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# FEKETE-SZEGÖ INEQUALITIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS RELATED WITH NEPHROID DOMAIN

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Abstract. The purpose of this paper is to consider coefficient estimates in a class of functions  $\mathcal{M}_{\alpha,\lambda}(q)$  consisting of analytic functions f normalized by f(0) = f'(0) - 1 = 0 in the open unit disk  $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  subordinating with nephroid domain, to derive certain coefficient estimates  $a_2, a_3$  and Fekete-Szegö inequality for  $f \in \mathcal{M}_{\alpha,\lambda}(q)$ . A similar result have been done for the function  $f^{-1}$ . Further application of our results to certain functions defined by convolution products with a normalized analytic function is given, and in particular we obtain Fekete-Szegö inequalities for certain subclasses of functions defined through neutrosophic Poisson distribution.

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**Keywords:** analytic function; starlike function; convex function; subordination; Fekete-Szegö inequality; neutrosophic Poisson distribution; Hadamard product.

### 1. INTRODUCTION

Let  $\mathscr{A}$  denote the class of all functions f(z) of the form

(1.1) 
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$\Delta := \{ z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1 \}$$

and  $\mathscr{S}$  be the subclass of  $\mathscr{A}$  consisting of univalent functions. A function  $f \in \mathscr{S}$  is said to be *starlike* in  $\Delta$  if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad (z \in \Delta)$$

and on the other hand, a function  $f\in \mathscr{S}$  is said to be  $\mathit{convex}$  in  $\Delta$  if and only if

$$\Re\Big(1+\frac{zf''(z)}{f'(z)}\Big)>0,\quad (z\in\Delta)$$

denoted by  $\mathscr{S}^*$  and  $\mathscr{C}$  respectively.

Let  $f_1$  and  $f_2$  be functions analytic in  $\Delta$ . Then we say that the function  $f_1$  is subordinate to  $f_2$  if there exists a Schwarz function w(z), analytic in  $\Delta$  with

w(0) = 0 and |w(z)| < 1  $(z \in \Delta)$ , such that  $f_1(z) = f_2(w(z))$   $(z \in \Delta)$ . We denote this subordination by

$$f_1 \prec f_2$$
 or  $f_1(z) \prec f_2(z)$   $(z \in \Delta)$ .

In particular, if the function  $f_2$  is univalent in  $\Delta$ , the above subordination is equivalent to  $f_1(0) = f_2(0)$  and  $f(\Delta) \subset f_2(\Delta)$ . The function  $q(z) = 1 + z - \frac{z^3}{3}$ maps  $\Delta$  onto the region bounded by the nephroid

$$\left((u-1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} = 0,$$

which is symmetric about the real axis and lies completely inside the right-half plane u > 0. Geometrically, a nephroid is the locus of a point on the circumference of a circle of radius  $\rho$  traversing positively the outside of a fixed circle of radius  $2\rho$ . It is an algebraic curve of degree six and is an epicycloid having two cusps. The plane curve nephroid was studied by Huygens and Tschirnhausen around 1679 in connection with the theory of caustics, a method of deriving a new curve based on a given curve and a point. In 1692, J. Bernoulli showed that the nephroid is the catacaustic (envelope of rays emanating from a specified point) of a cardioid for a luminous cusp. However, the name nephroid, which means kidney shaped, was first used by the English mathematician Richard A. Proctor in 1878 in his book "The Geometry of Cycloids". (For more details see [20] and references cited therein)

**Definition 1.1.** [20] Let  $\mathscr{S}^*(q)$  denote the class of analytic functions f in the unit disc  $\Delta$  normalized by f(0) = f'(0) - 1 = 0 and satisfying the condition that

(1.2) 
$$\frac{zf'(z)}{f(z)} \prec 1 + z - \frac{z^3}{3} =: q(z), \ z \in \Delta.$$

and  $\mathscr{C}(q)$  if

(1.3) 
$$\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + z - \frac{z^3}{3} =: q(z), \ z \in \Delta.$$

Further they proved by considering, q(z) as a holomorphic solution of the differential equation

$$\frac{zq'(z)}{q(z)} = 1 + z - \frac{z^3}{3}, \quad z \in \Delta, \ q(0) = 0, \ q'(0) = 1,$$

i.e.

(1.4) 
$$\Omega_n(z) = z \exp\left(\int_0^z \frac{q(\zeta^{n-1}) - 1}{\zeta} d\zeta\right) = z + \frac{z^n}{n-1} + \frac{z^{2n-1}}{2(n-1)^2} + \cdots, \quad z \in \Delta$$

plays the extremal role of the class  $\mathscr{S}_q^*$  as noted by Wani and Swaminathan [20]. Also

(1.5) 
$$\Upsilon_n(z) = \exp\left(\int_0^z \frac{q(\zeta^{n-1}) - 1}{\zeta} d\zeta\right) = z + \frac{z^n}{n(n-1)} + \frac{z^{2n-1}}{2(2n-1)(n-1)^2} + \cdots,$$
  
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 $z \in \Delta$ , plays the extremal role of the class  $\mathscr{C}_q$  as noted by Wani and Swaminathan [20]. It may be noted from (1.3) of Definition1.1 that the set  $q(\Delta)$  lies in the right half-plane and it is not a starlike domain with respect to the origin.

