k-th derivative of its shift $f(z+c)$ are investigated in this paper, where j,k are integers with $0 \leqslant j < k$. We show that when $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share one IM value and two partially shared values CM, the uniqueness result remains valid under some additional hypotheses. With one CM value and two partially shared values CM, a uniqueness theorem about the j-th derivative of $f(z)$ and the k-th derivative of its shift $f(z+c)$ is also proved.

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1. Introduction and main results

Nevanlinna value distribution theory of meromorphic functions has been extensively applied to the uniqueness theory of meromorphic functions, see [23]. Given a meromorphic function f, recall that a meromorphic function α is said to be a small functions of f, if $T(r,\alpha(z)) = S(r,f)$ where $S(r,f)$ is used to denote any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside of a set of r of finite logarithmic measure. Let $\hat{S}(f) = S(f)\bigcup\{\infty\}$. For each $a \in \hat{S}(f)$, we say that two meromorphic functions $f(z)$ and $g(z)$ share a IM(ignoring multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros, and we say that $f(z)$ and $g(z)$ share a CM(counting multiplicities) provided that $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities.

Rubel and Yang[20] considered the uniqueness of a nonconstant entire function when it shares two values with its first derivative. Mues, Steinmetz [17] and Gundersen [12] improved the result to the case of meromorphic functions and obtained the following result.

Theorem A.[20] Let f be a nonconstant meromorphic function, and let a and b be two distinct finite values. If f and f' share a and b CM, then $f \equiv f'$.

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Gundersen[12] showed, by a counter-example, that two shared values CM in Theorem A cannot be reduced to 1CM+1IM. However, 2CM is able to be replaced by 3IM, see[10, 17]. Moreover, Frank and Weissenborn[8] proved the conclusion is still valid by replacing f' by a higher order derivative $f^{(k)}$.

Theorem B.[8] Let f be a nonconstant entire function and $k \geqslant 2$ be a positive integer. If f shares two distinct finite values a and b CM with $f^{(k)}$, then $f \equiv f^{(k)}$.

Later on, there are many related results about the uniqueness of meromorphic functions with their first derivative f' or their k-th derivative $f^{(k)}$ [1, 2, 8, 21]. In recent decade, Halburd and Korhonen[13]and, independently, Chiang and Feng[6] developed a parallel difference version of classical Nevanlinna theory for meromorphic functions. Then, many scholars tried to investigate the uniqueness of a meromorphic function $f(z)$ taking into account with its shift $f(z+c)$ or difference operator $\Delta_c f(z) = f(z+c) - f(z)$ where c is a complex constant, see[14, 15, 18, 22]. For instance, Heittokangas et.al[14] considered the problem of a meromorphic function f of finite order with its shift $f(z+c)$ sharing two values CM and one value IM.

Theorem C.[14] Let $f(z)$ be a meromorphic function of finite order, and let $a_1, a_2, a_3 \in \widehat{S}(f)$ be three distinct periodic functions with period c, where $c \in \mathbb{C} \setminus \{0\}$ is a constant. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) \equiv f(z+c)$.

Regarding Theorem A and Theorem C, one may ask a question: What can be said when the shift or difference operator of a meromorphic function $f(z)$ shares some values with its derivative? For a transcendental entire function $f(z)$, Qi et.al[18] proved the uniqueness result still remains true if $f'(z)$ and $f(z+c)$ share two values CM.

Theorem D.[18] Let $f(z)$ be a transcendental entire function of finite order and a be a nonzero complex constant. If $f'(z)$ and $f(z+c)$ share 0, a CM, then $f'(z) \equiv f(z+c)$.

In 2018, Chen[4] considered the question above using the notation of partially shared values by some ingenious methods.

Definition 1.1. Denote by $E(a, f)$ the set of all zeros of $f - a$, where each zero with multiplicity m times is counted m times. Similarly, we denote by $\overline{E}(a, f)$ the set of zeros of $f - a$, where each zero is counted only once. If $\overline{E}(a, f) \subseteq \overline{E}(a, g)$, then we say that $f(z)$ partially shares a with $g(z)$. If $E(a, f) \subset E(a, g)$, then we can say that f and g partially share a CM.

Theorem E.[4] Let $f(z)$ be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \neq 0 \in \mathbb{C}$. If $\Delta_c f$ and $f(z)$ share value 1 CM and satisfy $E(0, f(z)) \subset E(0, \Delta_c f)$ and $E(\infty, \Delta_c f) \subset E(\infty, f(z))$, then $f(z) \equiv \Delta_c(f)$ for all $z \in \mathbb{C}$.

In [5], Chen et.al extended the result to the case of n-th order differences $\Delta_c^n f(z)$. More recently, for $f'(z)$ and $f(z+c)$, Qi et.al[19] proved the following result.

