Известия НАН Армении, Математика, том 57, н. 4, 2022, стр. 23 – 33.

SHARP WEIGHTED ESTIMATES FOR STRONG-SPARSE OPERATORS

G. MNATSAKANYAN

https://doi.org/10.54503/0002-3043-2022.57.4-23-33

Institute of Mathematics of Armenian National Academy of Sciences
E-mail: *mnatsakanyan_g@yahoo.com*

Abstract. We prove the sharp weighted-L^2 bounds for the strong-sparse operators introduced in [3]. The main contribution of the paper is the construction of a weight that is a lacunary mixture of dual power weights. This weight helps to prove the sharpness of the trivial upper bound of the operator norm.

MSC2020 numbers: 42B25; 42A82.
Keywords: weighted inequalities; sharp inequalities; sparse operators; power weights.

1. Introduction

The theory of weighted inequalities started with the seminal work of Muckenhoupt [10], where he proved that the Hardy-Littlewood maximal operator is bounded on $L^p(w)$, $1 < p < \infty$, for positive measurable $w : \mathbb{R} \to \mathbb{R}$ if and only if

$$[w]_{A_p} := \sup_I \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}\right)^{p-1} < \infty, \tag{1.1}$$

where the supremum is taken over all intervals and $|I|$ denotes the Lebesgue measure of the interval. If (1.1) holds, then w is said to be in the Muckenhoupt class A_p and the quantity $[w]_{A_p}$ is called its A_p characteristic. Later, Buckley [11] obtained the sharp dependence of the norm of the maximal operator on the A_p characteristic. Namely, he proved that

$$\|M\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\frac{1}{p}}, \tag{1.2}$$

$$\|M\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}, \tag{1.3}$$

and these are sharp in the sense of the theorems below.

The problem of the sharp dependence of the $L^2(w) \to L^2(w)$ norm of the Caldéron-Zygmund operator on the A_2 characteristic of w is known as the A_2-conjecture. It was first proved by Hytönen [7, 6]. A simpler proof was given by Lerner [8, 9] proving that the Caldéron-Zygmund operators can be dominated by the simple sparse operators. Later, it was proved that a number of operators in harmonic analysis admit pointwise or norm domination by the sparse operators

[12, 13, 2, 1, 8, 9]. On the other hand, L^p and weighted-L^p bounds for the sparse operators are fairly easy to obtain[1].

Let us have a family $\mathcal{S}$ of intervals in $\mathbb{R}$ and $0 < \gamma < 1$. $\mathcal{S}$ is called γ-sparse, or just sparse, if there exists pairwise disjoint subsets $E_A \subset A$, $A \in \mathcal{S}$, such that $|E_A| \geq \gamma |A|$. Let us set for an interval B

$$\langle f \rangle_B := \frac{1}{|B|} \int_B |f|, \quad M_B f := \sup_{A \text{ intervals: } A \supset B} \langle f \rangle_A.$$

For a sparse family $\mathcal{S}$, we define the sparse and the strong-sparse operators as

$$\mathcal{A}_{\mathcal{S}} f(x) := \sum_{A \in \mathcal{S}} \langle f \rangle_A \cdot \mathbf{1}_A(x), \tag{1.4}$$

$$\mathcal{A}^*_{\mathcal{S}} f(x) := \sum_{A \in \mathcal{S}} (M_A f) \cdot \mathbf{1}_A(x), \tag{1.5}$$

respectively. The sharp weighted bound for the sparse operator[1] is as follows

$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max(1, \frac{1}{p-1})}. \tag{1.6}$$

The strong-sparse operators were introduced by Karagulyan and the author in [3], where L^p and weak-L^1 estimates are proved in the setting of an abstract measure space with ball-basis. In this paper, we obtain the sharp dependence of the weighted-L^2 norm of the strong-sparse operator on the A_2 characteristic of the weight.

Theorem 1.1. *For an A_2 weight w we have the bound*

$$\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w) \to L^{2,\infty}(w)} \lesssim [w]_{A_2}^{\frac{3}{2}}. \tag{1.7}$$

The inequality is sharp in the following sense: there exist a sparse family $\mathcal{S}$ and a sequence of weights w_α such that

$$[w_\alpha]_{A_2} \to \infty, \text{ as } \alpha \to 0, \tag{1.8}$$

and for any function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(x)/x^{\frac{3}{2}} \to 0$ as $x \to \infty$, we have

$$\frac{\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w_\alpha) \to L^{2,\infty}(w_\alpha)}}{\phi([w_\alpha]_{A_2})} \to \infty, \text{ as } \alpha \to 0. \tag{1.9}$$

