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INFINITELY MANY SOLUTIONS FOR NEUMANN PROBLEMS ASSOCIATED TO NON-HOMOGENEOUS DIFFERENTIAL OPERATORS THROUGH ORLICZ-SOBOLEV SPACES

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Abstract. The goal of this paper is to establish the existence of an unbounded sequence of weak solutions for the following non-homogeneous Neumann problem

$$\begin{cases} -\operatorname{div}\big(a(x,|\nabla u(x)|)\nabla u(x)\big) + a(x,|u(x)|)u(x) = \lambda f\big(x,u(x)\big) & \text{for } x \in \Omega, \\ a\big(x,|\nabla u(x)|\big)\dfrac{\partial u}{\partial v}(x) = \mu g\big(\gamma(u(x))\big) & \text{for } x \in \partial\Omega. \end{cases}$$

To deal with the existence of the mention solutions, we use the variational methods and critical point theory, in an appropriate Orlicz-Sobolev space.

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1. Introduction

In this paper, we study the non-homogeneous Neumann problem

(1.1)
$$\begin{cases} -\operatorname{div}\big(a(x,|\nabla u(x)|)\nabla u(x)\big) + a(x,|u(x)|)u(x) = \lambda f\big(x,u(x)\big) & \text{for } x \in \Omega, \\ a\big(x,|\nabla u(x)|\big)\dfrac{\partial u}{\partial v}(x) = \mu g\big(\gamma(u(x))\big) & \text{for } x \in \partial\Omega. \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial v}$ is the outer unit normal derivative, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, λ and μ are real parameters with $\lambda > 0$ and $\mu \geq 0$, and the functions $a(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and γ will be specified later.

In recent years, quasilinear elliptic partial differential equations involving non-homogeneous differential operators are becoming increasingly important in applications in many fields of mathematics, such as approximation theory, mathematical physics (electrorheological fluids, nonlinear elasticity and plasticity), calculus of ariations, nonlinear potential theory, the theory of quasi-conformal mappings, differential geometry, geometric function theory, probability theory (for instance see [9, 16]). Another recent application which uses non-homogeneous differential operators can be found in the framework of image processing (see [5]). The study of nonlinear elliptic equations involving quasilinear homogeneous type operators is based on the theory of Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. In the case of non-homogeneous differential operators, the natural setting for this approach is the use of Orlicz-Sobolev spaces. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. Many properties of Orlicz-Sobolev spaces come in [1, 7, 8], [17]-[21]. The importance of Orlicz-Sobolev spaces arises from its applications to many different fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, nonlinear potential theory, quasiconformal mappings, differential geometry, geometric function theory, and probability theory, for instance see [9, 16]. Due to these, in recent years, the existence of solutions for the eigenvalue problems involving non-homogeneous operators in the divergence form, have been widely studied by many authors. They have studied the existence of solutions by means of variational methods and critical point theory, monotone operator methods, fixed point theory and degree theory, for instance see [2, 15, 17, 23]. For example, in [3] the author establish some new sufficient conditions under which the problem (1.1) possesses three weak solutions in the Orlicz-Sobolev space. In [2] employing variational methods and critical point theory, in an appropriate Orlicz-Sobolev setting, the existence of infinitely many solutions for Steklov problems associated to non-homogeneous differential operators was established. In [23] the authors establish the existence of infinitely many non-negative weak solutions to the non-homogeneous Neumann problem:

$$\begin{cases} -\mathrm{div}(a(|\nabla u(x)|)\nabla u(x))u(x) + a(|u(x)|)u(x) = \lambda h(x) f(u(x)) & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $f : \mathbb{R} \to \mathbb{R}$ and $h : \bar{\Omega} \to [0, +\infty)$ are continuous functions, λ is positive parameter, and the functions $a : (0, +\infty) \to \mathbb{R}$ is such that the mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \begin{cases} a(|t|)t & \text{for} \quad t \neq 0, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

is an odd and increasing homeomorphism $\mathbb{R}$ onto $\mathbb{R}$.

