Известия НАН Армении, Математика, том 57, н. 1, 2022, стр. 77 – 84.

ALGORITHM OF CALCULATION OF COMBINED COMMODITY OPTIONS VALUE

H. KECHEJIAN, V. K. OHANYAN AND V. G. BARDAKHCHYAN

https://doi.org/10.54503/0002-3043-2022.57.1-77-84

Freepoint Commodities, Stamford, CT, USA
Yerevan State University, Yerevan, Armenia[1]
E-mails: *hkechejian@hotmail.com; victoohanyan@ysu.am*
vardanbardakchyan@gmail.com

Abstract. General exotic commodity options involving more than one price process are modeled by an ordinary stochastic differential equation and mostly priced by either closed formula if one is derived or via Monte Carlo simulation. In this paper we derive some helpful simplification for general class of exotic switch options, with more than two commodity products, for less costly Monte Carlo simulation.

MSC2010 numbers: 97M30; 91B25; 65C30.

Keywords: commodity options; switch options; multivariate Gaussian distribution.

1. Introduction

The present paper continues the investigations begun in [1]-[6]. Commodity derivatives combined with other assets or containing several commodities generally require numerical techniques for pricing. For most cases with difference of two assets one uses Margrabe's formula (see [7]). Though this formula describes pricing for assets with predetermined dividend rate, one can derive analogous formula for case of two commodities (without any prescribed rate, but in complete market). The inspiration for this paper comes from market executed physical commodity sale/purchase contracts embedded with options. In one such case the physical commodity purchaser has the option of buying the commodity at the minimum of 2 future contracts. Denote these two future contracts prices by $F(t, T_1)$ and $G(t, T_2)$ at time t, where T_1 and T_2 are maturities of two contracts. Hereafter we always take risk free rate to be 0.

Note that the negative sign denotes cash out during the purchase we can write down the following equation dropping T_1 and T_2 but keeping in mind that $t_1 \le T_1 < t_2 \le T_2$.

$$-\min(G(t_2), F(t_1)) = -G(t_2) + \max(G(t_2) - F(t_1), 0) = -G(t_2) + (G(t_2) - F(t_1))^+$$

To see this explicitly for commodity market, consider the following derivative

$$V = E((G(t_2) - aF(t_1))^+)$$

[1]The research of the second author is partially supported by the Mathematical Studies Center at Yerevan State University.

with $G(t_2)$ and $F(t_1)$ representing different commodities market prices at different time, while E stands for expectation. This is a kind of exchange or switch option.

When working within Black-Scholes framework (with 1 driver for each process), the closed-form formula is the following

$$\begin{aligned} V &= x_2\Phi\left(\frac{-\ln a + \ln\frac{x_2}{x_1} + \frac{\sigma_1^2}{2}t_1 + \frac{\sigma_2^2}{2}t_2 - \sigma_1\sigma_2\rho t_1}{\sqrt{\sigma_2^2 t_2 + \sigma_1^2 t_1 - 2\sigma_1\sigma_2\rho t_1}}\right) \\ &- a x_1\Phi\left(\frac{-\ln a + \ln\frac{x_2}{x_1} - \frac{\sigma_1^2}{2}t_1 - \frac{\sigma_2^2}{2}t_2 + \sigma_1\sigma_2\rho t_1}{\sqrt{\sigma_2^2 t_2 + \sigma_1^2 t_1 - 2\sigma_1\sigma_2\rho t_1}}\right) \end{aligned} \tag{1.1}$$

With $x_2 = G(0)$ and $x_1 = F(0)$, σ_2^2 and σ_1^2 are respective variances, and ρ is instantaneous correlation of two commodities, thus correlation of two commodities is ρt_1.

Formula (1.1) naturally reduces to two special cases (thus it is generalization of both of them).