Recently, Raina and Sokol [15] have studied and obtained some coefficient inequalities for the class  $\mathscr{S}^*(z + \sqrt{1+z^2})$  and these results are further improved by Sokol and Thomas [19] further the Fekete-Szegö inequality for functions in the class  $\mathscr{C}(q)$  were obtained and in view of the Alexander result between the class  $\mathscr{S}^*(z + \sqrt{1+z^2})$ and  $\mathscr{C}(z + \sqrt{1+z^2})$ , the Fekete-Szegö inequality for functions in  $\mathscr{S}^*(z + \sqrt{1+z^2})$ were also obtained. For a brief history of Fekete-Szegö problem for the class of starlike, convex and various other subclasses of analytic functions, we refer the interested reader to [18]. Let  $\alpha \ge 0$ ,  $\lambda \ge 0$  and  $0 \le \rho < 1$  and  $f \in \mathscr{A}$ . We say that  $f \in M(\alpha, \lambda, \rho)$  if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} > \rho.$$

The class  $M(\alpha, \lambda, \rho)$  was introduced by Guo and Liu [4].

Motivated essentially by the aforementioned works, (see [15, 17] and [1]) in this paper we define the following class  $\mathscr{M}_{\alpha,\lambda}(q)$  given in Definition1.2. First,we shall find estimations of first few coefficients of functions f of the form (1.1) belonging to  $\mathscr{M}_{\alpha,\lambda}(q)$  and we prove the Fekete-Szegö inequality  $f \in \mathscr{M}_{\alpha,\lambda}(q)$  and also for  $f^{-1} \in \mathscr{M}_{\alpha,\lambda}(q)$ . Also we give applications of our results to certain functions defined through Poisson distribution .

Now, we define the following class  $\mathcal{M}_{\alpha,\lambda}(q)$ :

**Definition 1.2.** For  $\alpha \geq 0$ ,  $\lambda \geq 0$  a function  $f \in \mathscr{A}$  is in the class  $\mathscr{M}_{\alpha,\lambda}(q)$  if

$$\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\}$$

$$(1.6) \quad \prec \quad 1 + z - \frac{z^3}{3} = q(z); \ z = re^{i\theta} \in \Delta.$$

Note that by specializing the parameter we get the following subclasses based on nephroid domain (see [20]).

• 
$$\mathcal{M}_{0,0}(q) \equiv \mathscr{S}^*(q) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta \right\}$$
  
•  $\mathcal{M}_{0,1}(q) \equiv \mathscr{C}(q) = \left\{ f \in \mathcal{A} : \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta \right\}$   
•  $\mathcal{M}_{0,\lambda}(q) \equiv \mathcal{M}_{\lambda}(q)$   
 $= \left\{ f \in \mathcal{A} : (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta \right\}$   
•  $\mathcal{M}_{\alpha,0}(q) \equiv \mathscr{B}^{\alpha}(q) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\alpha} \prec q(z) = 1 + z - \frac{z^3}{3}, z \in \Delta \right\}$ .

## 2. A Coefficient Estimate

To prove our main result, we need the following:

**Lemma 2.1.** [8] If  $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$  is a function with positive real part in  $\Delta$ , then

$$|c_2 - vc_1^2| \le \begin{cases} -4v + 2, & \text{if } v \le 0, \\ 2, & \text{if } 0 \le v \le 1, \\ 4v - 2, & \text{if } v \ge 1. \end{cases}$$

When v < 0 or v > 1, the equality holds if and only if  $p_1(z)$  is  $\frac{1+z}{1-z}$  or one of its rotations. If 0 < v < 1, then equality holds if and only if  $p_1(z)$  is  $\frac{1+z^2}{1-z^2}$  or one of its rotations. If v = 0, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \le \eta \le 1)$$

or one of its rotations. If v = 1, the equality holds if and only if  $p_1$  is the reciprocal of one of the functions such that the equality holds in the case of v = 0.

Although the above upper bound is sharp, when 0 < v < 1, it can be improved as follows:

$$|c_2 - vc_1^2| + v|c_1|^2 \le 2 \quad (0 < v \le 1/2)$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \le 2 \quad (1/2 < v \le 1).$$

We also need the following:

**Lemma 2.2.** [3] If  $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$  is a function with positive real part in  $\Delta$ , then

$$|c_n| \le 2$$
 for all  $n \ge 1$  and  $|c_2 - \frac{c_1^2}{2}| \le 2 - \frac{|c_1|^2}{2}$ .

The class of all such functions with positive real part are denoted by  $\mathscr{P}$ .