In the present paper, motivated by the above facts and the papers [6, 13], under an appropriate oscillating behavior of the primitive of the nonlinearity and a suitable growth of the primitive of g at infinity, when $p^- := \inf_{x \in \Omega} p(x) > N$, the existence of infinitely many weak solutions for the problem (1.1) in the Orlicz-Sobolev space, is obtained, for all λ belonging to a precise interval and provided μ small enough (Theorem 3.1).

A special case of Theorem 3.1 is the following theorem, when $g(t) = 0$ for all $t \in \mathbb{R}$.

Theorem 1.1. *Let $p \in C(\overline{\Omega})$ with $p^- > N$ for all $x \in \overline{\Omega}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$ and assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^{\phi_0}} = 0 \qquad \text{and} \qquad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^{\phi^0}} = +\infty.$$

Then, the problem

$$\begin{cases} -div\big(\alpha(x, |\nabla u(x)|)\nabla u(x)\big) + \alpha(x, |u(x)|)u(x) = f(u(x)) & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

admits infinitely many weak solutions in $W^1L_\Phi(\Omega)$.

This paper is organized as follows. In section 2, the abstract critical point theorem (Theorem 2.1) is recalled. Moreover, some preliminaries and the abstract Orlicz-Sobolev spaces setting are presented. In Section 3, our main result is established, then some particular case and some example are presented.

2. Preliminaries

Our main tool is the following critical points theorem obtained in [4]. This result is a refinement of the variational principle of Ricceri [22].

Theorem 2.1. *Let X be a reflexive real Banach space, let $J, I : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that J is strongly continuous, sequentially weakly lower semicontinuous and coercive and I is sequentially weakly upper semicontinuous. For every $r > \inf_X J$, put*

$$\varphi(r) := \inf_{u \in J^{-1}(]-\infty, r[)} \frac{\left(\sup_{v \in J^{-1}(]-\infty, r[)} I(v)\right) - I(u)}{r - J(u)},$$

and

$$\overline{\gamma} := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X J)^+} \varphi(r).$$

Then we have the following.

(a) *For every $r > \inf_X J$ and every $\lambda \in]0, \frac{1}{J(r)}[$, the restriction of the functional $T_\lambda := J - \lambda I$ to $J^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of T_λ in X.*

(b) *If $\overline{\gamma} < +\infty$, then, for each $\lambda \in]0, \frac{1}{\overline{\gamma}}[$, the following alternative holds: either the functional $T_\lambda := J - \lambda I$ has a global minimum, or there exists a sequence $\{u_n\}$ of critical points (local minima) of T_λ such that $\lim_{n \to +\infty} J(u_n) = +\infty$.*

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either there exists a global minimum of J which is a local minimum of $T_\lambda := J - \lambda I$, or there exists a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of T_λ, with $\lim_{n\to+\infty} J(u_n) = \inf_X J$, which weakly converges to a global minimum of J.*

We begin by recalling some facts from the theory of Orlicz-Sobolev spaces that will be used in the present paper; for more details we refer the reader to Adams [1], Diening [8], Musielak [18], Rao and Ren [21], Rădulescu [19], Rădulescu and Repovš [20]. Suppose that the function $\alpha(x,t) : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ is such that the mapping $\varphi(x,t) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x,t) = \begin{cases} a(x,|t|)t & \text{for} \quad t \neq 0, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

satisfies the condition(φ) for all $x \in \Omega, \varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and

$$\Phi(x,t) = \int_0^t \varphi(x,s)ds, \qquad \forall\, x \in \bar{\Omega},\, t \geq 0$$

belongs to class Φ (see [18], p. 33), i.e., the function Φ satisfies the following conditions:

(Φ_1) for all $x \in \Omega$, $\phi(x,\cdot) : [0,+\infty) \to \mathbb{R}$ is a non-decreasing continuous function, with $\Phi(x,0) = 0$ and $\Phi(x,t) > 0$ whenever $t > 0$, $\lim_{t\to\infty} \Phi(x,t) = \infty$,

(Φ_2) for every $t \geq 0$, $\Phi(x,\cdot) : \Omega \to \mathbb{R}$ is a measurable function.

Since $\phi(x,\cdot)$ satisfies condition (ϕ), we deduce that $\Phi(x,\cdot)$ is convex and increasing from $\mathbb{R}^+$ to $\mathbb{R}^+$.

