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THE WALSH-FOURIER TRANSFORM ON THE REAL LINE

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Abstract. The element of the Walsh system, that is the Walsh functions map from the unit interval to the set $\{-1, 1\}$. They can be extended to the set of nonnegative reals, but not to the whole real line. The aim of this article is to give an Walsh-like orthonormal and complete function system which can be extended on the real line.

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1. The triadic field

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$. Let $\mathbf{F}$ denote the set of double infinite sequences

$$x = (x_n : n \in \mathbb{Z})$$

where $x_n = 0, 1$ or -1 and $x_n \to 0$ as $n \to -\infty$. Thus, to each $x \in \mathbf{F}$ with $x \neq 0$ there corresponds an integer $s(x) \in \mathbb{Z}$ such that

$$x_{s(x)} \neq 0 \text{ but } x_n = 0 \text{ for } n < s(x).$$

Let $x = (x_n : n \in \mathbb{Z})$ and $y = (y_n : n \in \mathbb{Z})$ be elements of $\mathbf{F}$. Define the sum of x and y by

$$x + y = ((x_n + y_n) \bmod 3 : n \in \mathbb{Z}).$$

Notice that $(\mathbf{F}, +)$ is an Abelian group. The algebra $\mathbf{F}$ is normed. Indeed, for $x = (x_n : n \in \mathbb{Z}) \in \mathbf{F}$ define

$$|x| := \sum_{n \in \mathbb{Z}} \frac{|x_n|}{3^{n+1}}.$$

It is easy to see that $|x| \geqslant 0$,

$$|x + y| \leqslant |x| + |y|.$$

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A character on $\mathbf{F}$ is a continuous complex-valued map which satisfies $(x, y \in \mathbf{F})$

$$\varphi(x+y) = \varphi(x)\varphi(y) \quad \text{and} \quad |\varphi(x)| = 1.$$

It is evident that $\varphi(0) = 1$. Let e_j denote the element $x = (x_n : n \in \mathbb{Z})$ which satisfies $x_j = 1$ for some $j \in \mathbb{Z}$ and $x_n = 0$ for $n \neq j \,(n \in \mathbb{Z})$. Since φ is continuous on $\mathbf{F}$ and $e_j \to 0$ in $\mathbf{F}$ as $j \to \infty$, we have

$$\varphi(e_j) \to \varphi(0) = 1$$

as $j \to \infty$. On the other hand,$e_j + e_j + e_j = 0$. Hence

$$1 = \varphi(0) = \varphi(e_j + e_j + e_j) = \varphi^3(e_j),$$
$$\varphi(e_j) = \sqrt[3]{1} = e^{\frac{2\pi i k}{3}}, k = -1, 0, 1.$$

Consequently, there is a sequence $y = \{y_j : j \in \mathbb{Z}, y_j = -1, 0, 1\}$ such that for every $-j-1 > s(y)$ and

$$\varphi_y(e_j) = e^{\frac{2}{3}\pi i y_{-j-1}}.$$

It is easy to show that

$$\varphi(x_j e_j) = (\varphi(e_j))^{x_j}.$$

Then from the continuity of φ we can write

$$\varphi_y(x) = \prod_{j\in\mathbb{Z}} (\varphi(e_j))^{x_j} = e^{\frac{2}{3}\pi i \sum\limits_{j\in\mathbb{Z}} y_{-j-1}x_j.} \qquad (x, y \in \mathbf{F}).$$

The functions $\varphi_y\,(y \in \mathbf{F})$ exhaut the character of the additive group $(\mathbf{F}, +)$.