**Lemma 2.3.** [7] If  $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$  is a function with positive real part in  $\Delta$ , then

$$|c_2 - vc_1^2| \le 2\max(1, |2v - 1|).$$

The result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

**Lemma 2.4.** [6] Let  $P(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$  be in  $\mathcal{P}$  then for any complex number  $\mu$ ,

$$\left|c_{2}-\mu \frac{c_{1}^{2}}{2}\right| \leq \max\{2, 2|\mu-1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2;\\ 2|\mu-1|, & elsewhere. \end{cases}$$

The result is sharp for the functions defined by  $P(z) = \frac{1+z^2}{1-z^2}$  or  $P(z) = \frac{1+z}{1-z}$ .

**Theorem 2.1.** Let  $\alpha \geq 0$  and  $\lambda \geq 0$ . If f(z) given by (1.1) belongs to  $\mathcal{M}_{\alpha,\lambda}(q)$ , then

$$|a_{2}| \leq \frac{1}{(1+\alpha)(1+\lambda)},$$
  

$$|a_{3}| \leq \frac{1}{(\alpha+2)(1+2\lambda)} \max\{1, \left| \left( \frac{\alpha^{2}+\alpha-2(\alpha+3)\lambda-2}{2((1+\alpha)(1+\lambda))^{2}} \right) \right| \}$$

**Proof.** If  $f \in \mathscr{M}_{\alpha,\lambda}(q)$ , then there is a Schwarz function w(z), analytic in  $\Delta$  with w(0) = 0 and |w(z)| < 1 in  $\Delta$  such that

(2.1) 
$$\begin{cases} \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1\right)\right] \end{cases}$$
$$= q(w(z)) = 1 + w(z) - \frac{(w(z))^3}{3}.$$

Define the function P(z) by

$$P(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots$$

it is easy to see that

(2.2) 
$$w(z) = \frac{P(z) - 1}{P(z) + 1} = \frac{1}{2} \left[ c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) z^3 + \cdots \right].$$

Since w(z) is a Schwarz function, we see that  $\Re(p_1(z)) > 0$  and  $p_1(0) = 1$ . Let us define the function p(z) by

$$p(z): = \left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1\right) \right] \right\}$$
  
(2.3) 
$$= 1 + b_1 z + b_2 z^2 + \cdots$$

In view of the equations (2.1), (2.2), (2.3), we have

(2.4) 
$$p(z) = q\left(\frac{P(z) - 1}{P(z) + 1}\right)$$

Hence (2.5)

$$1 + w(z) - \frac{(w(z))^3}{3} = 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right)z^3 - \frac{c_1^3}{24}z^3 \cdots$$
$$= 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{12}\right)z^3 + \cdots, \quad z \in \mathbb{D}.$$

Using (2.2) in (2.4), we get,

$$b_1 = \frac{c_1}{2}$$
 and  $b_2 = \frac{c_2}{2} - \frac{c_1^2}{4}$ .

A computation shows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3a_2)z^3 + \cdots$$

Similarly we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + \cdots$$

An easy computation shows that

$$\left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z}\right)^{\alpha} + \lambda \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1\right) \right] \right\}$$
  
= 1 + (1 + \alpha)(1 + \lambda)a\_2z + (\alpha + 2)(1 + 2\lambda)a\_3z^2  
+  $\left( \frac{\alpha^2 + \alpha}{2} - (\alpha + 3)(1 - 1) \right) a_2^2 z^2 + \cdots$ 

In view of the equation (2.3), we see that

(2.6) 
$$b_1 = (1+\alpha)(1+\lambda)a_2$$
  
(2.7)  $b_2 = (\alpha+2)(1+2\lambda)a_3 + \left(\frac{\alpha^2+\alpha}{2} - (\alpha+3)\lambda - 1\right)a_2^2$ 

or equivalently, we have

$$(2.8) \quad a_2 = \frac{c_1}{2(1+\alpha)(1+\lambda)},$$

$$a_3 = \frac{1}{(\alpha+2)(1+2\lambda)} \left( \frac{c_2}{2} - \frac{c_1^2}{4} \left[ 1 + \frac{\alpha^2 + \alpha - 2(\alpha+3)\lambda - 2}{2((1+\alpha)(1+\lambda))^2} \right] \right),$$

$$(2.9) = \frac{1}{2(\alpha+2)(1+2\lambda)} \left( c_2 - \frac{c_1^2}{2} \left[ 1 + \frac{\alpha^2 + \alpha - 2(\alpha+3)\lambda - 2}{2((1+\alpha)(1+\lambda))^2} \right] \right).$$

(2.9) 
$$= \frac{1}{2(\alpha+2)(1+2\lambda)} \left[ c_2 - \frac{1}{2} \left[ 1 + \frac{1}{2((1+\alpha)(1+\lambda))^2} \right] \right]$$
$$= \frac{1}{2(\alpha+2)(1+2\lambda)} \left( c_2 - vc_1^2 \right)$$

where

(2.10) 
$$v = \frac{1}{2} \left( 1 + \frac{\alpha^2 + \alpha - 2(\alpha + 3)\lambda - 2}{2\left((1 + \alpha)(1 + \lambda)\right)^2} \right).$$

