ՀԱՅԱՍՏԱՆԻ ԳԻՏՈՒԹՅՈՒՆՆԵՐԻ ԱԶԳԱՅԻՆ ԱԿԱԴԵՄԻԱ
НАЦИОНАЛЬНАЯ АКАДЕМИЯ НАУК АРМЕНИИ
NATIONAL ACADEMY OF SCIENCES OF ARMENIA
ДОКЛАДЫ ԶԵԿՈՒՅՑՆԵՐ REPORTS

Հատոր Том Volume 122 2022 № 1

MATHEMATICS

УДК 621.391.15
DOI: 10.54503/0321-1339-2022.122.1-7

H. K. Sahakyan, Zh. G. Margaryan

Independent Neighbourhoods of Sets in B^n Groups

(Submitted by academician S. K. Shoukourian 21/I 2022)

Keywords: *additive channels, perfect codes, first order neighbourhoods, independent neighbourhoods.*

Introduction. Saying B_2^n we mean the set of all vectors having the members of the set {0,1} with the length *n*. We define the sum of two vectors as the sum of corresponding members by modulo 2. For instance, 0110 + 1010 = 1100. It is obvious that $< B_2^n, + >$ is a group, having 00...0 as its unit, and $a^{-1} = a$. We define the norm of a vector as the sum of its elements.

Definition 1 (a good set). *We call the set* $B \subset B_2^n$ $(|B| \geq 2)$ *a good set for the set A, if for* $\forall\alpha, \beta \in B$ $(\alpha \neq \beta)$ *there* $\alpha + \beta \in A$.

For instance, the set *B* = {001,010,110} is good for *A* = {011,111,010}.

Definition 2 (a bad set). *The set* $C \subset B_2^n (|C| \geq 2)$ *is bad for the set A, if for* $\forall\alpha, \beta \in C$ $(\alpha \neq \beta)$ *there* $\alpha + \beta \notin A$:

For instance, the set *C* = {011,000,111} is bad for the set *A* = {010,101,110}.

Definition 3. *For a given set D, the following set*

$$D^* = \{x + y \mid x, y \in D, x \neq y\} \tag{1}$$

is called a neighbourhood of the first order, and it is denoted by the asterisk.*

Definition 4. *We call the given sets* $B, C \subset B_2^n$ *a good pair and we denote them by* $\langle B, C \rangle$ *if such a set A exists for which B is good and C is bad. And we consider C as a good complement for B.*

Property 1. Using the definition of a neighbourhood, we can define the idea of a good pair in another way:

$$\langle B, C \rangle \Leftrightarrow B^* \subset \overline{C^*}, \tag{2}$$

or in this way, which is the same:

$$\langle B, C \rangle \Leftrightarrow B^* \cap C^* = \emptyset. \tag{3}$$

That is, the neighbourhoods of *B* and *C* are independent. We can take as *A* any set that satisfies the condition $B^* \subset A \subset \overline{C^*}$. Note that a good pair is equivalent to an additive channel [1] (there does not exist $b_1, b_2 \in B$ and $c_1, c_2 \in C$ for which $b_1 + c_1 = b_2 + c_2$).

It is easily can be seen that $B^* = (B+b)^*$, thus we can always assume that $0 \in B$ and $0 \in C$.

Property 2. One can see from (3) that if $\langle B, C\rangle$ is a good pair, then the pair $\langle B', C'\rangle$ is a good pair as well, for any $B' \subset B$ and $C' \subset C$ sets.

Theorem 1. *The following is true for arbitrary sets B, C:*

$$\langle B, C\rangle \Leftrightarrow |B| \cdot |C| = |B + C|, \tag{4}$$

where $B + C = \{b + c | \; b \in B, c \in C\}$.