2. The Walsh-Fourier transform in L_1

For a given $f \in L_1(\mathbf{F})$ the Walsh-Fourier transform of f is the function on $\mathbf{F}$ will be defined by

$$\widehat{f}(y) := \frac{1}{9}\int\limits_{\mathbf{F}} f(x)\,\overline{\varphi}_y(x)\,d\mu(x) \qquad (y \in \mathbf{F}).$$

It is quite well-known that a clasical Walsh function maps from the unit interval to the set $\{-1, 1\}$ and also that it can be extended to the set of nonnegative real numbers (see e.g. [6], [4]). But cannot to the whole set of reals. Besides, the same situation hold for the Vilenkin functions (see e.g. [1]). Therefore, in order to involve the real line we must do something different. Basically the "problem" with the Walsh functions is that we can "stay" or "step right". We should be "able to step left" also. Next, we introduce Walsh-like functions on the real line as follows. It is easy to prove that every real number $y \in \mathbb{R}$ can be expressed by the following sum.

$$y = \sum_{k=-\infty}^{+\infty} \frac{y_k}{3^k},$$

where $y_k \in \{-1, 0, 1\}$ for all $k \in \mathbb{N}$. The digits $-1, 0, 1$ mean that we're going to the left, or getting nowhere, or we're going to the right by $1/3^k$. There is no convergence problems, since y is a finite real, and consequently, $y_k = 0$ for k's small enough ($\lim_{k\to-\infty} y_k = 0$). We can identify y be the sequence $(y_k, k \in \mathbb{Z})$. Unfortunately, this identification is not always a bijection, since for all $j \in \mathbb{N}$ the numbers

$$\begin{aligned}\frac{6j+1}{2\cdot 3^{n+1}} &= \frac{j}{3^n} + \frac{1}{2\cdot 3^{n+1}} = \frac{j}{3^n} + \frac{1}{3^{n+1}} + \sum_{k=n+2}^{+\infty} \frac{-1}{3^k} \\ &= \frac{j}{3^n} + \frac{0}{3^{n+1}} + \sum_{k=n+2}^{+\infty} \frac{1}{3^k}\end{aligned}$$

have two corresponding $-1, 0, 1$ sequences. The set of these numbers is called the set of triadic rationals. In this situation we choose the one terminates in -1's. Anyhow, the set of these reals is countable ($j, n \in \mathbb{N}$), and consequently of minor importance.

Define the addition $\oplus : \{-1, 0, 1\}^2 \to \{-1, 0, 1\}$ as the mod 3 addition. (E.g. $1 \oplus 1 = -1, (-1) \oplus (-1) = 1$.) Then define the addition $\oplus$ on $\mathbb{R}$ as $x \oplus y := (x_k \oplus y_k, k \in \mathbb{Z})$. The inverse operation is denoted by $\ominus$

Introduce the set of triadic intervals on $\mathbb{R}$. Let $n \in \mathbb{Z}$, and $t \in \mathbb{R}$. Then the set

$$I_n(t) := \{y \in \mathbb{R} : y_i = t_i \text{ for } i \leqslant n\}$$

is called a (triadic) interval. We also use the notation $I_{-\infty}(t) = \mathbb{R}$. It is easy to have

$$I_n(t) = \left[t_{(n)} - \frac{1}{2\cdot 3^n}, t_{(n)} + \frac{1}{2\cdot 3^n}\right) := \left[\frac{2k-1}{2\cdot 3^n}, \frac{2k+1}{2\cdot 3^n}\right),$$

where

$$t_{(n)} = \sum_{l=-\infty}^{n} t_l/3^l$$

and

$$k = \sum_{l=-\infty}^{n} t_l 3^{n-l}. \tag{2.1}$$

The Lebesgue measure of an interval: $\operatorname{mes}(I_n(t)) = 3^{-n}$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Introduce the generalized Walsh function in the following way. Let $x, y \in \mathbb{R}$, and

$$\omega(x, y) := r^{\sum_{k=-\infty}^{\infty} x_k y_{-k-1}}, \quad r = \exp\left(2\pi\imath\frac{1}{3}\right), \imath = \sqrt{-1}.$$

Some properties of the Walsh function:

$$|\omega| = 1, \quad \omega(x, y) = \omega(y, x), \quad \omega(x \oplus z, y) = \omega(x, y)\omega(z, y)$$

for $x, y, z \in \mathbb{R}$, and $x \oplus z$ is triadic irrational.