Theorem F.[19] Let $f(z)$ be a nonconstant meromorphic function of finite order, and $a \in \mathbb{C} \setminus \{0\}$. If $f'(z)$ and $f(z+c)$ share a CM, and satisfy $E(0, f(z+c)) \subset E(0, f'(z))$, $E(\infty, f'(z)) \subset E(\infty, f(z+c))$, then $f'(z) \equiv f(z+c)$. Further, $f(z)$ is a transcendental entire function.

In this paper we consider the uniqueness of $f^{(j)}(z)$ and the k-th derivative of shift $f(z+c)$ under the conditions of one shared value IM and two partially shared values $0, \infty$ CM. Actually, we obtain the following Theorem 1.1 by a different method from those mentioned above.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order, and let c be a nonzero finite complex number and j, k be integers with $0 \leqslant j < k$. Suppose that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a finite value $a \neq 0$ IM and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$. If $N(r, \frac{1}{f^{(j)}(z)}) + \overline{N}(r, \frac{1}{f(z)}) = S(r, f)$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

If we remove the hypothesis "$N(r, \frac{1}{f^{(j)}(z)}) + \overline{N}(r, \frac{1}{f(z)}) = S(r, f)$"and replace IM by CM, then the conclusion still holds.

Theorem 1.2. *Let $f(z)$ be a nonconstant meromorphic function of finite order, a be a nonzero finite complex number and j, k be integers with $0 \leqslant j < k$. If $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share a CM, and satisfy $E(0, f^{(j)}(z)) \subset E(0, f^{(k)}(z+c))$ and $E(\infty, f^{(k)}(z+c)) \subset E(\infty, f^{(j)}(z))$, then $f^{(j)}(z) \equiv f^{(k)}(z+c)$.*

2. Some lemmas

To prove our result, we recall some notations and results. Let k be a positive integer, we use $N_{k)}(r, \frac{1}{f-a})$ to denote the counting function of a points of f with multiplicity $\leqslant k$ and use $N_{(k+1}(r, \frac{1}{f-a})$ to denote the counting function of a points of f with multiplicity $> k$, where each a point is counted on the basis of its multiplicity. Similarly, we define $\overline{N}_{k)}(r, \frac{1}{f-a})$ and $\overline{N}_{(k+1}(r, \frac{1}{f-a})$ where in counting the a points of f we ignore the multiplicities.

Lemma 2.1. [6] *Let $f(z)$ be a meromorphic function of finite order and $c \in \mathbb{C}$. Then we have*

$$m(r, \frac{f(z+c)}{f(z)}) + m(r, \frac{f(z)}{f(z+c)}) = S(r, f),$$

where $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set E with finite linear measure.

Lemma 2.2. [23] *Let $f(z)$ be a nonconstant meromorphic function in the complex plane and k be a positive integer. Set*

$$\Psi(z)=\sum_{k=0}^{n}a_k(z)f^{(k)}(z),$$

where $a_k(z)(k=0,1,...,n)$ are all small functions of $f(z)$. Then

$$\begin{aligned}T(r,\Psi) &\leqslant T(r,f)+k\overline{N}(r,f)+S(r,f)\\ &\leqslant (k+1)T(r,f)+S(r,f),\\ N(r,\frac{1}{\Psi}) &\leqslant N(r,\frac{1}{f})+k\overline{N}(r,f)+S(r,f).\end{aligned}$$

Lemma 2.3. [6] *Let $f(z)$ be a nonconstant meromorphic function of finite order and $c\in\mathbb{C}$. Then*

$$T(r,f(z+c))=T(r,f)+S(r,f),$$

$$N(r,f(z+c))=N(r,f)+S(r,f),\qquad N(r,\frac{1}{f(z+c)})=N(r,\frac{1}{f(z)})+S(r,f),$$

and

$$\overline{N}(r,f(z+c))=\overline{N}(r,f)+S(r,f),\qquad \overline{N}(r,\frac{1}{f(z+c)})=\overline{N}(r,\frac{1}{f(z)})+S(r,f).$$

Lemma 2.4. *Suppose that $f(z)$ is a nonconstant meromorphic function of finite order in $|z|<R$ and $a_t(t=1,2,...,q)$ are $q(\geqslant 2)$ distinct finite complex numbers. Let j,k be integers with $0\leqslant j<k$. Then for $0<r<R$, we have*

$$m(r,f^{(j)}(z))+\sum_{t=1}^{q}m(r,\frac{1}{f^{(j)}(z)-a_t})\leqslant 2T(r,f^{(j)}(z))-N_{pair}(r,f)+S(r,f),$$

where

$$N_{pair}(r,f)=2N(r,f^{(j)}(z))-N(r,f^{(k)}(z+c))+N(r,\frac{1}{f^{(k)}(z+c)})+S(r,f).$$