For the function Φ, we define the *generalized Orlicz class*,

$$K_\Phi(\Omega) = \{u : \Omega \to \mathbb{R}, \text{ measurable; } \int_\Omega \Phi(x,|u(x)|)\,dx < \infty\}$$

and the *generalized Orlicz class*,

$$L^\Phi(\Omega) = \{u : \Omega \to \mathbb{R}, \text{ measurable; } \lim_{\lambda\to 0^+} \int_\Omega \Phi(x,\lambda|u(x)|)\,dx = 0\}$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$|u|_\Phi = \inf\left\{\mu > 0;\ \int_\Omega \Phi\left(x, \frac{|u(x)|}{\mu}\right) dx \leq 1\right\}$$

or the equivalent norm (the *Orlicz norm*)

$$|u|_{(\Phi)} = \sup\left\{\left|\int_\Omega u\,v\,dx\right|;\ v \in L^{\overline{\Phi}}(\Omega),\ \int_\Omega \overline{\Phi}(x,|v(x)|)\,dx \leq 1\right\}.$$

where $\overline{\Phi}$ denotes the *conjugate Young* function of Φ, that is,

$$\overline{\Phi}(x,t) = \sup_{s>0}\left\{t\,s - \Phi(x,s);\ s \in \mathbb{R}\right\}, \qquad \forall\, x \in \overline{\Omega},\, t \geq 0.$$

Furthermore, for Φ and $\overline{\Phi}$ conjugate Young functions, the Hölder type inequality holds true

$$\left|\int_\Omega u\,v\,dx\right| \leq B \times |u|_\Phi \times |v|_{\overline{\Phi}}, \qquad \forall\, u \in L^\Phi(\Omega),\, v \in L^{\overline{\Phi}}(\Omega), \tag{2.1}$$

where B is a positive constant (see [18, Theorem 13.13]). In this paper we assume that there exist two positive constants φ_0 and φ^0 such that

$$1 < \varphi_0 \leq \frac{t\,\varphi(x,t)}{\Phi(x,t)} \leq \varphi^0 < \infty, \qquad \forall\, x \in \overline{\Omega},\, t \geq 0. \tag{2.2}$$

The above relation implies that Φ satisfies the $\triangle_2$-*condition*, i.e.

$$\Phi(x, 2t) \leq K \times \Phi(x,t), \qquad \forall\, x \in \overline{\Omega},\, t \geq 0. \tag{2.3}$$

where K is a positive constant (see [17, Proposition 2.3]). Relation (2.3) and Theorem 8.13 in [18] imply that $L^\Phi(\Omega) = K_\Phi(\Omega)$. Furthermore, we assume that Φ satisfies the following condition:

$$\text{for each}\, x \in \overline{\Omega}, \text{ the function}\, [0,\infty) \ni t \to \Phi(x, \sqrt{t}) \text{ is convex.} \tag{2.4}$$

Relation (2.4) assures that $L^\Phi(\Omega)$ is an uniformly convex space and thus, a reflexive space (see [17], Proposition 2.2).

On the other hand, we point out that assuming that Φ and Ψ belong to class Φ and

$$\Psi(x,t) \leq K_1 \cdot \Phi(x, K_2.t) + h(x), \qquad \forall\, x \in \overline{\Omega},\, t \geq 0, \tag{2.5}$$

where $h \in L^1(\Omega)$, $h(x) \geq 0$ a.e. $x \in \Omega$ and K_1, K_2 are positive constants, then by Theorem 8.5 in [18] we have that there exists the continuous embedding $L^\Phi(\Omega) \subset L^\Psi(\Omega)$. Next, we define the *generalized Orlicz-Sobolev space*

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^\Phi(\Omega);\, \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega),\, i = 1, \ldots, N \right\}.$$

On $W^{1,\Phi}(\Omega)$ we define the equivalent norms

$$\|u\|_{1,\Phi} = |\,|\nabla u|\,|_\Phi + |u|_\Phi,$$

$$\|u\|_{2,\Phi} = \max\{|\,|\nabla u|\,|_\Phi, |u|_\Phi\},$$

$$\|u\| = \inf\left\{ \mu > 0;\, \int_\Omega \left[\Phi\left(x, \frac{|u(x)|}{\mu}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\mu}\right)\right] dx \leq 1 \right\}.$$

More precisely, for every $u \in W^{1,\Phi}(\Omega)$, we have

$$\|u\| \leq 2\|u\|_{2,\Phi} \leq 2\|u\|_{1,\Phi} \leq 4\|u\| \tag{2.6}$$

(see [17, Proposition 2.4]). The generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ endowed with one of the above norms is a reflexive Banach space.

