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ON THE INVERSE LQG HOMING PROBLEM

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Abstract. The problem of minimizing or maximizing the time spent by a controlled diffusion process in a given interval is known as LQG homing. The optimal control, when it is possible to obtain an explicit solution to such a problem, is often expressed as special functions. Here, the inverse problem is considered: we determine, under certain assumptions, the processes for which the optimal control is a simple power function. Moreover, the problem is extended to the case of jump-diffusion processes.

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1. Introduction

Let $\{X_u(t), t \geq 0\}$ be the one-dimensional controlled diffusion process defined by the stochastic differential equation

$$(1.1) \qquad \mathrm{d}X_u(t) = f[X_u(t)]\,\mathrm{d}t + b[X_u(t)]\,u[X_u(t)]\,\mathrm{d}t + \sigma[X_u(t)]\,\mathrm{d}B(t),$$

where $b(\cdot)$ is not identical to zero, $\sigma(\cdot)$ is non-negative and $\{B(t), t \geq 0\}$ is a standard Brownian motion. The random variable

$$(1.2) \qquad T(x) := \inf\{t > 0 : X_u(t) \notin (a,b) \mid X_u(0) = x \in (a,b)\}$$

is called a *first-passage time* in probability. The problem of finding the control $u^*(x)$ that minimizes the expected value of the cost function

$$(1.3) \qquad J(x) := \int_0^{T(x)} \left\{ \tfrac{1}{2}\, q[X_u(t)]\, u^2[X_u(t)] + \lambda \right\} \mathrm{d}t + K[X_u[T(x)]],$$

where $q(\cdot)$ is positive in (a,b) and λ is a real parameter, is a particular *LQG homing* problem; see Whittle [8]. If the parameter λ is positive (respectively, negative), then the optimizer wants the controlled process $X_u(t)$ to leave the interval (a,b) as soon (resp., late) as possible, taking the quadratic control costs and termination cost $K(\cdot)$ into account. Notice that the optimal control problem considered is time-invariant. In the general formulation, $\{X_u(t), t \geq 0\}$ can be an n-dimensional controlled diffusion process, and all the functions can depend explicitly on t. The cost function

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can also take the risk-sensitivity of the optimizer into account; see Whittle [9] or Kuhn [2] and Makasu [7]. Moreover, Lefebvre and Moutassim [6] considered the case when the uncontrolled process is a Wiener process with random parameters.

In addition to being of theoretical interest, LQG homing problems have many applications in various areas: financial mathematics, reliability theory, hydrology, etc. Recently, the author (Lefebvre [3]) has considered this type of problem in the context of epidemiology. More precisely, he considered a stochastic version of the classic three-dimensional model for the spread of epidemics due to Kermack and McKendrick. The aim was to end the epidemic as soon as possible. In practice, no one knows how long an epidemic will last. Therefore, the final time in this optimal control problem is indeed a random variable. See also Ionescu *et al.* [1].

Whittle [8] has shown that, under some conditions, it is sometimes possible to express the optimal control $u^*(x)$ in terms of a mathematical expectation for the uncontrolled process $\{X_0(t), t \geq 0\}$ obtained by setting $u[X_u(t)] \equiv 0$ in Eq. (1.1). However, solving the purely probabilistic problem is generally quite difficult, especially in two or more dimensions.

When an explicit solution to an LQG homing problem can be found, the optimal control is often expressed in terms of special functions or integrals that can only be evaluated numerically; see, for instance, Lefebvre [4]. Here, we consider the inverse problem: we will try to determine what are the problems for which the optimal control $u^*(x)$ is a simple power function, namely a constant, a linear function of x or proportional to $1/x$. Moreover, we assume that the functions $b(\cdot)$ and $q(\cdot)$ are also power functions.