Proof. Set $F(z)=\sum\limits_{t=1}^{q}\frac{1}{f^{(j)}(z)-a_t}$, then

$$G(z)=F(z)f^{(k)}(z+c)=\sum_{t=1}^{q}\frac{f^{(k)}(z+c)}{f^{(j)}(z)-a_t}.$$

It follows from the lemma of logarithmic derivatives that

$$m(r,G(z))=m(r,\sum_{t=1}^{q}\frac{f^{(k)}(z+c)}{f^{(j)}(z)-a_t})\leqslant\sum_{t=1}^{q}m(r,\frac{f^{(k)}(z+c)}{f^{(j)}(z)-a_t})+S(r,f)=S(r,f).$$

Thus

$$\begin{aligned}m(r,F(z))=m(r,G(z)\frac{1}{f^{(k)}(z+c)})\leqslant m(r,G(z))+\\ +m(r,\frac{1}{f^{(k)}(z+c)})=m(r,\frac{1}{f^{(k)}(z+c)})+S(r,f).\end{aligned}\tag{2.1}$$

Next, by Nevanlinna's first fundamental theorem, we get from (2.1) that

$$\begin{aligned} T(r, f^{(k)}(z+c)) &= T(r, \frac{1}{f^{(k)}(z+c)}) + O(1) \\ &= m(r, \frac{1}{f^{(k)}(z+c)}) + N(r, \frac{1}{f^{(k)}(z+c)}) + S(r,f) \\ &\geqslant m(r, F(z)) + N(r, \frac{1}{f^{(k)}(z+c)}) + S(r,f) \\ &= m(r, \sum_{t=1}^{q} \frac{1}{f^{(j)}(z) - a_t}) + N(r, \frac{1}{f^{(k)}(z+c)}) + S(r,f). \end{aligned} \tag{2.2}$$

Then by (2.2), we have

$$\begin{aligned} \sum_{t=1}^{q} m(r, \frac{1}{f^{(j)}(z) - a_t}) &= m(r, \sum_{t=1}^{q} \frac{1}{f^{(j)}(z) - a_t}) + O(1) \\ &\leqslant T(r, f^{(k)}(z+c)) - N(r, \frac{1}{f^{(k)}(z+c)}) + S(r,f). \end{aligned} \tag{2.3}$$

Hence, it is easy to deduce from (2.3) that

$$\begin{aligned} & m(r, f^{(j)}(z)) + \sum_{t=1}^{q} m(r, \tfrac{1}{f^{(j)}(z)-a_t}) \qquad (2.4) \\ & \leqslant m(r, f^{(j)}(z)) + T(r, f^{(k)}(z+c)) - N(r, \tfrac{1}{f^{(k)}(z+c)}) + S(r,f) \\ & = T(r, f^{(j)}(z)) - N(r, f^{(j)}(z)) + m(r, f^{(k)}(z+c)) + N(r, f^{(k)}(z+c)) - N(r, \tfrac{1}{f^{(k)}(z+c)}) \\ & + S(r,f) \leqslant T(r, f^{(j)}(z)) - N(r, f^{(j)}(z)) + m(r, f^{(j)}(z)) + m(r, \tfrac{f^{(k)}(z+c)}{f^{(j)}(z)}) \\ & + N(r, f^{(k)}(z+c)) - N(r, \tfrac{1}{f^{(k)}(z+c)}) + S(r,f) \\ & = 2T(r, f^{(j)}(z)) - 2N(r, f^{(j)}(z)) + N(r, f^{(k)}(z+c)) - N(r, \tfrac{1}{f^{(k)}(z+c)}) + S(r,f) \\ & = 2T(r, f^{(j)}(z)) - [2N(r, f^{(j)}(z)) - N(r, f^{(k)}(z+c)) + N(r, \tfrac{1}{f^{(k)}(z+c)})] + S(r,f). \end{aligned}$$

We use $N_p(r, \frac{1}{f^{(k)}-a})$ to denote the counting function of the zeros of $f - a$ where a $p-$ folds zero is counted m times if $m \leqslant p$ and p times if $m > p$.

Lemma 2.5. [24, Lemma 2.4] *Let f be a non-constant transcendental meromorphic function. If $f^{(k)} \not\equiv 0$, we have $N_p(r, \frac{1}{f^{(k)}}) \leqslant T(r, f^{(k)}) - T(r,f) + N_{p+k}(r, \frac{1}{f}) + S(r,f)$.*

Lemma 2.6. *Let f be a non-constant transcendental meromorphic function and $j \geqslant 0$ is an integer. If $\overline{N}(r, \frac{1}{f(z)}) = S(r,f)$, then $S(r, f^{(j)}) = S(r,f)$.*

Proof. By Lemma 2.5,

$$\begin{aligned} T(r,f) &\leqslant T(r, f^{(j)}) - N_p(r, \frac{1}{f^{(j)}}) + N_{1+j}(r, \frac{1}{f}) + S(r,f) \\ &\leqslant T(r, f^{(j)}) + (1+j)\overline{N}(r, \frac{1}{f}) + S(r,f) \leqslant T(r, f^{(j)}) + S(r,f). \end{aligned}$$

Also by Lemma 2.2, $T(r, f^{(j)}) \leqslant (j+1)T(r,f) + S(r,f)$. This completes the proof.