The Walsh-Fourier transform of an $f \in L_1(\mathbb{R})$ is defined by

$$\widehat{f}(y) := \frac{1}{9} \int_{\mathbb{R}} f(x) \overline{\omega}(x, y)\, dx \quad (y \in \mathbb{R}).$$

3. Inversion of the Walsh-Fourier transform

For each $f \in L_1(\mathbb{R})$ and $t > 0$ define the Walsh-Dirichlet integral by

$$S_t(x; f) := \int_{-t}^{t} \widehat{f}(y) \omega(x, y)\, dy.$$

By Fubin's theorem it is evident that

$$\begin{aligned} S_t(x; f) &= \int_{-t}^{t} \frac{1}{9} \left(\int_{\mathbb{R}} f(u) \overline{\omega}(u, y)\, du \right) \omega(x, y)\, dy \\ &= \frac{1}{9} \int_{\mathbb{R}} \left(\int_{-t}^{t} \overline{\omega}(u, y) \omega(x, y)\, dy \right) f(u)\, du = \frac{1}{9} \int_{\mathbb{R}} \left(\int_{-t}^{t} \omega(u \ominus x, y)\, dy \right) f(u)\, du \\ &= \frac{1}{9} \int_{\mathbb{R}} f(u) D_t(u \ominus x)\, du, \end{aligned}$$

where

$$D_t(x) = \int_{-t}^{t} \omega(x, y)\, dy$$

for $t \in \mathbb{R}_+$ and $f \in L_1(\mathbb{R})$.

Theorem 3.1. *Let $N \in \mathbb{Z}$. Then*

$$D_{\frac{3^N}{2}}(x) = 3^N \mathbb{I}_{I_{N-2}(0)}(x),$$

where $\mathbb{I}_E(x)$ is the characteristic function of the set E.

Proof. Let $x \in I_{N-2}(0)$ and $y \in \left[-\frac{3^N}{2}, \frac{3^N}{2}\right)$. Then $x_k = 0, k \leqslant N-2$ and $y_k = 0$, $k \leqslant -N$. Hence,

$$\omega(x, y) = e^{\frac{2}{3}\pi i \sum_{k=-\infty}^{\infty} x_k y_{-k-1}} = 1. \tag{3.1}$$

Now, we suppose that $x \notin I_{N-2}(0)$. Then there exists $l \in \mathbb{Z}$ such that $l \leqslant N-2$ and $x_l \in \{-1,1\}$. We can write

$$\begin{aligned}
& D_{\frac{3^N}{2}}(x) \\
= & \int_{-\frac{3^N}{2}}^{\frac{3^N}{2}} \omega(x,y)\,dy \\
= & \sum_{\substack{y_m\in\{-1,0,1\},\\ m\in\{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},-1)} e^{-\frac{2}{3}\pi i x_l} e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy \\
& + \sum_{\substack{y_m\in\{-1,0,1\},\\ m\in\{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},0)} e^{\frac{2}{3}\pi i x_l\cdot 0} e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy \\
& + \sum_{\substack{y_m\in\{-1,0,1\},\\ m\in\{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},1)} e^{\frac{2}{3}\pi i x_l} e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy \\
= & \sum_{\substack{y_m\in\{-1,0,1\},\\ m\in\{-N+1,\dots,-l-2\}}} \int_{I_{-l-1}(0,\dots,0,y_{-N+1},\dots,y_{-l-2},-1)} \left(e^{-\frac{2}{3}\pi i x_l} + e^{\frac{2}{3}\pi i x_l\cdot 0} + e^{\frac{2}{3}\pi i x_l}\right) \\
& \times e^{\frac{2}{3}\pi i \sum_{k=-\infty,k\neq l}^{N-2} x_k y_{-k-1}} dy
\end{aligned}$$

Since

$$e^{-\frac{2}{3}\pi i x_l} + e^{\frac{2}{3}\pi i x_l 0} + e^{\frac{2}{3}\pi i x_l} = 0, x_l \in \{-1,1\}$$

we obtain that

$$D_{\frac{3^N}{2}}(x) = 0, x \neq I_{N-2}(0)\,. \tag{3.2}$$

Combining (3.1) and (3.2) we complete the proof of Theorem 3.1. □

Now, we prove some inversion result for the Walsh-Fourier transform.