To solve an LQG homing problem, we can make use of dynamic programming: we define the value function

$$F(x) = \inf_{u[X_u(t)],\, 0 \leq t \leq T(x)} E\left[J(x)\right]. \tag{1.4}$$

We can show (see Whittle [8]) that the function F satisfies the *dynamic programming equation*

$$0 = \inf_{u(x)} \left\{ \frac{1}{2}\, q(x)\, u^2(x) + \lambda + f(x)\, F'(x) + b(x)\, u(x)\, F'(x) + \frac{\sigma^2(x)}{2}\, F''(x) \right\}. \tag{1.5}$$

We deduce from the above equation that the optimal control can be expressed as follows:

$$u^*(x) = -\frac{b(x)}{q(x)}\, F'(x). \tag{1.6}$$

Hence, substituting this expression for $u^*(x)$ into Eq. (1.5), we obtain that we must solve the non-linear second-order differential equation

$$0 = -\frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 + \lambda + f(x)F'(x) + \frac{\sigma^2(x)}{2}F''(x). \tag{1.7}$$

This equation is valid for $a < x < b$ and is subject to the boundary conditions

$$F(a) = K(a) \quad \text{and} \quad F(b) = K(b). \tag{1.8}$$

Notice that Eq. (1.7) is a Riccati equation for $G(x) := F'(x)$.

In the next section, we will assume that the optimal control $u^*(x)$ is a certain power function of x and we will try to determine the value of the functions $b(\cdot)$, $q(\cdot)$ and $K(\cdot)$ for which this power function is indeed the exact solution to the optimal control problem. Then, in Section 3, the inverse LQG homing problem will be extended to the case of jump-diffusion processes.

2. Inverse problem

The functions $b(\cdot)$ and $q(\cdot)$ in LQG homing problems are generally power functions. Actually, they are often assumed to be respectively a non-zero and a positive constant. Here, we assume that

$$b(x) = b_0 x^m \quad \text{and} \quad q(x) = q_0 x^n, \tag{2.1}$$

where $b_0 \neq 0$ and $q_0 > 0$. Moreover, $m, n \in \{0, 1, 2, \ldots\}$. If $a \geq 0$ in the interval (a, b), then n can be an odd integer; otherwise, it must be an even integer (including 0). In the case of the function $K(\cdot)$, it is often chosen to be identical to zero. In this paper, it can be any real function.

Case I. Assume first that the optimal control is a constant: $u^*(x) \equiv u_0$. An important special case is the one when $u^*(x) \equiv 0$. We then deduce from Eq. (1.6) that $F(x) \equiv F_0$. Therefore, this solution can only be the exact one if $\lambda = 0$. Moreover, we must have $K(a) = K(b) = F_0$. With these assumptions, it is actually obvious that the optimizer must not use any control, for any functions $b(\cdot)$, $q(\cdot)$, $f(\cdot)$ and $\sigma(\cdot)$.

Next, if $u^*(x) \equiv u_0 \neq 0$, we obtain that

$$F'(x) = -u_0 \frac{q(x)}{b(x)} = -\frac{u_0 q_0}{b_0} x^{n-m}. \tag{2.2}$$

Equation (1.7) becomes

$$0 = -\frac{1}{2}u_0^2 q_0 x^n + \lambda - f(x)\frac{u_0 q_0}{b_0}x^{n-m} - \frac{\sigma^2(x)}{2}\frac{u_0 q_0}{b_0}(n-m)x^{n-m-1}. \tag{2.3}$$

In the special case when $m = n$, $F'(x)$ is a constant and Eq. (2.3) is satisfied if and only if

$$f(x) = -\frac{1}{2} u_0 b_0 x^n + \lambda \frac{b_0}{u_0 q_0}. \tag{2.4}$$

Furthermore, the final cost must be given by

$$K(a) = -\frac{u_0 q_0 a}{b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 b}{b_0} + F_0, \tag{2.5}$$

where F_0 is a real constant.