In the following, we will use the norm $\|\cdot\|$ on $E := W^{1,\Phi}(\Omega)$ and we suppose that $\gamma : E \to L^\Phi(\Omega)$ is the trace operator.

The following lemma is useful in the proof of our results (see, for instance, Lemma 2.3 of [15]).

Lemma 2.1. *Let $u \in E$. Then*

$$\int_\Omega \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx \geq \|u\|^{\phi_0} \qquad \text{if} \quad \|u\| > 1; \tag{2.7}$$

$$\int_\Omega \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx \geq \|u\|^{\phi^0} \qquad \text{if} \quad \|u\| < 1. \tag{2.8}$$

We point out that assuming that Φ and Ψ belong to class Φ, satisfying relation (2.5) and $\inf_{x\in\Omega} \Phi(x, 1) > 0$, $\inf_{x\in\Omega} \Psi(x, 1) > 0$ then there exists the continuous embedding $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,\Psi}(\Omega)$.

In this paper we study the problem (1.1) in the particular case when Φ satisfies

$$M \times |t|^{p(x)} \leq \Phi(x, t), \qquad \forall\, x \in \overline{\Omega},\, t \geq 0, \tag{2.9}$$

where $p(x) \in C(\overline{\Omega})$ with $p^- > N$ for all $x \in \overline{\Omega}$, and $M > 0$ is a constant.

By the relation (2.9) we deduce that E is continuously embedded in $W^{1,p(x)}(\Omega)$ (see relation (2.5) with $\Psi(x, t) = |t|^{p(x)}$).

Moreover, as pointed out in [11] and [14], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and since $p^- > N$, we deduce that $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, E is compactly embedded in $C^0(\overline{\Omega})$, and there exists a constant $m > 0$ such that

$$\|u\|_\infty \leq m\, \|u\|, \quad \forall\, u \in E, \tag{2.10}$$

where $\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$.

Throughout the sequel, $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, and λ and μ are real parameters. Put

$$F(x, t) := \int_0^t f(x, \xi) d\xi \qquad \text{for all}\, (x, t) \in \Omega \times \mathbb{R},$$

and

$$G(t) := \int_0^t g(\xi) d\xi \qquad \text{for all}\, t \in \mathbb{R}.$$

We say that $u \in E$ is a weak solution of the problem (1.1) if

$$\int_\Omega \alpha(x, |\nabla u(x)|)\nabla u(x) \times \nabla v(x)\, dx + \int_\Omega \alpha(x, |u(x)|)\, u(x)\, v(x)\, dx$$
$$= \lambda \int_\Omega f(x, u(x))\, v(x)\, dx + \mu \int_{\partial\Omega} g\Big(\gamma(u(x))\Big)\gamma(v(x))\, d\sigma$$

for every $v \in E$.

3. Main results

We formulate our main result as follows. Let

$$A := \liminf_{\xi\to+\infty} \frac{\int_\Omega \max\limits_{|t|<\xi} F(x,t)dx}{\xi^{\phi_0}}, \qquad B := \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^{\phi^0}},$$

and

$$\lambda_1 := \frac{\int_\Omega \Phi(x,1)dx}{B}, \qquad \lambda_2 := \frac{1}{c^{\phi_0} A}, \tag{3.1}$$

where c is given by (2.10).

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \max_{|t|<\xi} F(x,t)dx}{\xi^{\phi_0}} < \frac{1}{c^{\phi_0}\int_\Omega \Phi(x,1)dx} \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^{\phi^0}}.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$, for each nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$G_\infty = \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^{\phi_0}} < +\infty,$$

and for each $\mu \in [0, \delta[$, with

$$\delta = \frac{1 - c^{\phi_0}\lambda A}{c^{\phi_0} G_\infty a(\partial\,\Omega)},$$

when $a(\partial\,\Omega) = \int_{\partial\,\Omega} d\sigma$, problem (1.1) admits a sequence of weak solutions which is unbounded in $E = W^1 L_\Phi(\Omega)$.