Theorem 3.2. *Let $f \in L_1(\mathbb{R})$ be W-continuous on $\mathbb{R}$. If $\widehat{f} \in L_1(\mathbb{R})$ then*

$$f(y) = \int_{\mathbb{R}} \widehat{f}(x)\,\omega(x,y)\,dx.$$

Proof. We can write

$$\begin{aligned}
& \int_{\mathbb{R}} \widehat{f}(x)\,\omega(x,y)\,dx \\
= & \int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx + \int_{\mathbb{R}\backslash\left[-\frac{3^n}{2},\frac{3^n}{2}\right]} \widehat{f}(x)\,\omega(x,y)\,dx := I + II.
\end{aligned} \tag{3.3}$$

Since $\widehat{f}$ is integrable for II we get

$$|II| \leqslant \int\limits_{R\backslash\left[-\frac{3^n}{2},\frac{3^n}{2}\right]} \left|\widehat{f}(x)\right| dx \to 0 \quad (n \to \infty) . \tag{3.4}$$

We can write

$$\int\limits_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx - f(y) = \frac{1}{9}\int\limits_{\mathbb{R}} f(y \oplus u)\, D_{\frac{3^n}{2}}(u)\,du - f(y)$$

$$= 3^{n-2} \int\limits_{I_{n-2}(0)} \left[f(y \oplus u) - f(y)\right] du.$$

Hence,

$$\left|\int\limits_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx - f(y)\right| \leqslant 3^{n-2} \int\limits_{I_{n-2}(0)} \left|f(y \oplus u) - f(y)\right| du.$$

Lrt $\varepsilon > 0$, fix $y \in \mathbb{R}$, and choose an integer $n > 0$ such that

$$\left|f(y \ominus u) - f(y)\right| < \varepsilon$$

for all $y \in \mathbb{R}$ which satisfy $u \in I_{n-2}(0)$. Then we obtain

$$\left|\int\limits_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \widehat{f}(x)\,\omega(x,y)\,dx - f(y)\right| < \varepsilon. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5) we complete the proof of Theorem 3.2. □

4. Generalized Walsh function

Theorem 4.1. *Let $n \in \mathbb{Z}$. Then the system*

$$\left\{3^{n/2}\omega\left(x, 3^{n+2}j\right), j \in \mathbb{Z}\right\}$$

is orthonormal and complete in $L_2(I_n(0))$.

Proof. Proof of the orthonormality. For the sake of brevity we prove Theorem 4.1 for $L_2(I_{-n}(0))$ instead of $L_2(I_n(0))$. That is, we discuss the system $\{3^{-n/2}\omega\left(x, 3^{-n+2}j\right), j \in \mathbb{Z}\}$. Recall that $I_{-n}(0) = [-3^n/2, 3^n/2)$. Since to see the normality it is trivial, then we can suppose that $j \neq k$, $j,k \in \mathbb{Z}$. We are to prove

$$\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \omega(x, 3^{-n+2}j)\bar{\omega}(x, 3^{-n+2}k)dx = 0. \tag{4.1}$$

We can write

$$\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \omega(x,3^{-n+2}j)\bar{\omega}(x,3^{-n+2}k)dx = \int_{-\frac{3^n}{2}}^{\frac{3^n}{2}} \omega(x,3^{-n+2}\left(j\ominus k\right))dx$$
$$= \quad D_{\frac{3^n}{2}}\left(3^{-n+2}\left(j\ominus k\right)\right).$$

It is easy to see that$3^{-n+2}\left(j\ominus k\right)\notin I_{n-2}\left(0\right) \quad (j\neq k)$. Then, from Theorem 3.1 we prove (4.1). The proof of the orthonormality is complete. Completeness is discused later. □

Define the Dirichlet kernel functions with respect to the system $(3^{n/2}\omega(x,3^{n+1}j)$, $j\in\mathbb{Z})$ as:

$$\mathbf{D}_N(x) := \sum_{\{j\in\mathbb{Z}:|j|<N\}} 3^n\omega(x,3^{n+2}j) \quad N\in\mathbb{N}, n\in\mathbb{Z}, x\in\mathbb{R}.$$

We prove a formula for the Dirichlet kernels $\mathbf{D}_{\frac{3^N+1}{2}}$.