3. Proof of Theorem 1.1

Without loss of generality, we assume that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share $a=1$ IM. For a general case, we can consider substituting $\frac{1}{a}f(z)$ for $f(z)$. Suppose on the contrary that $f^{(j)}(z)\not\equiv f^{(k)}(z+c)$.

Set $h(z)=f^{(j)}(z)$ and $g(z)=f^{(k)}(z+c)$. By the assumption that $E(0,h(z))\subset E(0,g(z))$ and $E(\infty,g(z))\subset E(\infty,h(z))$, we have

$$\frac{f^{(k)}(z+c)}{f^{(j)}(z)}=\frac{g(z)}{h(z)}=G(z), \tag{3.1}$$

where $G(z)$ is an entire function.
From (3.1), the lemma of logarithmic derivative and Lemma 2.1 it follows that
(3.2)

$$m(r,G(z))=m(r,\frac{f^{(k)}(z+c)}{f^{(j)}(z)})\leqslant m(r,\frac{f^{(k)}(z+c)}{f^{(k)}(z)})+m(r,\frac{f^{(k)}(z)}{f^{(j)}(z)})=S(r,f).$$

Since $G(z)$ is an entire function, we know that

$$N(r,G(z))=0. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$T(r,G(z))=m(r,G(z))+N(r,G(z))=S(r,f). \tag{3.4}$$

Set

$$F=\frac{1}{h}(\frac{g'}{g-1}-\frac{h'}{h-1})=\frac{g}{h}(\frac{g'}{g-1}-\frac{g'}{g})-(\frac{h'}{h-1}-\frac{h'}{h}). \tag{3.5}$$

From the lemma of logarithmic derivative again, (3.2) and (3.5) it follows that

$$m(r,F)=m(r,\frac{g}{h}(\frac{g'}{g-1}-\frac{g'}{g})-(\frac{h'}{h-1}-\frac{h'}{h}))=S(r,f). \tag{3.6}$$

By (3.5), we see that the possible poles of $F(z)$ can occur at the zeros of $h(z)$, the 1 points of $h(z)$ and $g(z)$, and the poles of $h(z)$ and $g(z)$. If z_0 is a 1 point of $h(z)$, then by a short calculation with Laurent series and (8) we know that z_0 is a simple pole of $F(z)$. And hence, the 1 points of $g(z)$ are also the simple pole of $F(z)$. If z_0 is a pole of $h(z)$ with multiplicity $p\geqslant 1$, by $E(\infty,g(z))\subset E(\infty,h(z))$, we have $F(z)=O((z-z_0)^{p-1})$. Similarly, the poles of $g(z)$ are not the poles of $F(z)$. Therefore, the poles of $F(z)$ can occur at the 1 point of $h(z)$, the 1 point of $g(z)$ and the zeros of $h(z)$. From (3.1), (3.4), the hypothesis $N(r,\frac{1}{f^{(j)}})=S(r,f)$ and h shares 1 IM with g, we can find that

$$\begin{aligned} N(r,F) &\leqslant \overline{N}(r,\frac{1}{h-1})+\overline{N}(r,\frac{1}{g-1})+N(r,\frac{1}{h})\\ &\leqslant N(r,\frac{1}{G-1})+N(r,\frac{1}{G-1})+N(r,\frac{1}{f^{(j)}})\\ &\leqslant 2T(r,G)+S(r,f)=S(r,f). \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we conclude that

$$T(r,F)=m(r,F)+N(r,F)=S(r,f). \tag{3.8}$$

If $F\equiv 0$, then by (3.5) we find that $g-1=A(h-1)$, with $A\neq 0$ being constant. We assert that $A=1$. Otherwise, if $A\neq 1$, then $m(r,\frac{1}{h})=\frac{1}{1-A}m(r,\frac{g}{h}-A)=S(r,f)$. Due to $N(r,\frac{1}{h})=N(r,\frac{1}{f^{(j)}})=S(r,f)$, it is easy to deduce that $T(r,\frac{1}{h})=m(r,\frac{1}{h})+N(r,\frac{1}{h})=S(r,f)$, and then by the first fundamental theorem, we have $T(r,h)=T(r,\frac{1}{h})+O(1)=S(r,f)$. Noting that $h=f^{(j)}$, by Lemma 2.6 we have $S(r,f)=S(r,h)$, and hence $T(r,h)=S(r,h)$, which is a contradiction. Then $F\not\equiv 0$. And so we can know from (3.5) and (3.8) that

$$m(r,h)\leqslant m(r,\frac{1}{F})+m(r,\frac{g'}{g-1}-\frac{h'}{h-1})\leqslant T(r,F)+S(r,f)=S(r,f). \tag{3.9}$$