Now, there are some conditions on the functions $f(\cdot)$ and $\sigma(\cdot)$ that must be satisfied for the uncontrolled process $\{X_0(t), t \geq 0\}$ to be a diffusion process. For the applications, the most important cases are the ones when $f(x) = f_0 x^p$ and $\sigma^2(x) = \sigma_0^2 x^r$, where $p \in \{-1, 0, 1\}$, $r \in \{0, 1, 2\}$, f_0 is a real constant and σ_0 is a positive constant.

Proposition 2.1. *Assume that $m = n \in \{0, 1, 2, \ldots\}$. If the conditions in Eq. (2.4) and Eq. (2.5) are satisfied, where n is such that the uncontrolled stochastic process $\{X_0(t), t \geq 0\}$ with $f(x)$ defined in Eq. (2.4) is a diffusion process, then the optimal control $u^*(x)$ is a non-zero constant u_0.*

Remark. (i) Notice that there is no explicit condition on the function $\sigma(x)$. (ii) When $n = 0$, the function $f(x)$ is a constant. Then, if $\sigma(x)$ is also a constant, $\{X_0(t), t \geq 0\}$ is a Wiener process. If $n = 1$ and $\sigma(x)$ is a constant, then $\{X_0(t), t \geq 0\}$ could be an Ornstein-Uhlenbeck process (if $u_0 b_0$ is positive). Finally, if $\lambda = 0$, $n = 1$ and $\sigma^2(x) = \sigma_0^2 x^2$, then the uncontrolled process is a geometric Brownian motion. We see that the optimal control is not equal to zero, even if $\lambda = 0$. This is due to the fact that $K(a) \neq K(b)$. (iii) When $n = 1$, we have $q(x) = q_0 x$. Because the function $q(x)$ is assumed to be positive in the interval (a, b), we must impose the additional condition $a \geq 0$. (iv) The Wiener process (or Brownian motion) is the basic diffusion process and the Ornstein-Uhlenbeck process is widely used in physics and biology, in particular. Geometric (or exponential) Brownian motion is the fundamental diffusion process in financial mathematics.

There are of course many *mathematical* cases that can be considered. However, the most frequent ones for the function $b(x)$ (respectively, $q(x)$) are those when $m = 0$ and $m = 1$ (resp., $n = 0$ and $n = 2$). Since we now assume that $m \neq n$, there are three important cases to examine: $(m, n) = (0, 2), (1, 0)$ and $(1, 2)$.

Firstly, with $(m, n) = (0, 2)$, Eq. (2.3) reduces to

$$0 = -\frac{1}{2} u_0^2 q_0 x^2 + \lambda - f(x) \frac{u_0 q_0}{b_0} x^2 - \sigma^2(x) \frac{u_0 q_0}{b_0} x. \tag{2.6}$$

Let $\lambda = 0$. Then, we can choose $f(x) \equiv f_0$ and $\sigma^2(x) = \sigma_0^2 x$, where f_0 is such that

$$f_0 = -\frac{1}{2} b_0 u_0 - \sigma_0^2. \tag{2.7}$$

Moreover, we must have $a \geq 0$ and the function $K(x)$ must satisfy the conditions

$$K(a) = -\frac{u_0 q_0 a^3}{3 b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 b^3}{3 b_0} + F_0, \tag{2.8}$$

where F_0 is a real constant.

Secondly, if we choose $(m, n) = (1, 0)$, Eq. (2.3) simplifies to

$$0 = -\frac{1}{2} u_0^2 q_0 + \lambda - f(x) \frac{u_0 q_0}{b_0} x^{-1} + \frac{\sigma^2(x)}{2} \frac{u_0 q_0}{b_0} x^{-2}. \tag{2.9}$$

The most interesting particular solution is when $\{X_0(t), t \geq 0\}$ is a geometric Brownian motion with $f(x) = f_0 x$ and $\sigma^2(x) = \sigma_0^2 x^2$. Then, the various parameters must be chosen so that

$$0 = -\frac{1}{2} u_0^2 q_0 + \lambda - f_0 \frac{u_0 q_0}{b_0} + \frac{\sigma_0^2}{2} \frac{u_0 q_0}{b_0}. \tag{2.10}$$