1. Whenever we have $G(t) = F(t)$ (meaning dealing with same commodity future but in different times), the formula reduces to forward starting option.

$$V = x\Phi\left(\frac{-\ln a + \frac{\sigma^2}{2}(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}}\right) - ax\Phi\left(\frac{-\ln a - \frac{\sigma^2}{2}(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}}\right)$$

2. Whenever $a = 1$; and $t_2 = t_1 = t$, (1.1) reduces to Margrabe's formula (with 0 dividends).

$$V = x_2\Phi\left(\frac{\ln\frac{x_2}{x_1} + \frac{\sigma_1^2}{2}t + \frac{\sigma_2^2}{2}t - \sigma_1\sigma_2\rho t}{\sqrt{\sigma_2^2 t + \sigma_1^2 t - 2\sigma_1\sigma_2\rho t}}\right) - x_1\Phi\left(\frac{\ln\frac{x_2}{x_1} - \frac{\sigma_1^2}{2}t - \frac{\sigma_2^2}{2}t + \sigma_1\sigma_2\rho t}{\sqrt{\sigma_2^2 t + \sigma_1^2 t - 2\sigma_1\sigma_2\rho t}}\right)$$

It is the case where both commodities have one correlated driver having correlation ρt (all parameters either known, or calibrated). This formula is quite easy to check (see [8], and compare it to Margrabe's formula).

When dealing with more than 2 commodities, we cannot derive closed form formula for general case. Attempts to simplify computation have been made. Some of applied models can be found in ([9, 10]). For mean reverting process model pricing see [8]. For spot price spread options modelling based on given forward curve dynamics see [11]. The simplifications here are done to make numerical analysis faster.

As closed-form formulas are intractable or cannot be explicitly derived by use of elementary functions, Monte Carlo simulation is used for the case of more than 2 commodities (see [12])

$$V_s = E(h(G(t_3), H(t_2), F(t_1))|F_s) \tag{1.2}$$

We consider an exact type of switch option. The aim of the paper is to provide an easier formula to simplify Monte Carlo simulation and make the process faster, as generally Monte Carlo simulation with even 3 commodities require more than 10^{9k} simulations at once, where k is multiplicity (order) of simulations.

If we could somehow reduce the dimension by one, we will make it 10^{5k} times faster. For one exact type of 3 commodity based derivative we give algorithm based on direct computation, and with computation of integral of bivariate normal distribution. We give all necessary formulas as well as order of computation. In this final form only 10^{2k} simulation are needed.

2. Statement of problem with three futures

As we deal with switch (exchange) options, we take exact form of function h(.) in (1.2), namely

$$\begin{aligned} V_s &= E(\max(G(t_3) - \min(F(t_1); H(t_2)), 0)|F_s) \\ &= E((G(t_3) - \min(F(t_1); H(t_2)))^+|F_s) \end{aligned}$$

If all three commodities G, F, H are driven with one Wiener process and can be brought to martingale form, we would like to have some integral formula involving bivariate normal distribution. Hereafter we will take $s = 0$, and use $V = V^0$ notation. Taking the processes to be

$$\begin{aligned} &F(t_1) = F(0)e^{-\sigma_1 Y_1 \sqrt{t_1} + t_1\theta_1}, \quad H(t_2) = H(0)e^{-\sigma_2 Y_2 \sqrt{t_2} + t_2\theta_2} \\ &G(t_3) = G(0)e^{-\sigma_3 Y_3 \sqrt{t_3} + t_3\theta_3}. \end{aligned} \tag{2.1}$$

Without loss of generality we assume $t_3 > t_2 > t_1$ and

$$(F(t_1), H(t_2), G(t_3)) \sim N\left(\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \Sigma = \begin{pmatrix} 1 & \frac{\rho_{12}\sqrt{t_1}}{\sqrt{t_2}} & \frac{\rho_{13}\sqrt{t_1}}{\sqrt{t_3}} \\ \frac{\rho_{12}\sqrt{t_1}}{\sqrt{t_2}} & 1 & \frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} \\ \frac{\rho_{13}\sqrt{t_1}}{\sqrt{t_3}} & \frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} & 1 \end{pmatrix}\right). \tag{2.2}$$

Vector $(F(t_1), H(t_2), G(t_3))$ has 3-dimensional normal distribution with mean vector μ and covariation matrix Σ, with $\rho_{12}, \rho_{13}, \rho_{23}$ are correlations (instantaneous) between respective commodities $(1 \to F; 2 \to H; 3 \to G)$. In general, making use of Monte Carlo simulation, you will need $3 \times 3 \times k$ simulations, to calculate approximate price.