Proof. Our aim is to apply Theorem 2.1 to problem (1.1). To this end, fix λ, μ and g satisfying our assumptions. For each $u \in E$, let the functionals $J, I : E \to \mathbb{R}$ be defined by

$$J(u) := \int_\Omega \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big) dx,$$

$$I(u) := \int_\Omega F(x, u(x))dx + \frac{\mu}{\lambda} \int_{\partial\,\Omega} G(\gamma(u(x)))\, d\sigma,$$

and put $T_{\lambda,\mu}(u) := J(u) - \lambda I(u)$. Similar arguments as those used in [17, Lemma 4.2] imply that $J \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle J'(u), v\rangle = \int_\Omega \alpha(x, |\nabla u(x)|)\nabla u(x) \times \nabla v(x)\, dx + \int_\Omega \alpha(x, |u(x)|)u(x)\, v(x)\, dx$$

for every $v \in E$. Also J is bounded from below. Moreover, $I \in C^1(E, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \int_\Omega f(x, u(x))v(x)\, dx + \frac{\mu}{\lambda} \int_{\partial\,\Omega} g\big(\gamma(u(x))\big)\gamma(v(x))\, d\sigma$$

for every $v \in E$.

So, with standard arguments, we deduce that the critical points of the functional T_λ are the weak solutions of problem (1.1). Let $\{\xi_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$, and

$$\lim_{n\to+\infty} \frac{\int_\Omega \max_{|t|<\xi_n} F(x,t)dx}{\xi_n^{\phi_0}} = A.$$

Put $r_n := \left(\frac{\xi_n}{c}\right)^{\phi_0}$, for all $n \in \mathbb{N}$. By Lemma 2.1 and this fact that $\max\{r_n^{1/\phi_0}, r_n^{1/\phi^0}\} = r_n^{1/\phi_0}$, we deduce

$$\{v \in E : J(v) < r_n\} \subseteq \left\{v \in E : \|v\| < r_n^{1/\phi_0}\right\} = \left\{v \in E : \|v\| < \frac{\xi_n}{c}\right\}.$$

Moreover, due to (2.10), we have $|v(x)| \le \|v\|_\infty \le c\|v\| \le \xi_n,\ x \in \Omega$. Hence,

$$\left\{v \in E : \|v\| < \frac{\xi_n}{c}\right\} \subseteq \{v \in E : \|v\|_\infty \le \xi_n\}.$$

Therefore, one has

$$\begin{aligned}\varphi(r_n) &\le \frac{\sup_{v\in J^{-1}(]-\infty, r_n[)} I(v)}{r_n} \le \frac{\int_\Omega \max_{|t|\le\xi_n} F(x,t)dx + \frac{\mu}{\lambda}\int_{\partial\Omega} \max_{|t|\le\xi_n} G(t)d\sigma}{\left(\frac{\xi_n}{c}\right)^{\phi_0}} \\ &\le c^{\phi_0} \frac{\int_\Omega \max_{|t|\le\xi_n} F(x,t)dx + \frac{\mu}{\lambda}\, a(\partial\Omega)\, G(\xi_n)}{\xi_n^{\phi_0}}, \qquad \text{for all } n \in \mathbb{N}.\end{aligned}$$

Then,

$$\overline{\gamma} \le \liminf_{n\to+\infty} \varphi(r_n) \le c^{\phi_0} A + \frac{\mu}{\lambda} c^{\phi_0}\, a(\partial\Omega) G_\infty < +\infty.$$

Now, let $\{\eta_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty} \eta_n = +\infty$, and

$$\lim_{n\to+\infty} \frac{\int_\Omega F(x,\eta_n)dx}{\eta_n^{\phi^0}} = B, \tag{3.2}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, define $w_n \in E$ by

$$w_n(x) = \eta_n, \qquad x \in \overline{\Omega}.$$

For any fixed $n \in \mathbb{N}$ large enough, due to the inequality

$$\Phi(x, \sigma\cdot t) \le \sigma^{\phi^0} \cdot \Phi(x,t), \qquad \forall\, x \in \overline{\Omega},\ t > 0,\ \sigma > 1$$