This time, λ could be any real number. Because the optimally controlled process $\{X_{u^*}(t), t \geq 0\}$ is also a geometric Brownian motion, with infinitesimal mean $(f_0 + b_0 u_0) x$, and geometric Brownian motions are strictly positive (or strictly negative), we should assume that $a > 0$. Moreover, the function $K(x)$ must satisfy the following conditions:

$$K(a) = -\frac{u_0 q_0 \ln(a)}{b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 \ln(b)}{b_0} + F_0, \tag{2.11}$$

for a certain constant F_0.

Thirdly, when $(m, n) = (1, 2)$, we deduce from Eq. (2.3) that

$$0 = -\frac{1}{2} u_0^2 q_0 x^2 + \lambda - f(x) \frac{u_0 q_0}{b_0} x - \frac{\sigma^2(x)}{2} \frac{u_0 q_0}{b_0}. \tag{2.12}$$

There are two interesting particular solutions: as above, if $\{X_0(t), t \geq 0\}$ is a geometric Brownian motion with $f(x) = f_0 x$ and $\sigma^2(x) = \sigma_0^2 x^2$, and if $\lambda = 0$, we must have

$$f_0 = -\frac{1}{2} (b_0 u_0 + \sigma_0^2) \tag{2.13}$$

and $a > 0$. Furthermore, $\{X_0(t), t \geq 0\}$ could be an Ornstein-Uhlenbeck process with infinitesimal mean $f_0 x$ and infinitesimal variance σ_0^2, where

$$f_0 = -\frac{b_0 u_0}{2} \ (< 0) \quad \text{and} \quad \sigma_0^2 = \frac{2 \lambda b_0}{u_0 q_0} \ (> 0). \tag{2.14}$$

Finally, in both cases the conditions

$$K(a) = -\frac{u_0 q_0 a^2}{2 b_0} + F_0 \quad \text{and} \quad K(b) = -\frac{u_0 q_0 b^2}{2 b_0} + F_0, \tag{2.15}$$

where F_0 is a real constant, must be satisfied.

Case II. Assume now that the optimal control $u^*(x)$ is linear: $u^*(x) = u_0\,x$, where $u_0 \neq 0$. Equation (1.6) then implies that

$$F'(x) = -u_0\,x\,\frac{q(x)}{b(x)} = -\frac{u_0\,q_0}{b_0}\,x^{n-m+1}, \tag{2.16}$$

so that Eq. (1.7) takes the form

$$0 = -\frac{1}{2}\,u_0^2\,q_0\,x^{n+2} + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x^{n-m+1} - \frac{\sigma^2(x)}{2}\,\frac{u_0\,q_0}{b_0}\,(n-m+1)\,x^{n-m}. \tag{2.17}$$

Here the special case is when $m = n+1$; then, $F'(x)$ is a constant and we find that Eq. (2.17) is satisfied if and only if (iff) the function $f(x)$ is such that

$$f(x) = -\frac{1}{2}\,u_0\,b_0\,x^{n+2} + \lambda\,\frac{b_0}{u_0\,q_0}. \tag{2.18}$$

As in Case I, the function $K(x)$ must satisfy the conditions in Eq. (2.5).

Proposition 2.2. *Assume that $m = n+1 \in \{1, 2, \ldots\}$. If the function $f(x)$ can be expressed as in Eq. (2.18), where n is such that the uncontrolled stochastic process $\{X_0(t), t \geq 0\}$ is a diffusion process, and if the final cost satisfies both conditions in Eq. (2.5), then the optimal control $u^*(x)$ is linear: $u^*(x) = u_0\,x$, where $u_0 \neq 0$.*

Remark. Again, we can choose any admissible function $\sigma(x)$. The most interesting case is when $n = 0$ and $\sigma^2(x) = \sigma_0^2\,x^2$.