Set

$$H(z)=\frac{g'(h-1)}{h'(g-1)}=(\frac{g'}{g-1}-\frac{g'}{g})\frac{g}{h'}(h-1). \tag{3.10}$$

It follows from the lemma of logarithmic derivative, (12) and (13) that

$$m(r,H)=m(r,\frac{g'(h-1)}{h'(g-1)})\leqslant m(r,\frac{g'}{g-1}-\frac{g'}{g})+m(r,\frac{g}{h'})+m(r,h-1)=S(r,f). \tag{3.11}$$

We now estimate the poles of $H(z)$. Obviously, the poles of $H(z)$ can only occur at the 1 point of g, the poles of h and g', and the zeros of h'. Since $h(z)$ and $g(z)$ share 1 IM, then by Laurent series we know that $H(z)$ is analytic at the 1 point of g. If h has a pole z_∞ with multiplicity $p\geqslant 2$, then by a short calculation with Laurent series and (3.10) we see that the poles of h are not poles of $H(z)$. Similarly, the poles of g' are not poles of $H(z)$. Let z_0 be a zero of h' with multiplicity q, if z_0 is also a zero of h (respectively $h-1$) with multiplicity $q+1$, then from these and (3.10) it is easy to see that z_0 is a pole of $H(z)$ with multiplicity at most q. Thus

$$N(r,H)\leqslant N_0(r,\frac{1}{h'})+S(r,f), \tag{3.12}$$

where $N_0(r,\frac{1}{h'})$ denotes the zeros of h' which are not zeros of $h-1$. From (3.11) and (3.12), we deduce that

$$T(r,H)=m(r,H)+N(r,H)\leqslant N_0(r,\frac{1}{h'})+S(r,f), \tag{3.13}$$

Next, we consider the simple poles of $h(z)$. Let z_0 be a simple pole of h. Since $E(\infty,h(z))\supset E(\infty,g(z))$, we need to discuss two cases:

Case 1. z_0 is not a simple pole of g. We set

$$h(z)=\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots \tag{3.14}$$

and

$$g(z)=b_0+b_1(z-z_0)+b_2(z-z_0)^2+\cdots, \tag{3.15}$$

where $a_j(j=-1,0,1,\cdots)$ and $b_j(j=0,1,\cdots)$ are the coefficients of the Laurent series of $h(z)$ and $g(z)$ respectively. Differentiating (3.14) and (3.15), we obtain

$$h'(z)=-\frac{a_{-1}}{(z-z_0)^2}+a_1+2a_2(z-z_0)+3a_3(z-z_0)^2+\cdots$$

and

$$g'(z)=b_1+2b_2(z-z_0)+3b_3(z-z_0)^2+\cdots.$$

By (3.10) it follows that

$$H(z)=\frac{g'(h-1)}{h'(g-1)}=\frac{[b_1+2b_2(z-z_0)+\cdots][\frac{a_{-1}}{z-z_0}+a_0-1+a_1(z-z_0)+\cdots]}{[-\frac{a_{-1}}{(z-z_0)^2}+a_1+2a_2(z-z_0)+\cdots][b_0-1+b_1(z-z_0)+\cdots]}.$$

Thus $H(z_0)=0$. If $H(z)\equiv 0$, then we have $g'(h-1)\equiv 0$. By integration, we can get $f(z)$ is a nonconstant polynomial, this contradicts with the fact that $f(z)$ is a transcendental function. Thus $H\not\equiv 0$, and so

$$N_{1)}(r,h)\leqslant N(r,\frac{1}{H}). \tag{3.16}$$

Case 2. z_0 is a simple pole of g. Similarly as in Case 1, let

$$h(z)=\frac{c_{-1}}{z-z_0}+c_0+c_1(z-z_0)+c_2(z-z_0)^2+\cdots$$

and

$$g(z)=\frac{d_{-1}}{z-z_0}+d_0+d_1(z-z_0)+d_2(z-z_0)^2+\cdots.$$

Then

$$h'(z)=-\frac{c_{-1}}{(z-z_0)^2}+c_1+2c_2(z-z_0)+3c_3(z-z_0)^2+\cdots$$

and

$$g'(z)=-\frac{d_{-1}}{(z-z_0)^2}+d_1+2d_2(z-z_0)+3d_3(z-z_0)^2+\cdots.$$

By (3.5), it follows that

$$F(z)=\frac{1}{h}(\frac{g'}{g-1}-\frac{h'}{h-1})=\frac{1}{\frac{c_{-1}}{z-z_0}+c_0+\cdots}(\frac{-\frac{d_{-1}}{(z-z_0)^2}+d_1+\cdots}{\frac{d_{-1}}{z-z_0}+d_0-1+\cdots}-\frac{-\frac{c_{-1}}{(z-z_0)^2}+c_1+\cdots}{\frac{c_{-1}}{z-z_0}+c_0-1+\cdots})$$