Therefore, we have

$$|a_2| \le \frac{1}{(1+\alpha)(1+\lambda)}$$

and by using the estimate

$$\frac{|c_2 - vc_1^2| \le 2\max(1, |2v - 1|)}{61}$$

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given in Lemma 2.3, we have

$$|a_{3}| \leq \frac{1}{(\alpha+2)(1+2\lambda)} \max\{1, |2 \times \frac{1}{2} \left(1 + \frac{\alpha^{2} + \alpha - 2(\alpha+3)\lambda - 2}{2((1+\alpha)(1+\lambda))^{2}}\right) - 1|\}$$
  
=  $\frac{1}{(\alpha+2)(1+2\lambda)} \max\{1, |\left(\frac{\alpha^{2} + \alpha - 2(\alpha+3)\lambda - 2}{2((1+\alpha)(1+\lambda))^{2}}\right)|\}$ 

**Remark 2.1.** Let  $\alpha = 0$  and  $\lambda \ge 0$ . If f(z) given by (1.1) belongs to  $\mathcal{M}_{\lambda}(q)$ , then

$$\begin{aligned} |a_2| &\leq \frac{1}{1+\lambda}, \\ |a_3| &\leq \frac{1}{2(1+2\lambda)} \max\{1, \left|\frac{3\lambda+1}{2(1+\lambda)^2}\right|\} = \frac{3\lambda+1}{4(1+2\lambda)(1+\lambda)^2}. \end{aligned}$$

**Remark 2.2.** Let  $\lambda = 0$ . If f(z) given by (1.1) belongs to  $\mathscr{B}^{\alpha}(q)$ , then

$$|a_2| \le \frac{1}{1+\alpha}$$
, and  $|a_3| \le \frac{1}{\alpha+2} \max\{1, \left|\left(\frac{\alpha^2 + \alpha - 2}{2(1+\alpha)^2}\right)\right|\}.$ 

**Remark 2.3.** (see [20]) Let  $\alpha = 0$  and  $\lambda = 0$ . If f(z) given by (1.1) belongs to  $\mathscr{S}^*(q)$ , then

$$|a_2| \le 1$$
, and  $|a_3| \le \frac{1}{2} \max\{1, \left|\frac{1}{2}\right|\} = \frac{1}{2}$ .

**Remark 2.4.** (see [20]) Let  $\alpha = 0$  and  $\lambda = 1$ . If f(z) given by (1.1) belongs to  $\mathscr{C}(q)$ , then

$$|a_2| \le \frac{1}{2}$$
,  $and$   $|a_3| \le \frac{1}{6} \max\{1, \left|\frac{1}{2}\right|\} = \frac{1}{6}$ .

**Theorem 2.2.** Let  $0 \le \mu \le 1$ ,  $\alpha \ge 0$  and  $\lambda \ge 0$ . If f(z) given by (1.1) belongs to  $\mathcal{M}_{\alpha,\lambda}(q)$ , then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2\xi} \left(-\frac{\gamma}{\tau^2}\right), & \text{if} \quad \mu \leq \sigma_1, \\\\ \frac{1}{\xi}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\\\ \frac{1}{2\xi} \left(\frac{\gamma}{\tau^2}\right), & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

where, for convenience,

$$\sigma_1 = \frac{-2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}; \sigma_2 = \frac{2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi}; \sigma_3 = \frac{2(\alpha+3)\lambda - \rho}{2\xi},$$

(2.11) 
$$\gamma := \rho - 2(\alpha + 3)\lambda + 2\mu\xi,$$

$$(2.12) \qquad \qquad \rho := \alpha^2 + \alpha - 2,$$

(2.13) 
$$\xi := (\alpha + 2)(1 + 2\lambda),$$

and

(2.14) 
$$\tau := (1 + \alpha)(1 + \lambda).$$

Further, if  $\sigma_1 \leq \mu \leq \sigma_3$ , then

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi} \left(1 + \frac{\gamma}{2\tau^2}\right) |a_2|^2 \le \frac{1}{\xi}.$$

If  $\sigma_3 \leq \mu \leq \sigma_2$ , then

$$|a_3 - \mu a_2^2| + \frac{\tau^2}{\xi} \left(1 - \frac{\gamma}{2\tau^2}\right) |a_2|^2 \le \frac{1}{\xi}.$$

These results are sharp.