Theorem 4.2. *Let $x\in I_n\left(0\right)$ and $N\in\mathbb{N}$. Then*

$$\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right) = \begin{cases} 3^{n+N}, & \text{if } x\in I_{n+N}\left(0\right) \\ 0, & \text{if } x\notin I_{n+N}\left(0\right) \end{cases}.$$

Proof. Integrate the function $|\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2$ on the interval $I_n\left(0\right)$. By the help of Theorem 4.1, that is, by orthonormality we have:

$$\int_{I_n(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = \int_{I_n(0)} \left|\sum_{\left\{j\in\mathbb{Z}:|j|<\frac{3^N+1}{2}\right\}} 3^n\omega(x,3^{n+2}j)\right|^2 dx$$
$$= \sum_{\left\{j,k\in\mathbb{Z}:|k|,|j|<\frac{3^N+1}{2}\right\}} 3^{2n}\int_{I_n(0)}\omega(x,3^{n+2}j)\bar{\omega}(x,3^{n+2}k)dx$$
$$= \sum_{\left\{j\in\mathbb{Z}:|j|<\frac{3^N+1}{2}\right\}} 3^{2n}\int_{I_n(0)} 1dx = 3^{n+N}.$$

It is easy to see that $\omega(x,3^{n+2}j)=1, x\in I_{n+N}\left(0\right), |j|<\frac{3^N+1}{2}$. On the other hand, this also gives:

$$3^{n+N} = \int_{I_n(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = \int_{I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx$$
$$+\int_{I_n(0)\setminus I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = 3^{n+N} + \int_{I_n(0)\setminus I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx.$$

This means:

$$\int_{I_n(0)\setminus I_{n+N}(0)} |\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)|^2dx = 0.$$

Consequently, the function $\mathbf{D}_{\frac{3^N+1}{2}}\left(x\right)$ equals with zero for almost every x on the set $I_n\left(0\right)\setminus I_{n+N}\left(0\right)$. Since the Walsh-like function $\omega(x,3^{n+2}j)$ is continuous, then so does the Dirichlet kernel $\mathbf{D}_{\frac{3^N+1}{2}}(x)$. That is, this function is the constant zero function on the set $I_n(0) \setminus I_{n+N}(0)$. $\square$

Define the Fourier coefficients of the integrable function $f : I_n(0) \to \mathbb{C}$ as

$$\hat{f}(j) := \int_{I_n(0)} f(x) 3^{n/2} \bar{\omega}(x, 3^{n+2}j) dx,$$

where $j \in \mathbb{Z}$. It is easy to have for the partial sums of the Fourier series

$$\begin{aligned} S_N f(y) \quad : &= \sum_{\{j \in \mathbb{Z}: |j| < N\}} \hat{f}(j) 2^{n/2} \omega(y, 3^{n+2}j) \\ &= \int_{I_n(0)} f(x) \mathbf{D}_N(y \ominus x) dx. \end{aligned}$$

Consequently,

$$S_{\frac{3^N+1}{2}} f(y) = 3^{n+N} \int_{I_{n+N}(y)} f(x) dx.$$

By this equality in the standard way one can prove that the system $(\omega(x, 3^{n+2}j), j \in \mathbb{Z})$ is complete in the Banach space of the integrable functions on the interval $I_n(0)$. It is also of interest, that the Dirichlet kernel are integer valued functions. The reason of this fact is that if $\omega(x, 3^{n+2}j)$ occurs as an addend in the kernel function, then so does its conjugate $\omega(x, -3^{n+2}j)$. The sum of these two things is 2 $(1+1=2)$ or -1 $(r + \bar{r} = -1)$.