Thus $F(z_0)=0$. If $F(z)\equiv 0$, then we have $g-1=t(h-1)$ with $t\neq 0$ constant. Similarly, we can assert that $t=1$, then $g\equiv h$, this contradicts with the assumption $g\not\equiv h$. Thus $F\not\equiv 0$, and so

$$N_{1)}(r,h)\leqslant N(r,\frac{1}{F}). \tag{3.17}$$

Combining (3.8), (3.13), (3.16) and (3.17), we have

$$\begin{aligned} N_{1)}(r,h) &\leqslant N(r,\frac{1}{F})+N(r,\frac{1}{H})\leqslant T(r,F)+T(r,H)\\ &\leqslant S(r,f)+N_0(r,\frac{1}{h'})+S(r,f)=N_0(r,\frac{1}{h'})+S(r,f). \end{aligned} \tag{3.18}$$

Since h and g share 1 IM, it follows from (3.1) and (3.4) that

$$\overline{N}(r,\frac{1}{h-1})\leqslant N(r,\frac{1}{G-1})\leqslant T(r,G)=S(r,f). \tag{3.19}$$

Combining (3.18), (3.19), the second fundamental theorem, $N(r,\frac{1}{h})=N(r,\frac{1}{f^{(j)}})=S(r,f)$ and $S(r,h)=S(r,f)$, we have

$$\begin{aligned} T(r,h) &\leqslant N(r,\frac{1}{h})+\overline{N}(r,h)+\overline{N}(r,\frac{1}{h-1})-N_0(r,\frac{1}{h'})+S(r,h)\\ &\leqslant S(r,f)+N_0(r,\frac{1}{h'})-N_0(r,\frac{1}{h'})+S(r,h)\\ &= S(r,f)+S(r,h)=S(r,h), \end{aligned} \tag{3.20}$$

which is impossible. Therefore, $f^{(j)}(z)\equiv f^{(k)}(z+c)$.

4. Proof of Theorem 1.2

Firstly, we prove that $T(r,f^{(j)}(z))$ and $T(r,f^{(k)}(z+c))$ can be restricted by each other. It follows from Lemma 2.2 that

$$\begin{aligned} T(r,f^{(j)}(z)) &\leqslant T(r,f(z))+j\overline{N}(r,f(z))+S(r,f(z))\\ &\leqslant (j+1)T(r,f(z))+S(r,f(z)) \end{aligned} \tag{4.1}$$

On the other hand, by Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned} T(r,f^{(k)}(z+c)) &= T(r,f^{(k)}(z))+S(r,f)\\ &\leqslant T(r,f(z))+k\overline{N}(r,f(z))+S(r,f(z))\\ &\leqslant (k+1)T(r,f(z))+S(r,f(z)). \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$S(r,f^{(j)}(z))=S(r,f^{(k)}(z+c))=S(r,f).$$

Set

$$H(z)=\frac{f^{(k)}(z+c)}{f^{(j)}(z)}, \tag{4.3}$$

From the assumption $E(0,f^{(j)}(z))\subset E(0,f^{(k)}(z+c))$ and $E(\infty,f^{(k)}(z+c))\subset E(\infty,f^{(j)}(z))$, we can deduce that $H(z)$ is an entire function. That is to say,

$$N(r,H(z))=0. \tag{4.4}$$

Case 1 If $H(z)\equiv 1$, then $f^{(j)}(z)\equiv f^{(k)}(z+c)$.

Case 2 We suppose on the contrary that the result of Theorem 1.2 is not valid, i.e., $H(z) \not\equiv 1$. By Lemma 2.1 and the lemma of logarithmic derivative, we know that

$$\begin{aligned} m(r, H(z)) &= m(r, \frac{f^{(k)}(z+c)}{f^{(j)}(z)}) \leqslant m(r, \frac{f^{(k)}(z+c)}{f^{(k)}(z)}) + \\ &+ m(r, \frac{f^{(k)}(z)}{f^{(j)}(z)}) = S(r,f). \end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we can obtain that

$$T(r, H(z)) = m(r, H(z)) + N(r, H(z)) = S(r,f). \tag{4.6}$$

Without loss of generality, we assume that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share $a = 1$ CM. For a general situation, we can consider replacing $f(z)$ by $\frac{1}{a}f(z)$. As a result of the hypothesis that $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share 1 CM, we find that

$$\begin{aligned} &\overline{N}(r, \frac{1}{f^{(j)}(z)-1}) \leqslant N(r, \frac{1}{\frac{f^{(k)}(z+c)}{f^{(j)}(z)} - 1}) = \\ &= N(r, \frac{1}{H-1}) \leqslant T(r,H) + S(r,f) = S(r,f). \end{aligned} \tag{4.7}$$

Secondly, we shall estimate the counting functions of the zeros of $f^{(j)}(z) - 1$ whose multiplicities are not less than 2.

