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SOLVABILITY OF QUADRATIC INTEGRAL EQUATIONS WITH SINGULAR KERNEL

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Abstract. In this paper, we discussed the existence and uniqueness of solution of the singular Quadratic integral equation (SQIE). The Fredholm integral term is assumed in position with singular kernel. Under certain conditions and new discussions, the singular kernel will tend to a logarithmic kernel. Then, using Chebyshev polynomial, a main of spectral relationships are stated and used to obtain the solution of the singular Quadratic integral equation with the logarithmic kernel and a smooth kernel. Finally, the Fredholm integral equation of the second kind is established and its solution is discussed, also numerical results are obtained and the error, in each case, is computed.

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1. Introduction

The numerous kinds of integral equations are necessary mathematical tools for describing knowledge models that appear in various areas of applied science, see [6, 22, 23]. Because of extensive application of integral equations and not having the exact solutions in many cases, numerical solution of integral equations has attracted researcher's attention to develop numerical method for approximating solution of these equations. Among these methods, we refer to Degenerate kernel method [29], Resolvent kernel method [2], Trapezoidal rule [3], Wavelet method [4, 8], Homotopy perturbation method [32, 18], Collocation method [12], Separation of variables method [1] and Meshless method [15]. A novel algorithm to get approximate solution of these equations is to express the solution as linear combination of orthogonal or nonorthogonal basis functions and polynomials such as Block-pulse functions [9, 21], Hat functions [10, 24], Bernoulli polynomials [11], Bessel polynomials [34], Chebyshev polynomials [7], Fibonacci polynomials [25], Orthonormal bernstein polynomials [26], and others [20, 33]

Mathematical modeling of many phenomena in the real world is led to a special kind of integral equations called Quadratic integral equations. Quadratic integral equations always arise in many problems of mathematical physics and chemical such as theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory, and the traffic theory and many other applications. Existence solution and numerical method to solve these type of integral equations have been studied in previous papers, see [14, 27]. Lately, Quadratic integral equations with singular kernels have taken a lot of attention because of their useful applications in describing many events and problems of the real world.

In this paper, we consider the Quadratic integral equation with singular kernel in the position. We use a numerical method to obtain the solution of the singular Quadratic integral equation, where the existence and the uniqueness of the solution of the integral equations can be discussed and proved using Picard's method. Consider the singular Quadratic integral equation,

$$\psi(x) = g(x) + \int_{-1}^{1} \xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi(y))\mathrm{d}y \int_{-1}^{1} p(x,y)\eta(y,\psi(y))\mathrm{d}y, \tag{1.1}$$

$$\begin{aligned} \xi\left(\left|\frac{y-x}{\lambda}\right|\right) &= \int_0^\infty \left(\frac{L(w)}{w}\right)\cos(\frac{y-x}{\lambda}w)\mathrm{d}w, \\ L(w) &= \frac{w+q}{1+w}, \quad q \geq 1, \quad \lambda \in (0,\infty) - \left\{\frac{\pi}{2}\right\}, \end{aligned} \tag{1.2}$$

where, the function $\psi(x)$ is unknown in the Banach space $L_2([-1,1]), -1 \leq x \leq 1$, the domain of integration with respect to the position $x \in [-1,1]$. $p(x,y)$ is given smooth function, the given function $g(x)$ belongs to the space $L_2([-1,1])$. $\xi\left(\left|\frac{y-x}{\lambda}\right|\right)$ is a discontinuous kernel in position.

This paper is divided into seven sections, In section two, the existence and unique solution of Eq. (1.1) is discussed and proved. In section three, we mentioned a theory explaining that the bad kernel takes the logarithmic form. While, in section four, we state some algebraic and integral formulas for the Chebyshev polynomials, also, we use Chebyshev polynomial method to obtain the solution of the singular Quadratic integral equation. In section five, a main theorem of spectral relationships for the Fredholm–integral equation of the second kind established and its solution is discussed. In section six, numerical results and estimated errors are computed. In section seven, general conclusions are deduced.

2. The existence and uniqueness of solution of the (SQIE)

In order to discuss the existence and uniqueness of solution of Eq. (1.1), we assume the following:

(i): A smooth function $p(x,y)$ satisfies $|p(x,y)| < k, \quad \forall x,y \in [-1,1]$ and k is a constant.

(ii): Bad behavior of the kernel $\xi\left(\left|\frac{y-x}{\lambda}\right|\right)$ satisfies the discontinuity condition

$$\left\{\int_{-1}^{1}\int_{-1}^{1}\left|\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\right|^{2} \mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{2}} = Q,$$

where Q is a small constant.