Theorem 1.2. *For an A_2 weight w we have the bound*

$$\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w) \to L^2(w)} \lesssim [w]_{A_2}^2. \tag{1.10}$$

The inequality is sharp in the following sense: there exist a sparse family $\mathcal{S}$ and a sequence of weights w_α such that

$$[w_\alpha]_{A_2} \to \infty, \text{ as } \alpha \to 0, \tag{1.11}$$

and for any function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(x)/x^2 \to 0$ as $x \to \infty$, we have

$$\frac{\|\mathcal{A}^*_{\mathcal{S}}\|_{L^2(w_\alpha) \to L^2(w_\alpha)}}{\phi([w_\alpha]_{A_2})} \to \infty, \text{ as } \alpha \to 0. \tag{1.12}$$

On the other hand, we have the following simple partial improvement for the strong bound. For this theorem we assume that all the intervals in the statement, proof and in the definition of the strong-sparse operator are dyadic.

Theorem 1.3. *Let the sparse family $\mathcal{S}$ be such that for any two $A, B \in \mathcal{S}$ either $A \subset B$ or $B \subset A$. Then, we have*

$$\|\mathcal{A}_{\mathcal{S}}^{*}\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}^{\frac{3}{2}}. \tag{1.13}$$

Looking at the definition of the strong-sparse operators, we see that $M_B f \leq Mf(x)$ for any $x \in B$. Thus, $M_B f \leq \langle Mf\rangle_B$ and we obtain

$$\mathcal{A}_{\mathcal{S}}^{*} f(x) \leq \mathcal{A}_{\mathcal{S}}(Mf). \tag{1.14}$$

Then, one can try to black-box the sharp weighted bounds (1.2), (1.3) and (1.6) for Theorem 1.1 and Theorem 1.2. As it will be shown in Section 2, the weighted weak-L^2 bound for the sparse operator is the same as for the strong one. Thus, Theorem 1.1 will not follow from such a black-box. Instead, we will decompose the operator according to the magnitude of the $M_B f$ for the sparse intervals B, then, we will use the weighted weak bound of the maximal function (1.2). We will do this in Section 2.

As for Theorem 1.2, we see that by black-boxing the above mentioned inequalities we trivially get the upper bound, i.e.

$$\begin{aligned}\|\mathcal{A}_{\mathcal{S}}^{*}\|_{L^2(w)\to L^2(w)} &\leq \|\mathcal{A}_{\mathcal{S}} \circ M\|_{L^2(w)\to L^2(w)} \\ &\lesssim \|\mathcal{A}_{\mathcal{S}}\|_{L^2(w)\to L^2(w)} \|M\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}^{2}.\end{aligned}$$

Thus, the interesting thing about Theorem 1.2 is to obtain the sharpness of this estimate. For that we will construct a weight which is a lacunary mixture of the dual power weights $x^{\alpha-1}$ and $x^{1-\alpha}$. We will do this in Section 3.

In Section 4, we will prove Theorem 1.3.

We say $a \lesssim b$ if there is an absolute constant c, maybe depending on the sparse parameter γ, such that $a \leq c \cdot b$. Furthermore, we say $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

2. The upper bound of Theorem 1.1

2.1. **A well-known property of A_∞ weights.** Following [5, 4], we say that w is an A_∞ weights if

$$[w]_{A_\infty} := \sup_I \frac{1}{w(I)} \int_I M(w\mathbf{1}_I)(x)dx < \infty. \tag{2.1}$$

It is well-known that any A_p weight is also an A_∞ weights and that a reverse Hölder inequality holds for in the latter class. The following theorem with sharp constants is due to Hytönen, Pérez and Rela[4].

Theorem 2.1. *If w is an A_∞ weight and $\epsilon = \frac{1}{4[w]_{A_\infty}}$, then $\langle w^{1+\epsilon}\rangle_I \leq 2\left(\langle w\rangle_I\right)^{1+\epsilon}$, for any interval I.*

This implies the following lemma.

Lemma 2.1. *For any cube Q and measurable subset $E \subset Q$, we have*

$$w(E) \leq 2w(Q)\left(\frac{|E|}{|Q|}\right)^{c/[w]_{A_\infty}},$$

where c is an absolute constant.