What we do, is simplification of formula through direct computation, which later yield to only 2 random variables to simulate. The nice part of it is that the derived formula does not need combined simulation of bivariate normal random variables. So there is no need to simulate $2 \times 2 \times k$, but rather $(1 + 1) \times k$, with the cost of other additional computations.

3. The main formula

Having the above framework, we can rewrite

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max\Big(G(0)e^{-\sigma_3 y_3\sqrt{t_3}+t_3\theta_3} - F(0)e^{-\sigma_1 y_1\sqrt{t_1}+t_1\theta_1}, \quad G(0)e^{-\sigma_3 y_3\sqrt{t_3}+t_3\theta_3} - H(0)e^{-\sigma_2 y_2\sqrt{t_2}+t_2\theta_2}\Big)\, dy_1 dy_2 dy_3. \tag{3.1}$$

We divide the region of integration into three parts.
1. $G(t_3) - F(t_1) \geq \max((G(t_3) - H(t_2), 0))$;
2. $G(t_3) - H(t_2) > \max(G(t_3) - F(t_1), 0)$;
3. $0 > \max(G(t_3) - F(t_1), G(t_3) - H(t_2))$.

These regions do not overlap, their union represent the whole domain of integration, and on the 3rd region the integral yields 0. Now back to the first and 2nd regions. The first region can be understood equivalently $G(t_3) > F(t_1)$ & $H(t_2) > F(t_1)$ The 2nd region can be understood as $G(t_3) > H(t_2)$ & $H(t_2) < F(t_1)$ So we can rewrite $V = V_1 + V_2$. The final formula will have the following complicated form, where Φ_2 stands for bivariate normal distribution function: for V_1

$$\begin{aligned} &P_1 \int_{-\infty}^{\infty} e^{A_{1,I}} \Phi_2(y_2 = f_{A,2}(y_1); y_3 = f_{A,3}(y_1); m_{A,II}, m_{A,III}, \sigma_{II}, \sigma_{III}, \rho_{II,III}) dy_1 \\ &-Q_1 \int_{-\infty}^{\infty} e^{B_{1,I}} \Phi_2(y_2 = f_{B,2}(y_1); y_3 = f_{B,3}(y_1); m_{B,II}, m_{B,III}, \sigma_{II}, \sigma_{III}, \rho_{II,III}) dy_1 \end{aligned} \tag{3.2}$$

with the components

$$P_1 = \frac{G(0)e^{t_3\theta_3}\sigma_{II}\sigma_{III}\sqrt{1-\rho_{II,III}^2}}{(2\pi)^{\frac{1}{2}}|\Sigma|^{\frac{1}{2}}}, \quad Q_1 = \frac{F(0)e^{t_1\theta_1}\sigma_{II}\sigma_{III}\sqrt{1-\rho_{II,III}^2}}{(2\pi)^{\frac{1}{2}}|\Sigma|^{\frac{1}{2}}} \tag{3.3}$$

$$\begin{aligned} f_{A,2}(y_1) = f_{B,2}(y_1) &= \frac{t_2\theta_2 - t_1\theta_1}{\sigma_2\sqrt{t_2}} + \frac{\sigma_1\sqrt{t_1}}{\sigma_2\sqrt{t_2}}y_1 + \frac{1}{\sigma_2\sqrt{t_2}}\ln\left(\frac{H(0)}{F(0)}\right) \\ f_{A,3}(y_1) = f_{B,3}(y_1) &= \frac{t_3\theta_3 - t_1\theta_1}{\sigma_3\sqrt{t_3}} + \frac{\sigma_1\sqrt{t_1}}{\sigma_3\sqrt{t_3}}y_1 + \frac{1}{\sigma_3\sqrt{t_3}}\ln\left(\frac{G(0)}{F(0)}\right) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \gamma_1 x_2 &= \left(\frac{(\rho_{12}\rho_{23} - \rho_{13})\sqrt{t_1}}{\sqrt{t_3}} + \frac{\sigma_3\sqrt{t_3}|\Sigma|}{y_1}\right)\beta_1 - \left(1 - \frac{\rho_{12}^2 t_1}{t_2}\right)\alpha_1 \\ \gamma_1 x_3 &= \alpha_1\beta_1 - \left(1 - \frac{\rho_{13}^2 t_1}{t_3}\right)\left(\frac{(\rho_{12}\rho_{23} - \rho_{13})\sqrt{t_1}}{\sqrt{t_3}} + \frac{\sigma_3\sqrt{t_3}|\Sigma|}{y_1}\right) \end{aligned}$$