When $m = n$, Eq. (2.17) becomes

$$0 = -\frac{1}{2}\,u_0^2\,q_0\,x^{n+2} + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x - \frac{\sigma^2(x)}{2}\,\frac{u_0\,q_0}{b_0}. \tag{2.19}$$

With $n = 0$, there are two important solutions: firstly, we can have

$$f(x) = \frac{\lambda\,b_0}{u_0\,q_0\,x} \quad \text{and} \quad \sigma^2(x) = -b_0\,u_0\,x^2, \tag{2.20}$$

provided that $b_0\,u_0 < 0$. If $\lambda = 0$, the uncontrolled process is then a geometric Brownian motion. Secondly, we can also have

$$f(x) = -\frac{b_0\,u_0}{2}\,x \quad \text{and} \quad \sigma^2(x) \equiv \frac{2\,\lambda\,b_0}{u_0\,q_0}\;(>0). \tag{2.21}$$

This time, if $b_0\,u_0 > 0$, $\{X_0(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process. Furthermore, in both cases Eq. (2.15) must be satisfied.

To conclude this part, let us consider the case when $m = n-1 \in \{0, 1, \ldots\}$; we deduce from Eq. (2.17) that

$$0 = -\frac{1}{2}\,u_0^2\,q_0\,x^{n+2} + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x^2 - \sigma^2(x)\,\frac{u_0\,q_0}{b_0}\,x. \tag{2.22}$$

If $\lambda = 0$ and $n = 1$, we can take $f(x) = f_0\,x$ and $\sigma^2(x) = \sigma_0^2\,x^2$, provided that $a > 0$, the conditions in Eq. (2.8) are satisfied and

$$f_0 = -\frac{1}{2}\,b_0\,u_0 - \sigma_0^2. \tag{2.23}$$

Case III. Lastly, suppose that $a > 0$ and that the optimal control $u^*(x)$ is inversely proportional to x: $u^*(x) = u_0/x$, where $u_0 \neq 0$. We then deduce from Eq. (1.6) that

$$F'(x) = -\frac{u_0}{x}\,\frac{q(x)}{b(x)} = -\frac{u_0\,q_0}{b_0}\,x^{n-m-1}. \tag{2.24}$$

It follows that Eq. (1.7) becomes

$$0 = -\frac{1}{2}\,u_0^2\,q_0\,x^{n-2} + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x^{n-m-1} - \frac{\sigma^2(x)}{2}\,\frac{u_0\,q_0}{b_0}\,(n-m-1)\,x^{n-m-2}. \tag{2.25}$$

The function $F'(x)$ is a constant when $n - m = 1$. Equation (2.25) is then satisfied iff

$$f(x) = -\frac{1}{2}\,u_0\,b_0\,x^{n-2} + \lambda\,\frac{b_0}{u_0\,q_0}. \tag{2.26}$$

As in the previous cases, the termination cost function $K(x)$ must satisfy both conditions in Eq. (2.5).

Proposition 2.3. *Assume that $m = n - 1 \in \{0, 1, 2, \ldots\}$ and that the function $f(x)$ can be expressed as in Eq. (2.26), where $n \geq 1$ is such that the uncontrolled stochastic process $\{X_0(t), t \geq 0\}$ is a diffusion process. If the two conditions in Eq. (2.5), with $a > 0$, are satisfied, then the optimal control $u^*(x)$ is inversely proportional to x: $u^*(x) = u_0/x$, where $u_0 \neq 0$.*

Remark. As above, we can choose any admissible function $\sigma(x)$. When $n = 1$ and $\lambda = 0$, we have

$$f(x) = -\frac{u_0\,b_0}{2\,x}. \tag{2.27}$$

Then, if $\sigma^2(x) \equiv \sigma_0^2$, the uncontrolled process could be a Bessel process, which is another important diffusion process. Moreover, if $n = 2$,

$$f(x) \equiv f_0 = -\frac{1}{2}\,u_0\,b_0 + \lambda\,\frac{b_0}{u_0\,q_0}. \tag{2.28}$$

If we choose $\sigma^2(x) \equiv \sigma_0^2$, then $\{X_0(t), t \geq 0\}$ is a Wiener process.