Proof. Let ϵ be as before.

$$\begin{aligned}
\int_E w &\leq \left(\int_E w^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}} \cdot |E|^{\epsilon/(1+\epsilon)} \quad \text{(Hölder)}\\
&\leq \langle w^{1+\epsilon}\rangle_Q^{\frac{1}{1+\epsilon}} \cdot |E|^{\epsilon/(1+\epsilon)} \cdot |Q|^{1/(1+\epsilon)}\\
&\leq 2\langle w\rangle_Q |E|^{\epsilon/(1+\epsilon)} \cdot |Q|^{1/(1+\epsilon)} \quad \text{(Reverse Hölder)}\\
&= 2w(Q)\left(\frac{|E|}{|Q|}\right)^{c/[w]_{A_\infty}}.
\end{aligned}$$

2.2. **The proof of the weak bound.** The idea is to group $M_B f$'s, $B \in \mathcal{S}$, according to their magnitude and estimate each group applying Lemma 2.1 and the weighted weak bound for the maximal operator (1.2). Denote $\alpha := \frac{1}{[w]_\infty}$, and for $\lambda > 0$ let

$$\begin{aligned}
A_0 &:= \{B \in \mathcal{S} \ : \ M_B f > \alpha\lambda\},\\
A_j &:= \{B \in \mathcal{S} \ : \ 2^{-j+1}\alpha\lambda \geq M_B f > 2^{-j}\alpha\lambda\},
\end{aligned}$$

for $j = 1, 2, \ldots$. Thus, A_j's partition $\mathcal{S}$. We write

$$\begin{aligned}
w\{\mathcal{A}^*_{\mathcal{S}} f > \lambda\} &\leq \sum_{j=0}^{\infty} w\Big\{\sum_{B \in A_j} (M_B f)\chi_B > \lambda 2^{-j/2} C\Big\}\\
&\leq w\left(\bigcup_{B\in A_0} B\right) + \sum_{j=1}^{\infty} w\Big\{\sum_{B\in A_j} \chi_B > \frac{1}{\alpha} 2^{j/2} C\Big\}\\
&\leq w\big\{Mf > \lambda\alpha\big\} + \sum_{j=1}^{\infty} 2w\left(\bigcup_{B\in A_j} B\right) \frac{|\{\sum_{B\in A_j} \chi_B > \frac{1}{\alpha}2^{j/2}C\}|^{c/[w]_\infty}}{|\bigcup_{B\in A_j} B|^{c/[w]_\infty}}\\
&\leq w\big\{Mf > \lambda\alpha\big\} + 2\sum_{j=1}^{\infty} w\left(\bigcup_{B\in A_j} B\right) 2^{-2^{j/2}C\alpha\cdot\frac{c}{\alpha}}\\
&\leq w\big\{Mf > \lambda\alpha\big\} + 2\sum_{j=1}^{\infty} w\big\{Mf > 2^{-j}\lambda\alpha\big\} 2^{-cC2^{j/2}}\\
&\lesssim \frac{[w]_{A_\infty}^2}{\lambda^2}\|M\|^2_{L^2\to L^{2,\infty}},
\end{aligned}$$

where the first line is due to the triangle inequality, the third inequality follows from Lemma 2.1 and the fourth one from the fact, that A_j is a sparse collection. It reimains to apply the bound (1.2) to get the upper bound of Theorem 1.1.

2.3. **The lower bound of Theorem 1.1.** Let $w = |x|^{\alpha-1}$ and $\sigma = |x|^{1-\alpha}$ be the dual power weights, $0 < \alpha < 1$. We know, for example from [11], that

$$[w]_{A_2} = [\sigma]_{A_2} \sim \frac{1}{\alpha}. \tag{2.2}$$

Let $\mathcal{S} := \{[0, 2^{-k}) : \text{ for } k \in \mathbb{N}\}$ be a sparse family. Then, we claim

$$\|\mathcal{A}^*_{\mathcal{S}}(\sigma \mathbf{1}_{[0,1)})\|_{L^{2,\infty}(w)} \sim [w]^{3/2}_{A_2} \|\sigma \mathbf{1}_{[0,1)}\|_{L^2(w)}. \tag{2.3}$$

The square of the right-hand side of (2.3) equals $\frac{1}{(2-\alpha)\alpha^3}$. On the other hand,

$$\begin{aligned} &\|\mathcal{A}^*_{\mathcal{S}}(\sigma \mathbf{1}_{[0,1)})\|^2_{L^{2,\infty}(w)} \geq \frac{1}{\alpha^2} w\{\mathcal{A}^*_{\mathcal{S}}(\sigma \mathbf{1}_{[0,1)}) > \frac{1}{\alpha}\} \\ &= \frac{1}{\alpha^2} w\{\sum_{k=1}^{\infty} \mathbf{1}_{[0,2^{-k})} \gtrsim \frac{1}{\alpha}\} = \frac{1}{\alpha^2} w([0, 2^{-\frac{c}{\alpha}})) \sim \frac{2^{-\frac{c}{\alpha}\alpha}}{\alpha^3}. \end{aligned}$$

So the proof of Theorem 1.1 is complete.