$$
\begin{aligned}
&\gamma_1 z_2 = \frac{(\rho_{12}\rho_{23}-\rho_{13})\sqrt{t_1}}{\sqrt{t_3}}\beta_1 - \left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)\alpha_1\\
&\gamma_1 z_3 = \alpha_1\beta_1 - \left(1-\frac{\rho_{13}^2 t_1}{t_3}\right)\frac{(\rho_{12}\rho_{23}-\rho_{13})\sqrt{t_1}}{\sqrt{t_3}}\\
&\alpha_1 := -\frac{\rho_{12}\sqrt{t_1}}{\sqrt{t_2}} + \frac{\rho_{13}\rho_{23}\sqrt{t_1 t_2}}{t_3}, \quad \beta_1 := -\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} + \frac{\rho_{12}\rho_{13}t_1}{\sqrt{t_3 t_2}}\\
&\gamma_1 := \left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} + \frac{\rho_{12}\rho_{13}t_1}{\sqrt{t_3 t_2}}\right)^2 - \left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right)
\end{aligned}
\tag{3.5}
$$

$$
\begin{gathered}
m_{A,II} = -y_1 x_2; m_{A,III} = -y_1 x_3\\
m_{B,II} = -y_1 z_2; m_{B,III} = -y_1 z_3
\end{gathered}
\tag{3.6}
$$

$$
\begin{gathered}
\sigma_{II} = \frac{\sqrt{|\Sigma|}}{1-\frac{\rho_{13}^2 t_1}{t_3}}\sqrt{1-\frac{\rho_{13}^2 t_1}{t_3} - \frac{\left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}}+\frac{\rho_{12}\rho_{13}t_1}{\sqrt{t_3 t_2}}\right)^2}{\left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)}}\\
\sigma_{III} = \frac{\sqrt{|\Sigma|}}{1-\frac{\rho_{12}^2 t_1}{t_2}}\sqrt{1-\frac{\rho_{12}^2 t_1}{t_2} - \frac{\left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}}+\frac{\rho_{12}\rho_{13}t_1}{\sqrt{t_3 t_2}}\right)^2}{\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right)}}\\
\rho_{II,III} = \frac{\rho_{23}t_2 - \rho_{12}\rho_{13}t_1}{\sqrt{(t_3-\rho_{13}^2 t_1)(t_2-\rho_{12}^2 t_1)}}
\end{gathered}
\tag{3.7}
$$

$$
\begin{aligned}
A_{1,I} = -\frac{1}{2|\Sigma|}&\left(y_1^2\left(1-\frac{\rho_{23}^2 t_2}{t_3}\right) - \right.\\
&y_1^2\left(\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right)x_2^2 + \left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)x_3^2 + \right.\\
&\left.\left.2\sqrt{\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right)\left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)}\left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}}+\frac{\rho_{12}\rho_{13}t_1}{\sqrt{t_3 t_2}}\right)x_2 x_3\right)\right)\\
B_{1,I} = -\frac{1}{2|\Sigma|}&\left(2|\Sigma|\sigma_1 y_1\sqrt{t_1} + y_1^2\left(1-\frac{\rho_{23}^2 t_2}{t_3}\right) - \right.\\
&y_1^2\left(\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right)z_2^2 + \left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)z_3^2 + \right.\\
&\left.\left.2\sqrt{\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right)\left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)}\left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}}+\frac{\rho_{12}\rho_{13}t_1}{\sqrt{t_3 t_2}}\right)z_2 z_3\right)\right),
\end{aligned}
\tag{3.8}
$$

where

$$
|\Sigma| = 1 + \frac{2\rho_{12}\rho_{13}\rho_{23}t_1}{t_3} - \frac{\rho_{13}^2 t_1}{t_3} - \frac{\rho_{23}^2 t_2}{t_3} - \frac{\rho_{12}^2 t_1}{t_2}
\tag{3.9}
$$