Definition 5. *We call the pair $\langle B, C\rangle$ completely good pair if $|B|\cdot|C|=2^n$. Respectively, we call C a completely good complement.*

Examples can be constructed by viewing Hamming codes [2] and Golay code [3] as a codes in additive channel. There are also examples not related to perfect codes:

$$\begin{aligned} B = \{&000000, 000001, 000010, 000011,\\ &000100, 000101, 000110, 000111,\\ &001000, 001001, 001010, 001011,\\ &010000, 010010, 100000, 100010\}\\ C = \{&000000, 101100, 110001, 011111\}\end{aligned}$$

$$|B||C| = 2^6.$$

Definition 6. *We call $C' \subset L(B)$ a partially good complement for B if $\langle B, C'\rangle$ is a good pair and $|B| \cdot |C'| = 2^r$, where r is the rank of B and L(B) is the linear span of B.*

It turns out that the problem of finding a partially good complement is equivalent to the problem of finding a completely good complement.

Theorem 2. *Let the set B ($|B| = 2^m$) be given; to have a completely good complement for B, it is necessary and sufficient to have a partially good complement for B.*

When constructing a good pair with given *B*, it is easily seen that number of existing C sets is dependent on the number of zero-sum subsets in set *B*, e.g., when $n = 3$ and $B = \{a, b, c, d\}$ if $a + b + c + d \neq 0$ then there is only one possible *C*, $C = \{0, a + b + c + d\}$. But if $a + b + c + d = 0$, then there are four possible *C* sets.

Definition 7. *We call $B_0 = \{x_1, x_2, ..., x_k\}$ a subset with zero sum for the set $B \subset B_2^n$ if:*

$$0 \notin B_0, \tag{5}$$

$$x_1 + x_2 + \ldots + x_k = 0.$$

Definition 8. *We denote the number of the zero-sum subsets of the given set $B \subset B_2^n$ by $t^B{}_k$.*

Obviously, t_k^B makes sense only if *k* is less than $|B|$. We take $t_0^B = 1$.

Property 3. One can easily see that $t_1^B = t_2^B = 0$.

Let us consider the following case: $B = B_2^n (n \geq 2)$. We will write t_k instead of $t_k^{B_2^n}$ for simplicity. The following equation is valid for $k > 2$:

$$t_k = \frac{C_{2^n-1}^{k-1} - (2^n - 1 - (k-2))t_{k-2} - t_{k-1}}{k} \quad . \tag{6}$$

Using (6), we can find all the t_k, because we know t_1 and t_2 (Property 3):

$$t_0 + t_1 + t_2 + t_3 + \cdots + t_{2^n-1} = 2^{2^n-1-n}. \tag{7}$$

It turns out that (7) is valid for the general case.

Theorem 3. *For any set $B \subset B_2^n$, the sum of the numbers t_k^B is*

$$t_0^B + t_1^B + \ldots + t_{|B|}^B = 2^{|B \setminus \{0\}| - \mathrm{rank}(B)} \quad \cdot \tag{8}$$

Yerevan State University

e-mails: hovhannes1417@gmail.com,

jiromr@mail.ru

H. K. Sahakyan, Zh. G. Margaryan

Independent Neighbourhoods of Sets in B^n Groups

The paper considers the structural problems and cardinality problems of pairs of non-empty subsets with certain restrictions with respect to the B_2^n space where the sum is by modulo 2. The described subsets are related, in many cases, to the construction of error-correcting codes on additive communication channels, and to the quantitative bounds of their cardinality.

Հ. Կ. Սահակյան, Ժ. Գ. Մարգարյան

Բազմությունների անկախ շրջակայքեր B^n խմբում

Դիտարկված են B_2^n տարածության ըստ երկու մոդուլի գումարման նկատմամբ, որոշակի սահմանափակումներով, ոչ դատարկ ենթաբազմությունների զույգերի հզորությունների և կառուցվածքային խնդիրներ։ Նկարագրված ենթաբազմությունները առնչվում և շատ դեպքերում օգնում են ադիտիվ կապի գծերում սխալների ուղղող կոդերի կառուցման և նրանց հզորությունների քանակական գնահատման խնդիրնե-րին։

О. К. Саакян, Ж. Г. Маргарян

Независимые окрестности множеств в группе B^n

Рассмотрены структурные проблемы и проблемы мощности пар непустых подмножеств с некоторыми ограничениями относительно пространства B_2^n, где сумма по модулю два. Описанные подмножества связаны во многих случаях с построением кодов исправления ошибок на аддитивных линиях связи и с количественной оценкой их мощности.

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