3. The lower bound of Theorem 1.2

3.1. **Construction of the weight.** Let $0 < \alpha < 1$ be small enough integer power of 2, i.e. $\alpha = 2^{-a}$ for large enough integer a. Let us define the weight $\sigma : \mathbb{R} \to [0, \infty)$ to be even and

(3.1)

$$\sigma(x) := \begin{cases} \frac{2^{2k(1-\alpha)}}{\alpha}(x - 2^{-(k+1)})^{1-\alpha}, \quad x \in [2^{-(k+1)}, (1+\alpha)2^{-(k+1)}) \text{ for } k \in \mathbb{N} \\ x^{\alpha-1}, \quad x \in [(1+\alpha)2^{-(k+1)}), (1-\alpha)2^{-k}) \text{ for } k \in \mathbb{N} \\ \frac{2^{2k(1-\alpha)}}{\alpha}(2^{-k} - x)^{1-\alpha}, \quad x \in [(1-\alpha)2^{-k}, 2^{-k}) \text{ for } k \in \mathbb{N} \\ x^{\alpha-1}, \quad x \in [\frac{1}{2}, \infty). \end{cases}$$

The dual weight to σ is $w(x) := \sigma(x)^{-1}$. We will prove that

$$\sup_I \frac{1}{|I|^2} \Big(\int_I w \Big) \cdot \Big(\int_I \sigma \Big) \sim \frac{1}{\alpha}, \tag{3.2}$$

that is, $\sigma \in A_2$ with $[\sigma]_{A_2} \sim \frac{1}{\alpha}$.

First, we show that (3.2) holds for dyadic intervals. Let us partition all dyadic intervals into three groups.

a. $I = [0, 2^{-k})$ for some $k \in \mathbb{N}_0$. Then, we compute

$$\int_{2^{-(k+1)}}^{2^{-k}} w(x)dx = \int_{(1+\alpha)2^{-(k+1)}}^{(1-\alpha)2^{-k}} x^{1-\alpha}dx + \alpha 2^{2k(\alpha-1)} \int_{(1-\alpha)2^{-k}}^{2^{-k}} (2^{-k}-x)^{\alpha-1}dx$$
$$+ \alpha 2^{2k(\alpha-1)} \int_{2^{-(k+1)}}^{(1+\alpha)2^{-(k+1)}} (x-2^{-(k+1)})^{\alpha-1}dx$$
$$= \frac{(1-\alpha)^{2-\alpha}2^{-(2-\alpha)k} - (1+\alpha)^{2-\alpha}2^{-(2-\alpha)(k+1)}}{2-\alpha}$$
$$+ \alpha 2^{2k(\alpha-1)} \cdot \frac{\alpha^\alpha(2^{-k\alpha} + 2^{-(k+1)\alpha})}{\alpha} = c(\alpha)2^{-k(2-\alpha)}. \tag{3.3}$$

In the above computations and below $c(\alpha)$ is a constant depending on α absolutely bounded and away from 0. It will be different at each occurence. Next, we have

$$\int_0^{2^{-k}} w(x)dx = \sum_{j=k}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} w(x)dx = \sum_{j=k}^{\infty} c(\alpha)2^{-j(2-\alpha)} = c(\alpha)\cdot 2^{-k(2-\alpha)}. \tag{3.4}$$

For σ we have

$$\int_{2^{-(k+1)}}^{2^{-k}} \sigma(x)dx = \int_{(1+\alpha)2^{-(k+1)}}^{(1-\alpha)2^{-k}} \sigma(x)dx + \int_{(1-\alpha)2^{-k}}^{2^{-k}} \sigma(x)dx + \int_{2^{-(k+1)}}^{(1+\alpha)2^{-(k+1)}} \sigma(x)dx$$
$$= \frac{(1-\alpha)^\alpha 2^{-k\alpha} - (1+\alpha)^\alpha 2^{-(k+1)\alpha}}{\alpha} + \frac{2^{2k(1-\alpha)}}{\alpha}\cdot$$
$$\cdot \left(\int_{2^{-k}(1-\alpha)}^{2^{-k}} (2^{-k}-x)^{1-\alpha}dx + \int_{2^{-(k+1)}}^{(1+\alpha)2^{-(k+1)}} (x-2^{-(k+1)})^{1-\alpha}dx \right)$$
$$= c(\alpha)2^{-k\alpha} + \frac{2^{2k(1-\alpha)}}{\alpha}\cdot\frac{\alpha^{2-\alpha}(2^{-k(2-\alpha)} + 2^{-(k+1)(2-\alpha)})}{2-\alpha}$$
$$= c(\alpha)\frac{2^{-k\alpha}}{\alpha} + \alpha 2^{-k\alpha} = c(\alpha)\frac{2^{-k\alpha}}{\alpha}. \tag{3.5}$$