**Proof.** Now by making use of (2.8) and (2.9), we get

$$a_{3} - \mu a_{2}^{2} = \frac{1}{(\alpha + 2)(1 + 2\lambda)} \left( \frac{c_{2}}{2} - \frac{c_{1}^{2}}{4} - \left( \frac{\alpha^{2} + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{8((1 + \alpha)(1 + \lambda))^{2}} \right) c_{1}^{2} \right)$$
$$= \frac{1}{2(\alpha + 2)(1 + 2\lambda)} \left( c_{2} - \frac{c_{1}^{2}}{2} \left( 1 + \frac{\alpha^{2} + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{2((1 + \alpha)(1 + \lambda))^{2}} \right) \right)$$
here

where

$$v := \frac{1}{2} \left( 1 + \frac{\alpha^2 + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{2\left((1 + \alpha)(1 + \lambda)\right)^2} \right)$$

That is simply

$$v := \frac{1}{2} \left( 1 + \frac{\rho - 2(\alpha + 3)\lambda + 2\mu\xi}{2\tau^2} \right) = \frac{1}{2} \left( 1 + \frac{\gamma}{2\tau^2} \right)$$

The assertion of Theorem 2.2 now follows by an application of Lemma 2.1.

To show that the bounds are sharp, we define the functions the functions  $F_{\eta}$  and  $G_{\eta}$   $(0 \le \eta \le 1)$ , respectively, with  $F_{\eta}(0) = 0 = F'_{\eta}(0) - 1$  and  $G_{\eta}(0) = 0 = G'_{\eta}(0) - 1$  by

$$\frac{z(F_{\eta})'(z)}{F_{\eta}(z)} \left(\frac{F_{\eta}(z)}{z}\right)^{\alpha} + \lambda \left[1 + \frac{z(F_{\eta})''(z)}{(F_{\eta})'(z)} - \frac{z(F_{\eta})'(z)}{F_{\eta}(z)} + \alpha \left(\frac{z(F_{\eta})'(z)}{F_{\eta}(z)} - 1\right)\right] = q \left(\frac{z(z+\eta)}{1+\eta z}\right),$$
$$z(G_{\eta})'(z) \left(G_{\eta}(z)\right)^{\alpha}$$

and

$$\frac{\overline{G_{\eta}(z)}}{G_{\eta}(z)} \left( \frac{\overline{z}}{z} \right)$$
$$+\lambda \left[ 1 + \frac{z(G_{\eta})'(z)}{(G_{\eta})'(z)} - \frac{z(G_{\eta})'(z)}{G_{\eta}(z)} + \alpha \left( \frac{z(G_{\eta})'(z)}{G_{\eta}(z)} - 1 \right) \right] = q \left( -\frac{z(z+\eta)}{1+\eta z} \right)$$

respectively. Clearly the functions  $K_q := q(z), F_\eta, G_\eta$  are members of  $\mathcal{M}_{\alpha,\lambda}(q)$ . If  $\mu < \sigma_1$  or  $\mu > \sigma_2$ , then the equality holds if and only if f is  $K_q$  or one of its rotations. When  $\sigma_1 < \mu < \sigma_2$ , then the equality holds if and only if f is  $K_q = q(z^2)$  or one of its rotations. If  $\mu = \sigma_1$  then the equality holds if and only if f is  $F_\eta$  or

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one of its rotations. If  $\mu = \sigma_2$  then the equality holds if and only if f is  $G_{\eta}$  or one of its rotations.

By making use of Lemma 2.3, we immediately obtain the following:

**Theorem 2.3.** Let  $0 \le \alpha \le 1$ , and  $0 \le \lambda \le 1$ . If  $f \in \mathcal{M}_{\alpha,\lambda}(q)$ , then for complex  $\mu$ , we have

$$|a_{3} - \mu a_{2}^{2}| \leq \frac{1}{(\alpha + 2)(1 + 2\lambda)} \max\left\{1, \left|\frac{\alpha^{2} + \alpha - 2 - 2(\alpha + 3)\lambda + 2\mu(\alpha + 2)(1 + 2\lambda)}{2((1 + \alpha)(1 + \lambda))^{2}}\right|\right\} \\ = \frac{1}{\xi} \max\left\{1, \left|\frac{\rho - 2(\alpha + 3)\lambda + 2\mu\xi}{2\tau^{2}}\right|\right\},$$

where  $\rho, \xi, \tau$  are as defined in (2.12), (2.13) and (2.14). The result is sharp.

**Remark 2.5.** (1) For the choice  $\alpha = 0$ , and  $\lambda = 1$ , Theorem 2.3, coincides with the result obtained for the class  $f \in \mathscr{C}(q)$  as

$$|a_3 - \mu a_2^2| \le \frac{1}{6} \max\left\{1, \left|\frac{3\mu}{2} - 1\right|\right\}.$$

(2) For the choices  $\alpha = 0$ , and  $\lambda = 0$ , Theorem 2.3 reduces to the result for the class  $f \in \mathscr{S}^*(q)$  (see [20]) as

$$|a_3 - \mu a_2^2| \le \frac{1}{2} \max\{1, |2\mu - 1|\}.$$

(3) For the choice of  $\alpha = 0$ , Theorem 2.3, reduces the result for the class  $f \in \mathscr{M}_{\lambda}(q)$  as

$$|a_3 - \mu a_2^2| \le \frac{1}{1+2\lambda} \max\left\{1, \left|\frac{-2 - 6\lambda + 4\mu(1+2\lambda)}{2(1+\lambda)^2}\right|\right\}$$