In the sequel we discuss the uniform convergence of these partial sums of the Fourier series of continuous functions. Denote by the (triadic) modulus of continuity of the function $f : I_n(0) \to \mathbb{C}$ by

$$w(I_n(0), N, f) := \sup_{h \in I_{n+N}, x \in I_n(0)} |f(x) - f(x \oplus h)| \quad (N \in \mathbb{N}).$$

If it does not cause any misunderstood, then we write $w(N, f)$ simply. This is a monotone decreasing nonnegative sequence. It is not difficult to prove, that a function f on $I_n(0)$ is continuous if and only if it modulus of continuity converges to zero. We have

$$\begin{aligned} &\left| S_{\frac{3^N+1}{2}} f(y) - f(y) \right| \leqslant 3^{n+N} \int_{I_{n+N}(y)} |f(x) - f(y)| \, dx \\ \leqslant &\left| 3^{n+N} \int_{I_{n+N}(0)} f(y \oplus h) - f(y) dh \right| \leqslant w(n+N, f). \end{aligned}$$

Remark 4.1. It seems also to be very interesting to discuss some other materials with respect to this system, and harmonic analysis. Dirichlet kernels $\mathbf{D}_N$, the norm (and pointwise) convergence of the partial sums S_N, the Fejér kernels and means. We suppose that there are many similarities with the ordinary Walsh system, since the function $\omega(x, 2^{n+1}j) + \omega(x, -2^{n+1}j)$ can take the values $+2$ and -1.

5. The Walsh-Fourier transform in $L_p\,(1<p\leqslant 2)$

It this section we obtain that in cases when $f\in L_p(\mathbb{R})\,(1<p\leqslant 2)$ the Walsh-Fourier transform is as a limit of truncated Walsh-Fourier transforms.

Theorem 5.1. *Let $f\in L_p(\mathbb{R})\,(1<p\leqslant 2)$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\widehat{f}(x,a):=\int_{-a}^{a} f(y)\,\omega(x,y)\,dy$$

converges in L_p norm. Moreover,

$$\int_{\mathbb{R}}\left|\int_{\mathbb{R}} f(y)\,\omega(x,y)\,dy\right|^{q}dx\leqslant 9\left(\int_{\mathbb{R}}|f(x)|^{p}\,dx\right)^{\frac{1}{p-1}}.$$

Proof. Set

$$\Delta_k^{(n)}:=\left[\frac{2k-1}{2\cdot 3^{n}},\frac{2k+1}{2\cdot 3^{n}}\right),\quad \alpha_k:=\int_{\Delta_k^{(n-2)}} f(u)\,du,$$

$$\Phi_m(x):=\sum_{|k|<m}\alpha_k\omega\left(x,3^{-n+2}k\right),$$

$$\Phi(x):=3^{-n/2}\Phi_m(x)=\sum_{|k|<m}\alpha_k 3^{-n/2}\omega\left(x,3^{-n+2}k\right),$$

where

$$m:=\left[a\cdot 3^{n-2}\right],a>0.$$

Applying Riesz's inequality [5] we can write

$$\left(\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}}|\Phi(x)|^{q}\,dx\right)^{1/q}\leqslant M^{\frac{2}{p}-1}\left(\sum_{|k|<m}|\alpha_k|^{p}\right)^{1/p},$$

where $M:=3^{-n/2}$. Hence,

$$\int_{-\frac{3^n}{2}}^{\frac{3^n}{2}}|\Phi_m(x)|^{q}\,dx\quad\leqslant\quad 9\left(\sum_{|k|<m}\int_{\Delta_k^{(n-2)}}|f(u)|^{p}\,du\right)^{\frac{1}{p-1}}\leqslant 9\left(\int_{-a}^{a}|f(u)|^{p}\,du\right)^{\frac{1}{p-1}}.$$

For any fixed $A<\frac{3^n}{2}$ we obtain

$$\int_{-A}^{A}|\Phi_m(x)|^{q}\,dx\leqslant 9\left(\int_{-a}^{a}|f(u)|^{p}\,du\right)^{\frac{1}{p-1}}. \tag{5.1}$$

It is easy to see that

$$\omega(x,y)=\omega\left(x,3^{-n+2}k\right)$$

if $x \in \left[-\frac{3^n}{2}, \frac{3^n}{2}\right) = I_{-n}(0)$ and $y \in \Delta_k^{(n-2)} := I_{n-2}(t)$. Indeed,

$$3^{-n+2}k = \sum_{l=-\infty}^{n-2} \frac{t_l}{3^l}, x = \sum_{k=-n+1}^{\infty} \frac{x_k}{3^k}.$$