(iii): The two known functions $\mu(y,\psi(y))$, $\eta(y,\psi(y))$ are bounded and satisfy:

(iii − 1) $\quad |\mu(y,\psi(y))| \leq N_1,\ |\eta(y,\psi(y))| \leq N_2$ s.t $N_1,\ N_2$ are a constants.

$$\text{(iii}-2) \quad |\mu(y,\psi_1(y)) - \mu(y,\psi_2(y))| \leq M_1(y)|\psi_1(y) - \psi_2(y)|,$$

$$|\eta(y,\psi_1(y)) - \eta(y,\psi_2(y))| \leq M_2(y)|\psi_1(y) - \psi_2(y)|,$$

where, $|M_1(y)| \leq l_1,\ |M_2(y)| \leq l_2, \quad (l_1,\ l_2$ are a constants).

Theorem 2.1. *Let the conditions* (i − iii) *are satisfied. If the condition*

$$4Qk[l_1N_2 + l_2N_1] < 1 \tag{2.1}$$

is satisfied, then the equation (1.1) *has a unique solution* $\psi(x)$ *in the space* $L_2([-1,1])$.

Proof. To prove the existence and uniqueness of solution of equation (1.1), we use the successive approximations method (*Picard's method*).
We assume the solution of Quadratic integral equation (1.1) approaches to the solution, which takes the following form:

(2.2)

$$\psi_n(x) = g(x) + \int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-1}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-1}(y))\mathrm{d}y,$$
$$\psi_0(x) = g(x).$$

All the functions $\psi_n(x)$ are continuous functions and $\psi_n(x)$ can be written as a sum of successive differences:

$$\psi_n(x) = \psi_0(x) + \sum_{i=1}^{n}(\psi_i(x) - \psi_{i-1}(x)).$$

This means that convergence of the sequence $\psi_n(x)$ is equivalent to convergence of the finite series $\sum_{i=1}^{n}(\psi_i(x) - \psi_{i-1}(x))$, the solution will be

$$\psi(x) = \lim_{n\to\infty}\psi_n(x),$$

i.e. if the finite series $\sum_{i=1}^{n}(\psi_i(x) - \psi_{i-1}(x))$ converges, then the sequence $\psi_n(x)$ will converge to $\psi(x)$.

Now, we must prove the following lemmas:

Lemma 2.1. *A sequence* $\{\psi_n(x)\}$ *is uniformly convergent to a continuous solution function* $\{\psi(x)\}$.

Proof. To prove the uniform convergence of $\{\psi_n(x)\}$, we shall consider the associated series

$$\sum_{n=1}^{\infty}(\psi_n(x)-\psi_{n-1}(x)).$$

From Eq. (2.2), for $n=1$, we obtain

$$\psi_1(x)-\psi_0(x)=\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_0(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_0(y))\mathrm{d}y,$$

and

$$|\psi_1(x)-\psi_0(x)|\le QN_1kN_2\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}y=4QN_1kN_2. \tag{2.3}$$

Now, we shall obtain an estimate for $\psi_n(x)-\psi_{n-1}(x);\quad n\ge 2$,

$$\begin{aligned}
\psi_n(x)-\psi_{n-1}(x)&=\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-1}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-1}(y))\mathrm{d}y\\
&-\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-2}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-2}(y))\mathrm{d}y,\\
&=\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-1}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-1}(y))\mathrm{d}y\\
&-\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-2}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-1}(y))\mathrm{d}y,\\
&+\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-2}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-1}(y))\mathrm{d}y\\
&-\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-2}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-2}(y))\mathrm{d}y,\\
&=\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)[\mu(y,\psi_{n-1}(y))-\mu(y,\psi_{n-2}(y))]\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_{n-1}(y))\mathrm{d}y\\
&+\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_{n-2}(y))\mathrm{d}y\int_{-1}^{1}p(x,y)[\eta(y,\psi_{n-1}(y))-\eta(y,\psi_{n-2}(y))]\mathrm{d}y.
\end{aligned}$$

Using conditions (i)-(iii), we have

$$\begin{aligned}
|\psi_n(x)-\psi_{n-1}(x)|\le& QkN_2\int_{-1}^{1}M_1(y)|\psi_{n-1}(y)-\psi_{n-2}(y)|\mathrm{d}y\int_{-1}^{1}\mathrm{d}y\\
&+QN_1k\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}M_2(y)|\psi_{n-1}(y)-\psi_{n-2}(y)|\mathrm{d}y,\\
\le& Ql_1kN_2\int_{-1}^{1}|\psi_{n-1}(y)-\psi_{n-2}(y)|\mathrm{d}y\int_{-1}^{1}\mathrm{d}y\\
&+QN_1kl_2\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}|\psi_{n-1}(y)-\psi_{n-2}(y)|\mathrm{d}y.
\end{aligned}$$