(4) For the choice of  $\lambda = 0$ , Theorem 2.3, reduces the result for  $f \in \mathscr{B}^{\alpha}(q)$ 

$$|a_3 - \mu a_2^2| \le \frac{2}{\alpha + 2} \max\left\{1, \left|\frac{\alpha^2 + \alpha - 2 + 2\mu(\alpha + 2)}{2(1 + \alpha)^2}\right|\right\}.$$

3. Coefficient inequalities for the function  $f^{-1}$ 

**Theorem 3.1.** If  $f \in \mathscr{M}_{\alpha,\lambda}(q)$  and  $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$  is the inverse function of f with  $|w| < r_0$  where  $r_0$  is greater than the radius of the Koebe domain of the class  $f \in \mathscr{M}_{\alpha,\lambda}(q)$ , then for any complex number  $\mu$ , we have

(3.1) 
$$|d_3 - \mu d_2^2| \leq \frac{1}{\xi} max \left\{ 1, \left| \frac{2\tau^2 + \rho - 2(\alpha + 3)\lambda + (4 + 2\mu)\xi}{\tau^2} - 1 \right| \right\},$$

where  $\rho, \xi, \tau$  are as defined in (2.12), (2.13) and (2.14).

**Proof.** As

(3.2) 
$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$$

is the inverse function of f, it can be seen that

(3.3) 
$$f^{-1}(f(z)) = f\{f^{-1}(z)\} = z.$$

From equations (1.1) and (3.3), it can be reduced to

(3.4) 
$$f^{-1}(z + \sum_{n=2}^{\infty} a_n z^n) = z$$

From (3.3) and (3.4), one can obtain

(3.5) 
$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2d_2 + d_3)z^3 + \dots = z.$$

By comparing the coefficients of z and  $z^2$  from relation (3.5), it can be seen that

$$(3.6) d_2 = -a_2, d_3 = 2a_2^2 - a_3.$$

From relations (2.8), (2.9), and (3.6)

(3.7) 
$$d_2 = -\frac{c_1}{2(1+\alpha)(1+\lambda)};$$

$$d_{3} = \frac{1}{2(\alpha+2)(1+2\lambda)}$$
(3.8)
$$\times \left(c_{2} - \frac{2\left((1+\alpha)(1+\lambda)\right)^{2} + 4(\alpha+2)(1+2\lambda) + \alpha^{2} + \alpha - 2(\alpha+3)\lambda - 2}{4\left((1+\alpha)(1+\lambda)\right)^{2}}c_{1}^{2}\right);$$

$$= \frac{1}{2\xi} \left(c_{2} - \frac{2\tau^{2} + 4\xi + \rho - 2(\alpha+3)\lambda}{2\tau^{2}}c_{1}^{2}\right);$$

and  $\rho, \xi, \tau$  are as defined in (2.12), (2.13) and (2.14). For any complex number  $\mu$ , consider

(3.9) 
$$d_3 - \mu d_2^2 = \frac{1}{2\xi} \left( c_2 - \frac{2\tau^2 + \rho - 2(\alpha + 3)\lambda + (4 + 2\mu)\xi}{2\tau^2} c_1^2 \right)$$

Taking modulus on both sides and by applying Lemma 2.3 on the right hand side of (3.9), one can obtain the result as in (3.1). Hence this completes the proof.  $\Box$ 

**Remark 3.1.** Suitably specializing the parameters in Theorem 3.1 one can easily state above result for the function classes  $\mathscr{M}_{0,\lambda}(q) \equiv \mathscr{M}_{\lambda}(q)$ ;  $\mathscr{M}_{\alpha,0}(q) \equiv \mathscr{B}^{\alpha}(q)$ ;  $\mathscr{M}_{0,0}(q) \equiv \mathscr{S}^{*}(q)$  and  $\mathscr{M}_{0,1}(q) \equiv \mathscr{C}(q)$ .

# 4. Application to functions defined by Neutrosophic Poisson Distribution

By letting  $\wp_N(z)$  as the neutrosophic Poisson distribution series we study the following results (for detaiss see[12, 14]). As is well known that the classical probability distributions only deals with specified data and specified parameter values, while neutrosophic probability distribution gives a more general and clear

ones. In fact, Neutrosophic Poisson distribution of a discrete variable X is a classical Poisson distribution of x with the imprecise parameter value. A variable X is said to have neutrosophic Poisson distribution if its probability with the value  $k \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$  is

$$NP(x=k) = \frac{(m_N)^k}{k!}e^{-m_N}, k = 0.1, 2, 3 \cdots$$

where the distribution parameter  $m_N$  is the expected value and the variance, that is to say,  $NE(x) = NV(x) = m_N$  for the neutrosophic statistical number N = d + I(refer to [5] and also see [14]the references cited). Define a power series whose coefficients are probabilities of neutrosophic Poisson distribution by

$$\Phi(m_N, z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!} e^{-m_N} z^n, \quad z \in \mathbb{D}.$$