Then, we have

$$\int_0^{2^{-k}} \sigma(x)dx = \sum_{j=k}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} \sigma(x)dx = \sum_{j=k}^{\infty} c(\alpha)\frac{2^{-j\alpha}}{\alpha} = c(\alpha)\frac{2^{-k\alpha}}{\alpha}. \tag{3.6}$$

Combining the two computations above, we have for (3.2)

$$2^{2k}\Big(\int_0^{2^{-k}} w\Big)\cdot\Big(\int_0^{2^{-k}} \sigma\Big) = c(\alpha)2^{2k}2^{-k(2-\alpha)}\frac{2^{-k\alpha}}{\alpha} \sim \frac{1}{\alpha}. \tag{3.7}$$

b. One of the following holds: for some $k \in \mathbb{N}_0$, $I \subset [2^{-(k+1)}, (1+\alpha)2^{-(k+1)})$, $I \subset [(1+\alpha)2^{-(k+1)}, (1-\alpha)2^{-k})$ or $I \subset [(1-\alpha)2^{-k}, 2^{-k})$. On these intervals, the weights w and σ are just rescaled versions of the power weights. Thus, we immediately have

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I \sigma\Big) \lesssim \frac{1}{\alpha}, \tag{3.8}$$

by the A_2 characteristic of the power weights (2.2).

c. $I \subset [2^{-(k+1)}, 2^{-k})$ and either $[(1-\alpha)2^{-k}, 2^{-k}) \subsetneq I$ or $[2^{-(k+1)}, (1+\alpha)2^{-(k+1)}) \subsetneq I$ for some $k \in \mathbb{N}_0$. This is the intermediate case between the above two. The computation for the choice of the last two conditions is identical, so we consider only one of them. Let $|I| = 2^{-m}$ so that $I = [2^{-k} - 2^{-m}, 2^{-k})$ and $k+2 \le m < k+a$, where we recall $\alpha = 2^{-a}$. We start calculating

$$\begin{aligned}
\int_{2^{-k}-2^{-m}}^{2^{-k}} w(x)dx &= \int_{2^{-k}-2^{-m}}^{(1-\alpha)2^{-k}} x^{1-\alpha}dx + \alpha 2^{2k(\alpha-1)} \int_{(1-\alpha)2^{-k}}^{2^{-k}} (2^{-k}-x)^{\alpha-1}dx \\
&= \frac{(1-2^{-a})^{2-\alpha}2^{-(2-\alpha)k} - 2^{-(2-\alpha)k}(1-2^{k-m})^{2-\alpha}}{2-\alpha} \\
&\quad + \alpha\cdot 2^{2k(\alpha-1)} \int_{2^{-k}(1-\alpha)}^{2^{-k}} (2^{-k}-x)^{\alpha-1}dx \\
&= c(\alpha,m)2^{-k(2-\alpha)}\Big((1+\frac{2^{k-m}-2^{-a}}{1-2^{k-m}})^{2-\alpha} - 1\Big) + \alpha^\alpha 2^{-k(2-\alpha)} \\
&= c(\alpha,m)2^{-k(2-\alpha)}2^{k-m} + \alpha^\alpha 2^{-k(2-\alpha)} = c(\alpha,m)2^{-k(2-\alpha)}.
\end{aligned}$$

As before $c(\alpha,m)$ is a positive constant bounded from above and away from 0. For σ we write

$$\begin{aligned}
\int_{2^{-k}-2^{-m}}^{2^{-k}} \sigma(x)dx &= \int_{2^{-k}-2^{-m}}^{(1-\alpha)2^{-k}} \sigma(x)dx + \int_{(1-\alpha)2^{-k}}^{2^{-k}} \sigma(x)dx \\
&= \frac{(1-\alpha)^\alpha 2^{-k\alpha} - 2^{-k\alpha}(1-2^{k-m})^\alpha}{\alpha} \\
&\quad + \frac{2^{2k(1-\alpha)}}{\alpha} \int_{2^{-k}(1-\alpha)}^{2^{-k}} (2^{-k}-x)^{1-\alpha}dx \\
&= c(\alpha,m)2^{-k\alpha}\frac{(1+\frac{2^{k-m}-2^{-a}}{1-2^{k-m}})^\alpha - 1}{\alpha} + \alpha 2^{-k\alpha} \\
&= c(\alpha,m)2^{-k\alpha}\cdot 2^{k-m} + 2^{-a}\cdot 2^{-k\alpha} = c(\alpha,m)2^{-k\alpha+k-m}.
\end{aligned}$$