Differentiating (4.3), we have

$$H'(z) = (\frac{f^{(k)}(z+c)}{f^{(j)}(z)})' = \frac{f^{(k+1)}(z+c)f^{(j)}(z) - f^{(k)}(z+c)f^{(j+1)}(z)}{[f^{(j)}(z)]^2}. \tag{4.8}$$

It follows from (4.3) and (4.8) that

$$\begin{aligned} \frac{H'(z)}{H(z)} &= \frac{f^{(k+1)}(z+c)f^{(j)}(z) - f^{(k)}(z+c)f^{(j+1)}(z)}{[f^{(j)}(z)]^2} \cdot \frac{f^{(j)}(z)}{f^{(k)}(z+c)} \\ &= \frac{f^{(k+1)}(z+c)f^{(j)}(z) - f^{(k)}(z+c)f^{(j+1)}(z)}{f^{(j)}(z)f^{(k)}(z+c)} \\ &= \frac{f^{(k+1)}(z+c)}{f^{(k)}(z+c)} - \frac{f^{(j+1)}(z)}{f^{(j)}(z)}. \end{aligned} \tag{4.9}$$

Let z_0 be a 1 point of $f^{(j)}(z)$ with multiplicity $m \geqslant 2$. Since $f^{(j)}(z)$ and $f^{(k)}(z+c)$ share 1 CM, we obtain that z_0 is also a 1 point of $f^{(k)}(z+c)$ with multiplicity $m \geqslant 2$. Then by (4.9) and calculation with Laurent series, we see that z_0 is also a zero of $\frac{H'(z)}{H(z)}$ with multiplicity at least $m-1$. Thus by Lemma 2.2 we can get

$$\begin{aligned} N_{(2}(r, \frac{1}{f^{(j)}(z)-1}) &\leqslant 2N(r, \frac{1}{\frac{H'}{H}}) \leqslant 2N(r,H) + 2N(r, \frac{1}{H'}) \\ &\leqslant 2N(r,H) + 2[N(r, \frac{1}{H}) + \overline{N}(r,H) + S(r,f)] \\ &\leqslant 6T(r,H) + S(r,f) = S(r,f). \end{aligned} \tag{4.10}$$

Together (4.7) with (4.10), we have

$$N(r,\frac{1}{f^{(j)}(z)-1})=\overline{N}(r,\frac{1}{f^{(j)}(z)-1})+N_{(2}(r,\frac{1}{f^{(j)}(z)-1})\leqslant S(r,f). \tag{4.11}$$

By the assumption that $E(0,f^{(j)}(z))\subset E(0,f^{(k)}(z+c))$ and $E(\infty,f^{(k)}(z+c))\subset E(\infty,f^{(j)}(z))$ again, we deduce that

$$N(r,\frac{1}{f^{(j)}(z)})-N(r,\frac{1}{f^{(k)}(z+c)})\leqslant 0,\quad N(r,f^{(k)}(z+c))-N(r,f^{(j)}(z))\leqslant 0. \tag{4.12}$$

From Lemma 2.4, we get

$$\begin{aligned} & m(r,f^{(j)}(z))+m(r,\frac{1}{f^{(j)}(z)})+m(r,\frac{1}{f^{(j)}(z)-1}) \\ & \leqslant 2T(r,f^{(j)}(z))-N_{pair}(r,f)+S(r,f). \end{aligned} \tag{4.13}$$

Adding $N(r,f^{(j)}(z))+N(r,\frac{1}{f^{(j)}(z)})+N(r,\frac{1}{f^{(j)}(z)-1})$ on both sides of (4.13) at the same time and by (4.12), we obtain

$$\begin{aligned} T(r,f^{(j)}(z)) &\leqslant N(r,f^{(j)}(z))+N(r,\frac{1}{f^{(j)}(z)})+N(r,\frac{1}{f^{(j)}(z)-1})-N_{pair}(r,f)+S(r,f) \\ &= N(r,\frac{1}{f^{(j)}(z)-1})+[N(r,\frac{1}{f^{(j)}(z)})-N(r,\frac{1}{f^{(k)}(z+c)})] \\ &+ [N(r,f^{(k)}(z+c))-N(r,f^{(j)}(z))]+S(r,f) \\ &\leqslant N(r,\frac{1}{f^{(j)}(z)-1})+S(r,f)\leqslant S(r,f), \end{aligned}$$

which yields a contradiction.