For  $f \in \mathcal{A}$ , we take the convolution operator \* and introduce the linear operator  $\Lambda : \mathcal{A} \to \mathcal{A}$  defined by

(4.1) 
$$\Lambda f(z) = \Phi(m_N, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(m_N)^{n-1}}{(n-1)!} e^{-m_N} a_n z^n$$
$$= z + \sum_{n=m+1}^{\infty} \Psi(m_N, n) a_n z^n,$$

where

$$\Psi_n := \Psi(m_N, n) = \frac{(m_N)^{n-1}}{(n-1)!} e^{-m_N}.$$

Specially

(4.2) 
$$\Psi_2 := m_N e^{-m_N}, \quad \Psi_3 := \frac{(m_N)^2}{2} e^{-m_N}.$$

For the application of the results given in the previous section, we define the class  $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$ , in the following way:

$$\mathscr{M}^{\varphi}_{\alpha,\lambda}(q) := \{ f \in \mathscr{A} \quad \text{and} \quad (f * \varphi) \in \mathscr{M}_{\alpha,\lambda}(q) \}$$

where

$$\varphi(z) = z + \sum_{n=2}^{\infty} \varphi_n z^n, \quad (\varphi_n > 0); \quad (f * \varphi) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n$$

and  $\mathcal{M}_{\alpha,\lambda}(q)$  is given by Definition 1.2 and \* denote the convolution or Hadamard product of two series. We define the class  $\mathcal{M}^m_{\alpha,\lambda}(q)$  in the following way:

$$\mathscr{M}^m_{\alpha,\,\lambda}(q) := \{ f \in \mathscr{A} \quad \text{and} \quad \Lambda f \in \mathscr{M}_{\alpha,\,\lambda}(q) \}$$

where  $\mathcal{M}_{\alpha,\lambda}(q)$  is given by Definition 1.2.

In following theorem, we obtain the coefficient estimate for functions in the class  $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$ , from the corresponding estimate for functions in the class  $\mathscr{M}_{\alpha,\lambda}(q)$ .

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Applying Theorem 2.2 for the function  $(f * \varphi)(z) = z + \varphi_2 a_2 z^2 + \varphi_3 a_3 z^3 + \cdots$ , we get the next Theorems 4.1 and 4.2 after an obvious change of parameter  $\mu$ .

**Theorem 4.1.** Let  $0 \le \alpha \le 1$ , and  $0 \le \lambda \le 1$ . If  $f \in \mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$ , then for complex  $\mu$ , we have

$$|a_{3}-\mu a_{2}^{2}| = \frac{1}{(\alpha+2)(1+2\lambda)\varphi_{3}} max \left\{ 1, \left| \frac{\alpha^{2}+\alpha-2-2(\alpha+3)\lambda}{2((1+\alpha)(1+\lambda))^{2}} + \frac{\mu(\alpha+2)(1+2\lambda)\varphi_{3}}{((1+\alpha)(1+\lambda)\varphi_{2})^{2}} \right| \right\}.$$

**Theorem 4.2.** Let  $0 \le \mu \le 1$ ,  $\alpha \ge 0$ ,  $\lambda \ge 0$  and  $\varphi_n > 0$ . If f(z) given by (1.1) belongs to  $\mathscr{M}^{\varphi}_{\alpha,\lambda}(q)$ , then

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \begin{cases} \frac{1}{2\xi\varphi_3} \left(-\frac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \leq \sigma_1, \\\\ \frac{1}{\xi\varphi_3}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\\\ \frac{1}{2\xi\varphi_3} \left(\frac{\gamma_2}{\tau^2}\right), & \text{if} \quad \mu \geq \sigma_2, \end{cases} \end{aligned}$$

where, for convenience,  $\gamma_2 := \rho - 2(\alpha + 3)\lambda + 2\mu\xi\frac{\varphi_3}{\varphi_2^2}$ ,

$$\sigma_1 := \frac{\varphi_2^2}{\varphi_3} \left[ \frac{2(\alpha+3)\lambda - \rho - 2\tau^2}{2\xi} \right], \sigma_2 = \frac{\varphi_2^2}{\varphi_3} \left[ \frac{2\tau^2 + 2(\alpha+3)\lambda - \rho}{2\xi} \right]$$

and  $\rho, \xi, \tau$  are as defined in(2.12)(2.13) and (2.14).

Now ,we obtain the coefficient estimate for  $f \in \mathscr{M}^m_{\alpha,\lambda}(q)$ , from the corresponding estimate for  $f \in \mathscr{M}_{\alpha,\lambda}(q)$ . Applying Theorem 2.2 for the function  $\Lambda f = z + \Psi_2 a_2 z^2 + \Psi_3 a_3 z^3 + \cdots$ , we get the following Theorems 4.3 and 4.4 after an obvious change of the parameter  $\mu$  as in above theorems.