Here, in the penultimate equality we used the the Taylor expansion

$$(1+x)^{\beta}-1\sim\beta x,\text{for } 0<x<1. \tag{3.9}$$

Thus, for (3.2) we have

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I \sigma\Big)=2^{2m}\cdot c(\alpha,m)2^{-k\alpha+k-m}\cdot 2^{-k(2-\alpha)}\sim 2^{m-k}\lesssim 2^{a}=\frac{1}{\alpha}.$$

We conclude, that the dyadic A_2 characteristic of w is $\frac{c}{\alpha}$. It is important here, that the supremum is attained at a large number of dyadic intervals and not only on one chain.

We turn to the case of a general interval I. First of all, the arguments for the case **b** are also true for all intervals I due to the A_2 characteristic of power weights. On the other hand, if I can be covered by a dyadic interval of a comparable size, then again (3.8) holds. Otherwise, let k be such that $I\subset[0,2^{-(k-1)})$, $I\not\subset[0,2^{-(k+1)})$ and $|I|\lesssim 2^{-k}$. We distinguish two cases.

(i) One of the following holds: $(1+\alpha)2^{-(k+1)}\in I$, $(1-\alpha)2^{-k}\in I$, $(1+\alpha)2^{-k}\in I$, $(1-\alpha)2^{-(k-1)}\in I$. All four cases are similar, so we only consider the second one. For σ we have

$$\int_I \sigma(x)dx\sim 2^{-k(\alpha-1)}|I|. \tag{3.10}$$

As for w we write

$$\int_I w(x)dx\sim\big((1-\alpha)2^{-k}-l(I)\big)2^{-k(1-\alpha)}+\int_{(1-\alpha)2^{-k}}^{r(I)} w(x)dx, \tag{3.11}$$

where $l(I)$ and $r(I)$ are the left and right endpoints of I.

(i.1) If $r(I)<(1-\alpha)2^{-k}+\alpha 2^{-(k+1)}$, then we have

$$\int_I w(x)dx\sim |I|2^{-k(1-\alpha)}, \tag{3.12}$$

and so

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I\sigma\Big)\lesssim 1. \tag{3.13}$$

(i.2) If $(1-\alpha)2^{-k}+\alpha 2^{-(k+1)}<r(I)$, then using the computation in (3.3), we have

$$\int_{(1-\alpha)2^{-k}}^{r(I)} w(x)dx\lesssim 2^{-k(2-\alpha)}. \tag{3.14}$$

Hence, we obtain

$$\frac{1}{|I|^2}\Big(\int_I w\Big)\cdot\Big(\int_I\sigma\Big)\lesssim\frac{1}{|I|^2}2^{-k(2-\alpha)}\cdot|I|2^{-k(1-\alpha)}\lesssim\frac{2^{-k}}{|I|}\lesssim\frac{1}{\alpha},$$

where the last step is due to $l(r)<(1-\alpha)2^{-k}<(1-\alpha)2^{-k}+\alpha 2^{-(k+1)}<r(I)$.

(ii) Let us have $(1-\alpha)2^{-k} \notin I$, $(1+\alpha)2^{-k} \notin I$ and $2^{-k} \in I$. Without loss of generality we can assume $r(I) - 2^{-k} \leq 2^{-k} - l(I)$. Then, we have $|I| \sim (2^{-k} - l(I))$. Furthermore,

$$\int_I \sigma(x)dx \sim \int_{l(I)}^{2^{-k}} \sigma(x)dx, \quad \text{and} \quad \int_I w(x)dx \sim \int_{l(I)}^{2^{-k}} w(x)dx.$$

Thus, as w and σ are just power weights on $[(1-\alpha)2^{-k}, 2^k)$, and the estimate (3.2) holds.