(see [17]), we have definitively,

$$J(w_n) = \int_\Omega \Phi(x,\eta_n)dx \le \eta_n^{\phi^0} \int_\Omega \Phi(x,1)dx.$$

On the other hand, from the definition of I, we infer

$$I(w_n) = \int_\Omega F(x, w_n(x))dx + \frac{\mu}{\lambda}\int_{\partial\,\Omega} G(\gamma(w_n))\, d\sigma$$
$$= \int_\Omega F(x,\eta_n)dx + \frac{\mu}{\lambda}\int_{\partial\,\Omega} G(\eta_n)\, d\sigma = \int_\Omega F(x,\eta_n)dx + \frac{\mu}{\lambda}\, a(\partial\,\Omega)G(\eta_n)$$

and so

$$T_\lambda(w_n) = J(w_n) - \lambda I(w_n) \le \eta_n^{\phi^0}\int_\Omega \Phi(x,1)dx - \lambda\left[\int_\Omega F(x,\eta_n)dx + \frac{\mu}{\lambda}\, a(\partial\,\Omega)G(\eta_n)\right].$$

Now, consider the following cases.

If $B < +\infty$, let

$$\epsilon \in \left]0, B - \frac{\int_\Omega \Phi(x,1)dx}{\lambda}\right[.$$

From (3.2), there exists ν_ϵ such that

$$\int_\Omega F(x,\eta_n)dx > (B-\epsilon)\,\eta_n^{\phi^0}, \qquad \text{for all}\, n > \nu_\epsilon,$$

and so

$$T_\lambda(w_n) < \eta_n^{\phi^0}\int_\Omega \Phi(x,1)dx - \lambda\left[(B-\epsilon)\,\eta_n^{\phi^0} + \frac{\mu}{\lambda}\, G(\eta_n)a(\partial\,\Omega)\right]$$
$$= \eta_n^{\phi^0}\left[\int_\Omega \Phi(x,1)dx - \lambda(B-\epsilon)\right] - \mu\, G(\eta_n)a(\partial\,\Omega).$$

Since $\int_\Omega \Phi(x,1)dx - \lambda(B-\epsilon) < 0$, one has $\lim_{n\to+\infty} T_\lambda(w_n) = -\infty$.

If $B = +\infty$, fix

$$M > \frac{\int_\Omega \Phi(x,1)dx}{\lambda}.$$

From (3.2), there exists ν_M such that

$$\int_\Omega F(x,\eta_n)dx > M\,\eta_n^{\phi^0}, \qquad \text{for all}\, n > \nu_M,$$

and moreover

$$T_\lambda(w_n) < \eta_n^{\phi^0}\int_\Omega \Phi(x,1)dx - \lambda\left[M\,\eta_n^{\phi^0} + \frac{\mu}{\lambda}\, G(\eta_n)a(\partial\,\Omega)\right]$$
$$= \eta_n^{\phi^0}\left[\int_\Omega \Phi(x,1)dx - \lambda M\right] - \mu\, G(\eta_n)a(\partial\,\Omega).$$

Since $\int_\Omega \Phi(x,1)dx - \lambda M < 0$, this leads to $\lim_{n\to+\infty} T_\lambda(w_n) = -\infty$.

Taking into account that

$$\left]\frac{\int_\Omega \Phi(x,1)dx}{B}, \frac{1}{c^{\phi_0}A}\right[\subseteq \left]0, \frac{1}{\bar{\gamma}}\right[,$$

and that T_λ does not possess a global minimum, from part (b) of Theorem 2.1, there exists an unbounded sequence $\{u_n\}$ of critical points, and our conclusion is achieved. $\square$

Remark 3.1. *If in Theorem 3.1, we assume $f(x,0)=0$ for a.e. $x\in\Omega$, then the weak solutions obtained are nonnegative.*