Therefore, $H(z)\equiv 1$. Then we have $f^{(j)}(z)\equiv f^{(k)}(z+c)$.

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Список литературы

[1] H. H. Amer, Al-Khaladi, “Meromorphic functions that share one finite value CM or IM with their k-th derivatives”, Results Math. **63**, 95 - 105 (2013).

[2] K. S. Charak, R. J. Korhonen, G. Kumar, “A note on partial sharing of values of meromorphic functions with their shifts”, J. Math. Anal. Appl. **435**(2), 1241 -1248 (2016).

[3] C. X. Chen,, R. R. Zhang, Z. B. Huang, “Uniqueness problems on difference operators of meromorphic functions”, Science Asia, **46**, 361 – 367 (2020).

[4] S. J. Chen, “On uniqueness of meromorphic functions and their difference operator with partially shared values”, Comput. Methods Funct. Theory **18**, 529 - 536 (2018).

[5] S. J. Chen, A. Z. Xu, Lin, X.Q.: Further Results on Meromorphic Functions and Their nth Order Exact Differences with Three Shared Values. Communications in Mathematical Research 35(3), 283-288 (2019)

[6] Y. M. Chiang, S. J. Feng, “On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane”, Ramanujian J. **16**, 105 – 129 (2008).

[7] B. M. Deng, M. L. Fang, D. Liu, “Unicty of meromorphic functions concering shared functions with their difference”, Bull. Korean. Math. Soc, **56** (6), 1511 – 1524 (2019).

[8] G. Fank, G. Weissenborn, “Meromorphic funktionens, die mit einer ihrer Ableitungen Werte teilen”, Complex Variables Theory Appl. **7**, 33 – 43 (1986).

[9] G. G. Gundersen, “Meromorphic functions that share three or four values”, J. Lond. Math. Soc. **20**, 457 – 466 (1979).
[10] G. G. Gundersen, “Meromorphic functions that share finite values with their derivative”, J. Math. Anal. Appl. **75**, 441 – 446 (1980).
[11] G. G. Gundersen, “Meromorphic functions that share four values”, Trans. Amer. Math. Soc. **277** (2), 545 – 567 (1983).
[12] G. G. Gundersen, “Meromorphic functions that share two finite values with their derivative”, Pacific J. Math. **105**, 299 – 309 (1983).
[13] R. G. Halburd, R. J. Korhonen, “Difference analogue of the lemma on the logarithmic derivative witth applications to difference equations”, J. Math. Anal. Appl. **314**, 477 – 487 (2006).
[14] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, J. Zhang, “Value sharing results for shifts of meromorphic function and sufficient conditions for periodicity”, J. Math. Anal. Appl. **355**, 352 – 363 (2009).
[15] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, “Uniqueness of meromorphic functions sharing values with their shifts”, Complex Var. Etliptic Equ. **56**, 81 – 92 (2011).
[16] F. L*ü*, W. R. L*ü*, “Meromrophic functions sharing three values with their difference operators”, Comput. Methods. Funct. Theory **17**(3), 395 – 403 (2017).
[17] E. Mues, N. Steinmetz, “Meromorphic Funktionen, die mit ihrer Ableitung Werte teilen”, Manuscr. Math. **29**, 195 – 206 (1979).
[18] X. G. Qi, N. Li, L. Z. Yang, “Uniqueness of meromorphic functions concerning their differences and solutions of difference PainlevE Equations”, Comput. Methods Funct. Theory **18**, 567 – 582 (2018).
[19] X. G. Qi, L. Z. Yang, “Uniqueness of meromorphic functions concerning their shifts and derivatives”, Comput. Methods Funct. Theory **20**, 159 – 178 (2020).
[20] L. A. Rubel, C. C. Yang, “Values shared by an entire function and its derivative”, In: Buckholtz JD, Suffridge TJ (eds) Complex Analysis, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg **599**, 101 – 103 (1977).
[21] J. Wang, “Uniqueness of entire function sharing a small function with its derivative”, J. Math. Anal. Appl. **362** (2), 387 – 392 (2010).
[22] D. M. Wei, Z. G. Huang, “Uniqueness of meromorphic functions concerning their difference operators and derivatives”, Advances in Difference Equations 2020, **480** (2020).
[23] C. C. Yang, H. X. Yi, Uniquenenss Theory of Meromorphic Functions, Kluwer Academic, Dordrecht (2003).
[24] J. L. Zhang, L. Z. Yang, “Some results related to a conjecture of R. Bruck”, J. Inequal. Pure Appl.Math., **8**, Art. 18 (2007).

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