Let us finally consider the particular case when $(m, n) = (0, 2)$. The conditions in Eq. (2.15) must then be satisfied. Moreover, Eq. (2.25) reduces to

$$0 = -\frac{1}{2}\,u_0^2\,q_0 + \lambda - f(x)\,\frac{u_0\,q_0}{b_0}\,x - \frac{\sigma^2(x)}{2}\,\frac{u_0\,q_0}{b_0}. \tag{2.29}$$

There are two interesting solutions: firstly, we can have $f(x) = f_0/x$ and $\sigma^2(x) \equiv \sigma_0^2$, with

$$f_0 = -\frac{1}{2}\left(u_0\,b_0 + \sigma_0^2\right) + \lambda\,\frac{b_0}{u_0\,q_0}. \tag{2.30}$$

Hence, $\{X_0(t), t \geq 0\}$ could be a Bessel process. Secondly, we can also have

$$\lambda = \frac{1}{2}\,u_0^2\,q_0 \quad \text{and} \quad f(x) = -\frac{\sigma^2(x)}{2\,x}. \tag{2.31}$$

An important case is the one for which $\sigma^2(x) = \sigma_0^2 x^2$, so that $f(x) = -\sigma_0^2 x/2$. The uncontrolled process is then a geometric Brownian motion. If $\sigma^2(x) \equiv \sigma_0^2$, then $\{X_0(t), t \geq 0\}$ could again be a Bessel process.

3. Jump-diffusion processes

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate α, and $\{Y_i, i = 1, 2, \ldots\}$ be independent and identically distributed (i.i.d.) random variables having the common probability density function $f_Y(y)$. In this section, we will extend the inverse LQG homing problem to the case when $\{X_u(t), t \geq 0\}$ is a controlled jump-diffusion process defined by

$$\begin{aligned} X_u(t) &= X_u(0) + \int_0^t \{f[X(s)] + b[X_u(s)]\,u[X_u(s)]\}\,\mathrm{d}s \\ &\quad + \int_0^t \sigma[X_u(s)]\,\mathrm{d}B(s) + \sum_{i=1}^{N(t)} Y_i. \end{aligned} \tag{3.1}$$

The stochastic processes $\{N(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ are assumed to be independent. Jump-diffusion processes are widely used in financial mathematics, among other fields.

The ordinary differential equation satisfied by the value function $F(x)$ becomes an integro-differential equation (see Lefebvre [5]):

$$\begin{aligned} 0 = &-\frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 + \lambda + f(x)\,F'(x) + \frac{\sigma^2(x)}{2}F''(x) \\ &+ \alpha\left\{\int_{-\infty}^{\infty} F(x+y)\,f_Y(y)\,\mathrm{d}y - F(x)\right\}. \end{aligned} \tag{3.2}$$

Moreover, because there can now be an overshoot, the boundary conditions become

$$F(x) = K(x) \quad \text{if } x \notin (a, b). \tag{3.3}$$

In the case when the jump size is a constant ϵ, so that $f_Y(y)$ becomes the Dirac delta function $\delta(y-\epsilon)$, the above integro-differential equation is reduced to a differential-difference equation:

$$0 = -\frac{1}{2}\frac{b^2(x)}{q(x)}[F'(x)]^2 + \lambda + f(x)\,F'(x) + \frac{\sigma^2(x)}{2}F''(x) + \alpha\,[F(x+\epsilon) - F(x)]\,. \tag{3.4}$$

Although Eq. (3.2) is obviously difficult to solve explicitly, by choosing the functions f_Y and K appropriately, the variety of problems for which the optimal control $u^*(x)$ is a power of x is very large. We will present below various examples of such problems. The same cases for $u^*(x)$ as in the preceding section will be considered. In financial mathematics, an example of an LQG homing problem might consist in finding the optimal investment policy when the investor decides to sell his/her shares of a given company the first time they reach a certain level, which is a random time. An optimal solution that is a simple power function is very easy to implement.