Indeed, define

$$f_+(x,t)=\begin{cases} f(x,t) & \text{if} \quad t\geq 0,\\ 0 & \text{if} \quad t<0,\end{cases}$$

and consider the following problem

(3.3)

$$\begin{cases} -\mathrm{div}\big(\alpha(x,|\nabla u(x)|)\nabla u(x)\big)+\alpha(x,|u(x)|)u(x)=\lambda f_+\big(x,u(x)\big) & \text{for a.e. } x\in\Omega,\\ \alpha\big(x,|\nabla u(x)|\big)\dfrac{\partial u}{\partial\nu}(x)=\mu g\big(\gamma(u(x))\big) & \text{for a.e. } x\in\partial\Omega.\end{cases}$$

If $\bar{u}$ is weak solution of problem (3.3), then one has

$$\begin{aligned}\int_\Omega \alpha(x,|\nabla\bar{u}(x)|)\nabla\bar{u}(x)\,\nabla v(x)\,dx+\int_\Omega \alpha(x,|\bar{u}(x)|)\,\bar{u}(x)\,v(x)\,dx\\ =\lambda\int_\Omega f_+(x,\bar{u}(x))\,v(x)\,dx+\mu\int_{\partial\Omega} g\Big(\gamma(\bar{u}(x))\Big)\gamma(v(x))\,d\sigma\end{aligned}$$

for every $v\in E$. Arguing by a contradiction and setting $A=\{x\in\Omega:\bar{u}(x)<0\}$ one has $A\neq\emptyset$. Put $\bar{u}^-=\min\{\bar{u},0\}$. One has $\bar{u}^-\in E$, in fact from, [12, Lemma 7.6] one has

$$\nabla\bar{u}^-=\begin{cases} 0 & u\geq 0;\\ \nabla\bar{u} & u<0.\end{cases}$$

So, taking into account that $\bar{u}$ is a weak solution and by choosing $v=\bar{u}^-$ one has

$$\begin{aligned}\int_A \alpha(x,|\nabla\bar{u}(x)|)\nabla\bar{u}(x)\,\nabla\bar{u}(x)\,dx+\int_A \alpha(x,|\bar{u}(x)|)\,\bar{u}(x)\,\bar{u}(x)\,dx\\ =\lambda\int_A f_+(x,\bar{u}(x))\,\bar{u}(x)\,dx+\mu\int_{\partial\Omega} g\Big(\gamma(\bar{u}(x))\Big)\gamma(\bar{u}^-(x))\,d\sigma\leq 0,\end{aligned}$$

that is, $\|\bar{u}\|_{W^{1,\Phi}(A)}=0$ which is an absurd. Hence, our claim is proved.

We also observe that, when $f(x,t)\geq 0$ for a.e. $x\in\Omega$ and for all $t\in\mathbb{R}$, the same conclusion holds (see [10, Lemma 1.1]).

Theorem 3.2. *Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function. Assume that*

$$\liminf_{\xi\to+\infty}\frac{\int_\Omega \max_{|t|<\xi}F(x,t)dx}{\xi^{\phi_0}}<\frac{1}{c^{\phi_0}\int_\Omega\Phi(x,1)dx}\limsup_{\xi\to+\infty}\frac{\int_\Omega F(x,\xi)dx}{\xi^{\phi^0}}.$$

Then, for each $\lambda\in]\lambda_1,\lambda_2[$, where λ_1 and λ_2 are given in (3.1), problem

$$\begin{cases} -div\big(\alpha(x,|\nabla u(x)|)\nabla u(x)\big)+\alpha(x,|u(x)|)u(x)=\lambda f\big(x,u(x)\big) & \text{for } x\in\Omega,\\ \dfrac{\partial u}{\partial\nu}(x)=0 & \text{for } x\in\partial\Omega.\end{cases}$$

admits a sequence of weak solutions which is unbounded in $W^1L_\Phi(\Omega)$.

Remark 3.2. *We observe that the role of functions f and g can be reversed. For instance, we can study the following problem*

$$\begin{cases} -div\big(\alpha(x,|\nabla u(x)|)\nabla u(x)\big)+\alpha(x,|u(x)|)u(x)=\mu g(u(x)) & \textit{for } x\in\Omega,\\ \alpha\big(x,|\nabla u(x)|\big)\dfrac{\partial u}{\partial \nu}(x)=\lambda f\big(x,u(x)\big) & \textit{for } x\in\partial\Omega. \end{cases}$$

and obtain a sequence of weak solutions providing an oscillating behavior of f for a suitable interval of parameters λ. It is enough to substitute in the proof of Theorem 3.1 the functional I with the following

$$\tilde{I}(u):=\int_{\partial\Omega}F(x,\gamma(u(x)))\,d\sigma+\frac{\mu}{\lambda}\int_{\Omega}G(u(x))dx.$$

In particular, the case $\mu=0$ can be investigated.