Each term (3.3)-(3.9) should be calculated precisely in the following order. First fix some y_1. Then do the calculation like this (3.9) $\to$ (3.7) $\to$ (3.3)(3.5) $\to$ (3.4)(3.6)(3.8). Here (3.6) and (3.7) represents all parameters of both bivariate normal distributions. Note that they change with the change of y_1. Before computing (3.3) one can calculate integrals first. As (3.3) does not depend on y_1. For calculating integral just make y_1 change through some interval (with small steps note also that one cannot say that interval $(-3\sigma, 3\sigma)$, as σ changes with y_1). We obtain

(3.10)

$$V_2 = P_2 \int_{-\infty}^{\infty} e^{C_{2,I}} \Phi_2(y_1 = f_{C,1}(y_2); y_3 = f_{C,3}(y_2); m_{D,I}, m_{D,III}, \sigma_{C,I}, \sigma_{C,III}, \rho_{C,I,III}) dy_2$$
$$-Q_2 \int_{-\infty}^{\infty} e^{D_{2,I}} \Phi_2(y_1 = f_{D,1}(y_2); y_3 = f_{D,3}(y_2); m_{D,II}, m_{D,III}, \sigma_{C,II}, \sigma_{C,III}, \rho_{C,I,III}) dy_2$$

with the components

$$P_2 = \frac{G(0) e^{t_3\theta_3} \sigma_{C,I} \sigma_{C,III} \sqrt{1-\rho_{C,I,III}^2}}{(2\pi)^{\frac{1}{2}} |\Sigma|^{\frac{1}{2}}}$$
$$Q_2 = \frac{H(0) e^{t_2\theta_2} \sigma_{C,I} \sigma_{C,III} \sqrt{1-\rho_{C,I,III}^2}}{(2\pi)^{\frac{1}{2}} |\Sigma|^{\frac{1}{2}}} \tag{3.11}$$

$$f_{C,1}(y_2) = f_{D,3}(y_2) = \frac{t_3\theta_3 - t_2\theta_2}{\sigma_3\sqrt{t_3}} + \frac{\sigma_2\sqrt{t_2}}{\sigma_3\sqrt{t_3}} y_1 + \frac{1}{\sigma_3\sqrt{t_3}} \ln\left(\frac{G(0)}{H(0)}\right)$$
$$f_{C,3}(y_2) = f_{D,1}(y_2) = \frac{t_1\theta_1 - t_2\theta_2}{\sigma_1\sqrt{t_1}} + \frac{\sigma_2\sqrt{t_2}}{\sigma_1\sqrt{t_1}} y_1 + \frac{1}{\sigma_2\sqrt{t_2}} \ln\left(\frac{F(0)}{H(0)}\right) \tag{3.12}$$

and

$$\gamma_2 v_1 = \beta_2 \left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} + \frac{\rho_{12}\rho_{13} t_1}{\sqrt{t_3 t_2}} + \frac{\sigma_3\sqrt{t_3}|\Sigma|}{y_2}\right) - \left(1 - \frac{\rho_{12}^2 t_1}{t_2}\right) \alpha_2$$
$$\gamma_2 v_3 = \beta_2\alpha_2 - \left(1 - \frac{\rho_{23}^2 t_2}{t_3}\right)\left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} + \frac{\rho_{12}\rho_{13} t_1}{\sqrt{t_3 t_2}} + \frac{\sigma_3\sqrt{t_3}|\Sigma|}{y_2}\right)$$
$$\gamma_2 w_1 = \beta_2 \left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} + \frac{\rho_{12}\rho_{13} t_1}{\sqrt{t_3 t_2}}\right) - \left(1 - \frac{\rho_{12}^2 t_1}{t_2}\right) \alpha_2$$
$$\gamma_2 w_3 = \beta_2\alpha_2 - \left(1 - \frac{\rho_{23}^2 t_2}{t_3}\right)\left(-\frac{\rho_{23}\sqrt{t_2}}{\sqrt{t_3}} + \frac{\rho_{12}\rho_{13} t_1}{\sqrt{t_3 t_2}}\right)$$
$$\alpha_2 = \alpha_1; \beta_2 := \frac{(\rho_{12}\rho_{23} - \rho_{13})\sqrt{t_1}}{\sqrt{t_3}}$$
$$\gamma_2 := \left(\frac{(\rho_{12}\rho_{23} - \rho_{13})\sqrt{t_1}}{\sqrt{t_3}}\right)^2 - \left(1 - \frac{\rho_{12}^2 t_1}{t_2}\right)\left(1 - \frac{\rho_{23}^2 t_2}{t_3}\right) \tag{3.13}$$