Case I. If $u^*(x) \equiv 0$, so that $F(x) \equiv F_0$, we saw in Section 2 that we must then have $\lambda = 0$ and $K(a) = K(b) = F_0$, so that the value of the optimal control was obvious. However, in the case of jump-diffusion processes, we can have $u^*(x) \equiv 0$ in non-trivial problems. Indeed, assume that $[a, b] = [0, 1]$ and that $Y \sim \mathrm{U}[-2, 2]$; that is, Y is uniformly distributed on the interval $[-2, 2]$. Then, we have $X[T(x)] \in (-2, 0]$ or $[1, 3)$. Let us define

$$I(x) = \int_{-\infty}^{\infty} F(x+y) f_Y(y)\,\mathrm{d}y. \tag{3.5}$$

We may write that

$$\begin{aligned} I(x) &= \frac{1}{4}\left\{\int_{-2}^{-x} K(x+y)\,\mathrm{d}y + \int_{-x}^{1-x} F(x+y)\,\mathrm{d}y + \int_{1-x}^{2} K(x+y)\,\mathrm{d}y\right\} \\ &= \frac{1}{4}\left\{\int_{-2}^{-x} K(x+y)\,\mathrm{d}y + F_0 + \int_{1-x}^{2} K(x+y)\,\mathrm{d}y\right\}. \end{aligned} \tag{3.6}$$

Let $K(0) = K(1) = F_0$ (as required), but $K(x) \equiv F_1$ if $x \in (-2, 0)$ or $x \in (1, 3)$. We have $I(x) = \frac{1}{4}(3F_1 + F_0)$. We therefore may state that Eq. (3.2) is satisfied if and only if

$$\lambda + \frac{3\alpha}{4}(F_1 - F_0) = 0. \tag{3.7}$$

Thus, when $F_1 \neq F_0$, so that $\lambda \neq 0$ as well, the optimal strategy is nevertheless to use no control at all. This example can obviously be generalized.

Remark. The function $K(x)$ is not necessarily continuous. In fact, it is natural to have a possibly different final cost when there is an overshoot.

Next, in the case when $u^*(x) \equiv u_0 \neq 0$, the value function must be of the form

$$F(x) = \kappa \frac{x^{n-m+1}}{n-m+1} + F_0, \tag{3.8}$$

where

$$\kappa := -\frac{u_0 q_0}{b_0}. \tag{3.9}$$

For the sake of brevity and simplicity, we will assume that $n = m$, so that $F(x) = \kappa x + F_0$. We take again $[a, b] = [0, 1]$, and we choose $K(x) = F(x)$, for $x \notin (0, 1)$. Then, if $Y \sim \mathrm{U}[-2, 2]$ (as above), we calculate

$$I(x) = \frac{1}{4}\int_{-2}^{2}[\kappa(x+y) + F_0]\,\mathrm{d}y = \kappa x + F_0, \tag{3.10}$$

so that we return to the case when there are no jumps, that is, $\alpha = 0$. Instead, let us take $Y \sim \mathrm{U}(0,1)$, which implies that there are only positive jumps. With this choice, we have

$$I(x) = \int_0^1 [\kappa(x+y) + F_0]\,\mathrm{d}y = \kappa\left(x + \frac{1}{2}\right) + F_0 = F(x) + \frac{\kappa}{2}. \tag{3.11}$$

It follows that Eq. (3.2) reduces to

$$0 = -\frac{1}{2}\frac{b_0^2}{q_0}\kappa^2 + \lambda + f(x)\kappa + \frac{\alpha\kappa}{2} = \lambda + \kappa\left(\frac{1}{2}b_0 u_0 + f(x) + \frac{\alpha}{2}\right). \tag{3.12}$$

Therefore, $f(x)$ must be a constant f_0 such that the above equation is satisfied, and we can choose any admissible infinitesimal variance $\sigma^2(x)$. In particular, the continuous part of the process $\{X_u(t), t \geq 0\}$ could be a controlled Brownian motion.