Example 3.1. Let $\Omega=\{(x,y)\in\mathbb{R}^2; x^2+y^2<3\}$. Define

$$\varphi(x,y,t)=p(x,y)\frac{|t|^{p(x,y)-2}}{\log(1+|t|)}t \qquad \text{for} \quad t\neq 0 \quad \text{and} \quad \varphi(x,y,0)=0,$$

where $p(x,y)=x^2+y^2+3$ for all $(x,y)\in\Omega$. Some simple computations imply

$$\Phi(x,y,t)=\frac{|t|^{p(x,y)}}{\log(1+|t|)}+\int_0^{|t|}\frac{s^{p(x,y)}}{(1+s)(\log(1+s))^2}\,ds,$$

and the relations (φ), (Φ_1), and (Φ_2) are verified. For each $x\in\overline{\Omega}$ fixed, by Example 3 on p. 243 in [7], we have

$$p(x,y)-1\le\frac{t\varphi(x,y,t)}{\Phi(x,y,t)}\le p(x,y), \qquad \forall\, t\ge 0.$$

Thus, the relation (2.2) holds true with $\varphi_0=p^--1=2$ and $\varphi^0=p^+:=\sup_{(x,y)\in\Omega}p(x,y)=6$. Next, Φ satisfies the condition (2.9) since

$$\Phi(x,y,t)\ge t^{p(x,y)-1}, \qquad \forall\,(x,y)\in\overline{\Omega},\ t\ge 0.$$

Finally, we point out that trivial computations imply that $\dfrac{d^2(\Phi(x,y,\sqrt{t}))}{dt^2}\ge 0$ for all $(x,y)\in\overline{\Omega}$ and $t\ge 0$. Thus, the relation (2.4) is satisfied. Consider

(3.4)
$$\begin{cases} -\mathrm{div}\left(p(x,y)\dfrac{|\nabla u|^{p(x,y)-2}}{\log(1+|\nabla u|)}\nabla u\right)+p(x,y)\dfrac{|u|^{p(x,y)-2}}{\log(1+|u|)}u=f(x,y,u) & \text{for } (x,y)\in\Omega,\\ p(x,y)\dfrac{|\nabla u|^{p(x,y)-2}}{\log(1+|\nabla u|)}\dfrac{\partial u}{\partial\nu}=\dfrac{1}{1+(\gamma(u(x,y)))^2} & \text{for } (x,y)\in\partial\Omega. \end{cases}$$

where

$$f(x,y,t)=\begin{cases} f^*(x,y)t^6\Big(7+\sin(\ln(|t|))-7\cos(\ln(|t|))\Big) & \text{if } (x,y,t)\in\partial\Omega\times(\mathbb{R}-\{0\}),\\ 0 & \text{if } (x,y,t)\in\partial\Omega\times\{0\}, \end{cases}$$

where $f^* : \partial\Omega \to \mathbb{R}$ is a non-negative continuous function and $t \in \mathbb{R}$. A direct calculation shows

$$F(x,y,t) = \begin{cases} f^*(x,y)t^7\Big(1-\cos(\ln(|t|))\Big) & \text{if} \quad (x,y,t) \in \partial\Omega \times (\mathbb{R}-\{0\}), \\ 0 & \text{if} \quad (x,y,t) \in \partial\Omega \times \{0\}. \end{cases}$$

So,

$$\liminf_{\xi \to +\infty} \frac{\displaystyle\int_{\partial\Omega} \max_{|t|<\xi} F(x,y,t)\, d\sigma}{\xi^2} = 0$$

and

$$\limsup_{\xi \to +\infty} \frac{\displaystyle\int_{\partial\Omega} F(x,y,\xi)\, d\sigma}{\xi^6} = +\infty.$$

Hence, using Theorem 3.1, the problem (3.4) admits infinitely many weak solutions in E.

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