$$m_{C,I} = -y_2 v_1;\ m_{C,III} = -y_2 v_3$$
$$m_{D,I} = -y_2 w_1;\ m_{D,III} = -y_2 w_3 \tag{3.14}$$

$$
\sigma_{C,I} = \frac{\sqrt{|\Sigma|}}{1-\frac{\rho_{23}^2 t_2}{t_3}} \sqrt{1-\frac{\rho_{23}^2 t_2}{t_3} - \frac{\left(\frac{(\rho_{12}\rho_{23}-\rho_{13})\sqrt{t_1}}{\sqrt{t_3}}\right)^2}{\left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)}}
$$

$$
\sigma_{C,III} = \frac{\sqrt{|\Sigma|}}{1-\frac{\rho_{12}^2 t_1}{t_2}} \sqrt{1-\frac{\rho_{12}^2 t_1}{t_2} - \frac{\left(\frac{(\rho_{12}\rho_{23}-\rho_{13})\sqrt{t_1}}{\sqrt{t_3}}\right)^2}{\left(1-\frac{\rho_{23}^2 t_2}{t_3}\right)}} \tag{3.15}
$$

$$
\rho_{C,II,III} = \frac{-(\rho_{12}\rho_{23}-\rho_{13})\sqrt{t_1 t_2}}{\sqrt{(t_3-\rho_{23}^2 t_2)\,(t_2-\rho_{12}^2 t_1)}}
$$

$$
\begin{aligned}
C_{2,I} = -\frac{1}{2|\Sigma|}\Bigg(& y_2^2\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right) - \\
& y_2^2\Bigg(\left(1-\frac{\rho_{23}^2 t_2}{t_3}\right)v_1^2 + \left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)v_3^2 + \\
& 2\sqrt{\left(1-\frac{\rho_{23}^2 t_2}{t_3}\right)\left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)}\frac{(\rho_{12}\rho_{23}-\rho_{13})\sqrt{t_1}}{\sqrt{t_3}}v_1 v_3\Bigg)\Bigg) \\
D_{2,I} = -\frac{1}{2|\Sigma|}\Bigg(& 2|\Sigma|\sigma_2 y_2\sqrt{t_1} + y_2^2\left(1-\frac{\rho_{13}^2 t_1}{t_3}\right) - \\
& y_2^2\Bigg(\left(1-\frac{\rho_{23}^2 t_2}{t_3}\right)w_1^2 + \left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)w_3^2 + \\
& 2\sqrt{\left(1-\frac{\rho_{23}^2 t_2}{t_3}\right)\left(1-\frac{\rho_{12}^2 t_1}{t_2}\right)}\frac{(\rho_{12}\rho_{23}-\rho_{13})\sqrt{t_1}}{\sqrt{t_3}}w_1 w_3\Bigg)\Bigg)
\end{aligned} \tag{3.16}
$$

Generate y_2. Then make the calculation like this (3.9) $\rightarrow$ (3.15)(3.13) $\rightarrow$ (3.11) $\rightarrow$ (3.12)(3.14)(3.16).