Then

$$\begin{aligned} \omega\left(x, 3^{-n+2}k\right) &= e^{\frac{2}{3}\pi i(x_{-n+1}t_{n-2}+x_{-n+2}t_{n-3}+\cdots)} \\ &= e^{\frac{2}{3}\pi i(x_{-n+1}y_{n-2}+x_{-n+2}y_{n-3}+\cdots)} = \omega(x,y). \end{aligned}$$

Hence

$$\Phi_m(x) = \sum_{|k|<m} \left(\int_{\Delta_k^{(n-2)}} f(u)\, du \right) \omega\left(x, 3^{-n+2}k\right)$$

$$= \sum_{|k|<m} \int_{\Delta_k^{(n-2)}} f(u)\,\omega(x,u)\, du = \int_{(-2m-3)/(2\cdot 3^{n-2})}^{(2m-1)/\left(2\cdot 3^{n-2}\right)} f(u)\,\omega(x,u)\, du,$$

$$\left| \Phi_m(x) - \int_{-a}^{a} f(u)\,\omega(x,u)\, du \right|$$

$$\leqslant \int_{-a}^{(-2m-3)/\left(2\cdot 3^{n-2}\right)} |f(u)|\, du + \int_{(2m-1)/(2\cdot 3^{n-2})}^{a} |f(u)|\, du \to 0$$

as $n \to \infty$. Then from (5.1) we obtain

$$\int_{-A}^{A} \left| \int_{-a}^{a} f(u)\,\omega(x,u)\, du \right|^q dx \leqslant 9 \left(\int_{-a}^{a} |f(u)|^p\, du \right)^{\frac{1}{p-1}}.$$

Consequently, when $A \to \infty$ we have

$$\int_{-\infty}^{\infty} \left| \int_{-a}^{a} f(u)\,\omega(x,u)\, du \right|^q dx \leqslant 9 \left(\int_{-a}^{a} |f(u)|^p\, du \right)^{\frac{1}{p-1}}. \tag{5.2}$$

Set

$$f(x,a) := f(x)\,\mathbb{I}_{(-\infty,-a]\cup[a,\infty)}(x).$$

For $b > a$ we have

$$\int_{\mathbb{R}} \left| \int_{-b}^{b} f(y,a)\,\omega(x,y)\, dy \right|^q dx \leqslant 9 \left(\int_{-b}^{b} |f(y,a)|^p\, dy \right)^{\frac{1}{p-1}}$$

$$= 9 \left(\int_{[-b,b]\setminus[-a,a]} |f(y)|^p\, dy \right)^{\frac{1}{p-1}} \to 0$$

as $a, b \to \infty$. On the other hand,

$$\begin{aligned}
&\int\limits_{\mathbb{R}} \left| \int\limits_{-b}^{b} f(y,a)\,\omega(x,y)\,dy \right|^q dx \\
= &\int\limits_{\mathbb{R}} \left| \int\limits_{-b}^{b} f(y)\,\omega(x,y)\,dy - \int\limits_{-a}^{a} f(y)\,\omega(x,y)\,dy \right|^q dx \\
= &\int\limits_{\mathbb{R}} \left| \widetilde{f}(x,b) - \widetilde{f}(x,a) \right|^q dx \to 0
\end{aligned}$$

as $a, b \to \infty$. Hence, there exists a function $\widetilde{f} \in L_p(\mathbb{R})$ such that

$$\lim_{a\to\infty} \left\| \widetilde{f}(\cdot,a) - \widetilde{f}(\cdot) \right\|_p = 0. \tag{5.3}$$

Since (see (5.2))

$$\int\limits_{-\infty}^{\infty} \left| \widetilde{f}(x,b) \right|^q dx \leqslant 9 \left(\int\limits_{-b}^{b} |f(u)|^p\,du \right)^{\frac{1}{p-1}}$$

from (5.3) we conclude that

$$\int\limits_{-\infty}^{\infty} \left| \int\limits_{-\infty}^{\infty} f(y)\,\omega(x,y) \right|^q dx \leqslant 9 \left(\int\limits_{-\infty}^{\infty} |f(u)|^p\,du \right)^{\frac{1}{p-1}}.$$

Theorem 5.1 is proved. □

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