Case II. We make the following assumptions: the optimal control $u^*(x)$ is of the form $u^*(x) = u_0 x$, where $u_0 \neq 0$, the interval $[a,b]$ is $[0,1]$ and $m = n = 0$. We have

$$F(x) = \kappa\frac{x^2}{2} + F_0 \quad \text{for } x \in (0,1). \tag{3.13}$$

As above, we choose $K(x) = F(x)$ for $x \notin (0,1)$. With $Y \sim \mathrm{U}(-2,2)$, we obtain that

$$I(x) = \frac{1}{4}\int_{-2}^{2}\left[\kappa\frac{(x+y)^2}{2} + F_0\right]\mathrm{d}y = F(x) + \frac{2\kappa}{3}. \tag{3.14}$$

Then, Eq. (3.2) becomes

$$0 = -\frac{1}{2}q_0 u_0^2 x^2 + \lambda + f(x)\kappa x + \frac{\kappa}{2}\sigma^2(x) + \frac{2\alpha\kappa}{3}. \tag{3.15}$$

There are numerous important processes for which the above equation holds, including the cases when the continuous part of $\{X_0(t), t \geq 0\}$ is an Ornstein-Uhlenbeck process, or a geometric Brownian motion.

Case III. Suppose that $[a,b] = [1,2]$, $m = 0$, $n = 1$ and $u^*(x) = u_0/x$, where $u_0 \neq 0$. The value function $F(x)$ becomes

$$F(x) = \kappa x + F_0. \tag{3.16}$$

With $K(x) = F(x)$ for $x \notin (1,2)$ and $Y \sim \mathrm{U}(0,1)$, we have

$$I(x) = \int_0^1 [\kappa(x+y) + F_0]\,\mathrm{d}y = F(x) + \frac{\kappa}{2}. \tag{3.17}$$

Hence, Eq. (3.2) is

$$0 = -\frac{q_0 u_0^2}{2x} + \lambda + f(x)\kappa + \frac{\alpha\kappa}{2}. \tag{3.18}$$

The continuous part of $\{X_0(t), t \geq 0\}$ could be a Bessel process.

4. Conclusion

In this paper, we obtained various explicit and exact solutions to LQG homing problems for important one-dimensional diffusion processes by considering the inverse problem. Instead of trying to find the solution to the appropriate non-linear second-order differential equation satisfied by the value function, from which the optimal control follows at once, we looked for problems for which the optimal control $u^*(x)$ was either a constant, a linear function of x or inversely proportional to x. We saw that there are indeed interesting problems for which the exact solution is simple.

We could have considered other cases, but the aim was to present solutions to realistic problems involving important diffusion processes, such as the Wiener process and geometric Brownian motion, rather than purely mathematical examples. Moreover, we could of course consider other particular forms for the optimal control; for instance, the case when $u^*(x)$ is a quadratic function of x is of interest.

Finally, in Section 3 we presented an extension of the inverse LQG homing problem to the important case of jump-diffusion processes. Although the equation satisfied by the value function is much more complicated, we saw that it is possible to find many interesting examples for which the optimal control u_0^* is a constant or a power of x.

As a sequel to this paper, we could consider multidimensional LQG homing problems, either for diffusion or jump-diffusion processes. There are few such problems that have been solved explicitly and exactly so far in two or more dimensions, because the equation satisfied by the value function is then a non-linear partial differential (or integro-differential) equation. Therefore it would indeed be interesting to find important problems that actually have simple solutions.

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