For  $\Psi_2$  and  $\Psi_3$  given by (4.2) Theorems 4.1 and 4.2 reduces to the following:

**Theorem 4.3.** Let  $0 \le \alpha \le 1$ , and  $0 \le \lambda \le 1$ . If  $f \in \mathscr{M}^m_{\alpha,\lambda}(q)$ , then for complex  $\mu$ , we have

$$|a_{3} - \mu a_{2}^{2}| = \frac{2}{(\alpha + 2)(1 + 2\lambda)m_{N}^{2}e^{-m_{N}}} \times \\ \times \max\left\{1, \left|\frac{\alpha^{2} + \alpha - 2 - 2(\alpha + 3)\lambda}{2((1 + \alpha)(1 + \lambda))^{2}} + \frac{\mu(\alpha + 2)(1 + 2\lambda)}{2((1 + \alpha)(1 + \lambda))^{2}e^{-m_{N}}}\right|\right\}$$

**Theorem 4.4.** Let  $0 \le \mu \le 1$ ,  $\alpha \ge 0$ ,  $\lambda \ge 0$  and  $\psi_n > 0$ . If f(z) given by (1.1) belongs to  $\mathscr{M}^m_{\alpha,\lambda}(q)$ , then

$$|a_{3} - \mu a_{2}^{2}| \leq \begin{cases} \frac{1}{\xi m_{N}^{2} e^{-m_{N}}} \left(-\frac{\gamma_{2}}{\tau^{2}}\right), & \text{if} \quad \mu \leq \sigma_{1}, \\ \frac{2}{\xi m_{N}^{2} e^{-m_{N}}}, & \text{if} \quad \sigma_{1} \leq \mu \leq \sigma_{2} \\ \frac{1}{\xi m_{N}^{2} e^{-m_{N}}} \left(\frac{\gamma_{2}}{\tau^{2}}\right), & \text{if} \quad \mu \geq \sigma_{2}, \end{cases}$$

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where  $\gamma_2 := \rho - 2(\alpha + 3)\lambda + \frac{\mu\xi}{e^{-m_N}}$ , for convenience we write

$$\sigma_1 := e^{-m_N} \left[ \frac{2\tau^2 + 2(\alpha + 3)\lambda - \rho}{2\xi} \right], \sigma_2 = e^{-m_N} \left[ \frac{2\tau^2 + 2(\alpha + 3)\lambda - \rho}{2\xi} \right]$$

and  $\rho, \xi, \tau$  are as defined in (2.12), (2.13) and (2.14).

A variable  $\mathcal{X}$  is said to be Poisson distributed if it takes the values  $0, 1, 2, 3, \cdots$ with probabilities  $e^{-m}$ ,  $m\frac{e^{-m}}{1!}$ ,  $m^2\frac{e^{-m}}{2!}$ ,  $m^3\frac{e^{-m}}{3!}$ , ... respectively, where m is called the parameter. Thus

$$P(\mathcal{X} = r) = \frac{m^r e^{-m}}{r!}, \ r = 0, 1, 2, 3, \cdots$$

In [13], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad z \in \Delta,$$

where m > 0. By ratio test the radius of convergence of above series is infinity. Using the Hadamard product, Porwal[13] (see also, [1, 9, 10] introduced a new linear operator  $\mathcal{I}^m(z) : \mathcal{A} \to \mathcal{A}$  defined by

$$\mathcal{I}^{m}f = \mathcal{K}(m, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, = z + \sum_{n=2}^{\infty} \psi_n(m) a_n z^n, \qquad z \in \Delta,$$

where Since,  $\mathcal{I}^m f = z + \sum_{n=2}^{\infty} \psi_m a_n z^n$ , where  $\psi_n = \frac{m^{n-1}}{(n-1)!} e^{-m}$ , we have

(4.3) 
$$\psi_2 = me^{-m} \quad and \quad \psi_3 = \frac{m^2}{2}e^{-m}$$

**Remark 4.1.** Suitably specializing the parameters in Theorems 4.3 and 4.4 one can easily state the results for the function classes associated with neutrosophic Poisson distribution and Poisson distribution as listed below:

(1) 
$$\mathscr{M}_{0,\lambda}^{m}(q) \equiv \mathscr{M}_{\lambda}^{m}(q)$$
  
(2)  $\mathscr{M}_{0,0}^{m}(q) \equiv \mathscr{S}_{m}^{*}(q)$   
(3)  $\mathscr{M}_{\alpha,0}(q) \equiv \mathscr{B}^{\alpha}(q)$  and  
(4)  $\mathscr{M}_{0,1}^{m}(q) \equiv \mathscr{C}^{m}(q)$ 

which are new and not been studied sofar.

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