Once you take inputs and combine them into formula (3.1), the remaining part is usage general Cholesky decomposition techniques (for 2 variable case it takes extremely simple form, see for example Box-Muller transform [12]), and generate 2 correlated variables in the form given in V_1 and V_2. This is exactly what is done in (3.4) and (3.12). Note, however, that one needs only one standard normal variable in each part. So first generate 2 normal random variables with prescribed bivariate distribution, then use each one in the integral.

Concluding algorithm:

1. Calibrate the parameters of the model parameters in (2.1) (either using least square or other techniques), plus find correlations from (2.2) (with correlation matrices estimation techniques).
2. Calculate (3.9) $\rightarrow$ (3.4) $\rightarrow$ (3.3)(3.5).
3. Generate y_1 running through some interval.
4. Calculate for each y_1 (3.4)(3.6)(3.8).
5. Then calculate normal distrbution values from (3.2) first and second integral. Multiply them by exponents in (3.8), and interval length of each step of change of y_1. Sum up them to compute integrals.

6. Find the value of (3.2).
7. Use (3.9) and further compute. (3.15)(3.13) $\rightarrow$ (3.11).
8. Generate y_2 running through some interval.
9. Calculate for each y_2 (3.12)(3.14)(3.16).
10. The same as in step 5, but for the second part (3.10) and using (3.16).
11. Find the value of (3.10).
12. Sum (3.2) and (3.10) to find (3.1).

Note that you need to generate only 1 variable in each case. In given form one needs to generate $\sim 10^{2k}$. However more computation should be carried out at each step. So the general result can be formulated as follows.

Theorem 3.1. *The price of (1.2) with given properties of h(.) function can be found by formula* $V = V_1 + V_2$, *where* V_1 *and* V_2 *can be calculated by (3.2) and (3.10).*

Remark. The described algorithm shows reduction in size of simulations needed for pricing by up to 10^{7k} times.

Список литературы

[1] H. Kechejian, V. K. Ohanyan, “Tolling contracts”, Proceedings of the 6-th working conference “Reliability and optimization of structural systems”, 231 -– 236 (2012).
[2] H. Kechejian, V. K. Ohanyan, V. G. Bardakhchyan, “Tolling contracts with two driving prices”, Russian Journal of Mathematical Research, Series A, **1**, 14 -– 19 (2015).
[3] H. Kechejian, V. K. Ohanyan, V. G. Bardakhchyan, “Geometric average options for several futures”, Vestnik of Kazan State Power Engineering University, **3** (31), 171 -– 180 (2016).
[4] H. Kechejian, V. K. Ohanyan, V. G. Bardakhchyan, “Monte Carlo method for geometric average options on several futures”, Russian Journal of Mathematical Research, Series A, **1**, 10 -– 15 (2016).
[5] H. Kechejian, V. K. Ohanyan, V. G. Bardakhchyan, “Gas storage valuation based on spot prices”, Modeling of Artificial Intelligence, **5** (1), 22 -— 28 (2018).
[6] H. Kechejian, V. K. Ohanyan, V. G. Bardakhchyan, “On Poisson Mixture of lognormal distributions”, Lobachevski Journal of Mathematics, **41** (3), 340 – 348 (2020).
[7] W. Margrabe, “The value of an option to exchange one asset for another”, Journal of Finance, **33**, n. 1, 177 – 186 (1978).
[8] R. Carmona, V. Durrleman, “Pricing and Hedging Spread Options in a Log-Normal Model”, Working Paper, Department of Operations Research and Financial Engingeering, Princeton University, SIAM Review **45** (4) (2003).
[9] I. Karatzas, S. Shreve, Mathematical Finance, Springer Verlag, New York (2000).
[10] G. Poitras, “Spread options, exchange options, and arithmetic Brownian motion”, The Journal of Futures Markets, **18**, 487 – 517 (1998).
[11] S. Hikspoors, S. Jaimungal, “Energy spot price models and spread options pricing”, International Journal of Theoretical and Applied Finance, **10** (7), 1111 – 1135 (2007).
[12] P. Glasserman, Monte Carlo Methods in Financial Engineering, Springer (2004).

Поступила 14 мая 2021

После доработки 6 августа 2021

Принята к публикации 20 сентября 2021