3.2. **Construction of the sparse family.** Let us take the following sparse family:

$$\mathcal{S} := \{[2^{-k} - 2^{-j}, 2^{-k}) \ : \ \text{for all } k, j \in \mathbb{N} \text{ and } j \geq a + k\}. \tag{3.15}$$

We also denote $B_{k,j} := [2^{-k} - 2^{-j}, 2^{-k})$. Using (3.6), we have

$$M_{B_{k,j}}(\sigma) \sim 2^k \int_0^{2^{-k}} \sigma(x)dx \sim \frac{2^{k(1-\alpha)}}{\alpha}, \tag{3.16}$$

and the corresponding strong-sparse operator is

$$\mathcal{A}^*_{\mathcal{S}} f(x) := \sum_{k=1}^{\infty} \frac{2^{k(1-\alpha)}}{\alpha} \sum_{j=a+k}^{\infty} \mathbf{1}_{B_{j,k}}(x). \tag{3.17}$$

3.3. **The lower bound.** We claim that

$$\int_0^1 \mathcal{A}^*_{\mathcal{S}}(\sigma)(x)^2 w(x)dx \sim \frac{1}{\alpha^4} \int_0^1 \sigma(x)dx. \tag{3.18}$$

By (3.17), we can write

$$\int_0^1 S^*(\sigma)(x)^2 w(x)dx \sim \sum_{k=1}^{\infty} \frac{2^{2k(1-\alpha)}}{\alpha^2} \int_0^1 \Big(\sum_{j=k+a}^{\infty} \mathbf{1}_{B_{k,j}}(x) \Big)^2 w(x)dx. \tag{3.19}$$

We make a change of variables in the integral and see that it realizes the sharp constant for the regular sparse operator. Putting $y = \frac{x-(1-\alpha)2^{-k}}{2^{-k}\alpha}$, we can write

$$\int_0^1 \Big(\sum_{j=k+a}^{\infty} \mathbf{1}_{B_{k,j}}(x) \Big)^2 w(x)dx = \alpha 2^{\alpha k - 2k} \int_0^1 \Big(\sum_{j=1}^{\infty} \mathbf{1}_{[0,2^{-j})}(y) \Big)^2 y^{\alpha-1} dy$$
$$\sim \alpha 2^{\alpha k - 2k} \cdot \frac{1}{\alpha^3} = \frac{2^{\alpha k - 2k}}{\alpha^2},$$

where the penultimate estimate is a direct computation. Plugging this into (3.19), we obtain

$$\int_0^1 \mathcal{A}^*_{\mathcal{S}}(\sigma)(x)^2 w(x)dx \sim \sum_{k=1}^{\infty} \frac{2^{2k(1-\alpha)}}{\alpha^2} \cdot \frac{2^{\alpha k - 2k}}{\alpha^2} \sim \frac{1}{\alpha^5} \sim \frac{1}{\alpha^4} \int_0^1 \sigma.$$

This finishes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

We can assume that the intervals in the sparse family are in some bounded interval, and the general case will follow by a limiting argument. Let us enumerate the intervals of the sparse family $\mathcal{S}$.

$$B_1 \supset B_2 \supset \cdots \supset B_k \supset \cdots .$$

Let $g \in L^2(w)$. We inductively choose $\pi(B_i) \supset B_i$ such that it is the largest interval with $M_{B_i}(g) \leq 2\langle g\rangle_{\pi(B_i)}$ and $\pi(B_i) \subset \pi(B_{i-1})$. We can enumerate $\{\pi(B_i)\}$ by $A_1 \supsetneq A_2 \supsetneq \ldots$. Note, that there can be many B_i with $\pi(B_i) = A_j$. Moreover, recalling that A_i are dyadic we see that $\{A_i\}_i$ is again a sparse family.

Consider the following function

$$\tilde{g}(x) = \begin{cases} \frac{1}{|A_i \setminus A_{i+1}|}\int_{A_i\setminus A_{i+1}} g, & x \in A_i \setminus A_{i+1} \text{ for some } i \in \mathbb{N}, \\ g(x), \text{ otherwise.} \end{cases}$$

First of all, it is clear that for all i

$$\int_{A_i} g = \int_{A_i} \tilde{g}. \tag{4.1}$$

Let $B \in \mathcal{S}$ be such that $A_i = \pi(B)$. Then, $A_{i+1} \subsetneq B$ due to the choice of $\pi(B)$. Then, by (4.1) and by the definition of $\tilde{g}$, we have

$$\begin{aligned} \langle g\rangle_{A_i} &= \frac{1}{|A_i|}\Big(\int_{A_{i+1}} g + \int_{A_i\setminus A_{i+1}} g\Big) \lesssim \frac{1}{|A_i|}\int_{A_{i+1}} \tilde{g} + \frac{1}{|B\setminus A_{i+1}|}\int_{B\setminus A_{i+1}} \tilde{g} \\ &\lesssim \frac{1}{|B|}\int_B \tilde{g} = \langle \tilde{g}\rangle_B. \end{aligned}$$

We conclude, that for all x

$$\mathcal{A}^*_{\mathcal{S}} g(x) \lesssim \mathcal{A}_{\mathcal{S}}\tilde{g}(x). \tag{4.2}$$

We turn to the norm of $\tilde{g}$.

$$\begin{aligned} \int_{\mathbb{R}} \tilde{g}^2 w &= \sum_i \int_{A_i\setminus A_{i+1}} \tilde{g}^2 w + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \\ &\leq \sum_i \left(\frac{1}{|A_i\setminus A_{i+1}|}\int_{A_i\setminus A_{i+1}} g\right)^2 w(A_i\setminus A_{i+1}) + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \\ &\leq \sum_i \frac{w(A_i\setminus A_{i+1})\cdot\sigma(A_i\setminus A_{i+1})}{|A_i\setminus A_{i+1}|^2}\int_{A_{i+1}\setminus A_i} g^2 w + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \\ &\leq \sum_i \frac{w(A_i)\cdot\sigma(A_i)}{|A_i|^2}\int_{A_{i+1}\setminus A_i} g^2 w + \int_{\mathbb{R}\setminus\cup(A_i\setminus A_{i+1})} g^2 w \lesssim [w]_{A_2}\int_{\mathbb{R}} g^2 w. \end{aligned}$$

Combining the last estimate, (4.2) and the sparse bound (1.6) we conclude

$$\|\mathcal{A}^*_{\mathcal{S}} g\|_{L^2(w)} \lesssim \|\mathcal{A}_{\mathcal{S}}\tilde{g}\|_{L^2(w)} \lesssim [w]_{A_2}\|\tilde{g}\|_{L^2(w)} \lesssim [w]_{A_2}^{\frac{3}{2}}\|g\|_{L^2(w)}.$$

And the proof of Theorem 1.3 is complete.

Acknowledgments. The author is grateful to his advisor Grigori Karagulyan for posing this problem, for introducing him to the field of Harmonic Analysis and for his continuous support.

Список литературы

[1] M. T. Lacey, “An elementary proof of the A_2 bound”, Israel Journal of Mathematics **217**, no. 1, 181 – 195 (2017).

[2] G. A. Karagulyan, An Abstract Theory of Singular Operators, Transaction of the American Mathematical Society, **372** (2016).

[3] G. Karagulyan and G. Mnatsakanyan, “On a weak type estimate for sparse operators of strong type”, Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), **54**, 216 – 221 (2019).

[4] T. Hytönen, C. Pérez and E. Rela, Sharp Reverse Hölder Property for A_∞ Weights on Spaces of Homogeneous Type, Journal of Functional Analysis, **263**, pages ???????? (2012).

[5] T. Hytönen, C. Pérez, “Sharp weighted bounds involving A_∞”, Analysis and Partial Differential Equations **6**, pages ???????? (2011).

[6] T. Hytönen, “The A_2 theorem: remarks and complements”, Harmonic Analysis and Partial Differential Equations **612**, pages ???????? (2012).

[7] T. Hytönen, “The sharp weighted bound for general Caldéron-Zygmund operators”, Annals of Mathematics **175**, pages ?????????? (2010).

[8] A. K. Lerner, “A simple proof of the A_2 conjecture”, Internartonal Mathematics Research Notices, 3159 – 3170 (2013).

[9] A. K. Lerner, “On estimate of Caldéron-Zygmund operators by dyadic positive operators”, Journal d'Analyse Mathématique, **121**, 141 – 161 (2012).

[10] B. Muckenhoupt, “Weighted norm inequalities for the Hardy maximal function”, Transactions of the American Mathematical Society, **165**, 207 – 226 (1972).

[11] S. M. Buckley, “Estimates for operator norms on weighted spaces and reverse Jensen inequalities”, Transactions of the American Mathematical Society, **340**, no. 1, 253 – 272 (1993).

[12] J. Conde-Alonso and G. Rey, “A pointwise estimate for positive dyadic shifts and some applications”, Mathematische Annalen **365** (2016).

[13] F. Di Plinio and A. K. Lerner, “On weighted norm inequalities for the Carleson and Walsh-Carleson operator”, Journal of the London Mathematical Society, **90**, no. 3, 654 – 674 (2014).

Поступила 01 декабря 2021

После доработки 25 марта 2022

Принята к публикации 10 апреля 2022