Putting $n = 2$, then using (2.3), we get

$$|\psi_2(x) - \psi_1(x)| \leq Ql_1kN_2 \int_{-1}^{1} |\psi_1(y) - \psi_0(y)|\mathrm{d}y \int_{-1}^{1} \mathrm{d}y$$
$$+ QN_1kl_2 \int_{-1}^{1} \mathrm{d}y \int_{-1}^{1} |\psi_1(y) - \psi_0(y)|\mathrm{d}y \leq 4^2Q^2l_1k^2N_1N_2^2 + 4^2Q^2N_1^2k^2l_2N_2,$$
$$\leq 4QkN_1N_2 \left(4Qk[l_1N_2 + l_2N_1]\right).$$

$$\begin{aligned}|\psi_3(x) - \psi_2(x)| \leq & Ql_1kN_2 \int_{-1}^{1} |\psi_2(y) - \psi_1(y)|\mathrm{d}y \int_{-1}^{1} \mathrm{d}y \\ & + QN_1kl_2 \int_{-1}^{1} \mathrm{d}y \int_{-1}^{1} |\psi_2(y) - \psi_1(y)|\mathrm{d}y, \\ \leq & 4Ql_1kN_2 \left(4QkN_1N_2 \left(4Qk[l_1N_2 + l_2N_1]\right)\right) \\ & + 4QN_1kl_2 \left(4QkN_1N_2 \left(4Qk[l_1N_2 + l_2N_1]\right)\right), \\ \leq & \left(4QkN_1N_2\right) \left(4Qk[l_1N_2 + l_2N_1]\right) \times \left(4Qk[l_1N_2 + l_2N_1]\right).\end{aligned}$$

Repeating this technique, we obtain the general estimate for the terms of the series:

$$\begin{aligned}|\psi_n(x) - \psi_{n-1}(x)| \leq & \left(4QkN_1N_2\right) \left(4Qk[l_1N_2 + l_2N_1]\right) \times \left(4Qk[l_1N_2 + l_2N_1]\right) \times \dots \\ & \times \left(4Qk[l_1N_2 + l_2N_1]\right) \leq \left(4Qk[l_1N_2 + l_2N_1]\right)^n.\end{aligned}$$

Since $4Qk[l_1N_2 + l_2N_1] < 1$, then the uniform convergence of

$$\sum_{n=1}^{\infty} (\psi_n(x) - \psi_{n-1}(x)),$$

is proved and so the sequence $\{\psi_n(x)\}$ is uniformly convergent.

$$\begin{aligned}\psi(x) &= \lim_{n\to\infty} \left(g(x) + \int_{-1}^{1} \xi\left(\left|\frac{y-x}{\lambda}\right|\right) \mu(y, \psi_n(y))\mathrm{d}y \int_{-1}^{1} p(x,y)\eta(y, \psi_n(y))\mathrm{d}y \right), \\ &= g(x) + \int_{-1}^{1} \xi\left(\left|\frac{y-x}{\lambda}\right|\right) \mu(y, \psi(y))\mathrm{d}y \int_{-1}^{1} p(x,y)\eta(y, \psi(y))\mathrm{d}y.\end{aligned}$$

Thus, the existence of a solution of equation (1.1) is proved.

Lemma 2.2. *The function $\psi(x)$ represents a unique solution of Quadratic integral equation (1.1).*

Proof. To prove the uniqueness of Eq. (1.1), let $\phi(x)$ be another continuous solution of Eq. (1.1). Then

$$\phi(x) = g(x) + \int_{-1}^{1} \xi\left(\left|\frac{y-x}{\lambda}\right|\right) \mu(y, \phi(y))\mathrm{d}y \int_{-1}^{1} p(x,y)\eta(y, \phi(y))\mathrm{d}y; \qquad x \in [-1, 1],$$

and

$$\begin{aligned}
\phi(x)-\psi_n(x) &= \int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\phi(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\phi(y))\mathrm{d}y\\
&-\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_n(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_n(y))\mathrm{d}y,\\
&=\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\phi(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\phi(y))\mathrm{d}y\\
&-\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_n(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\phi(y))\mathrm{d}y,\\
&+\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_n(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\phi(y))\mathrm{d}y\\
&-\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_n(y))\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\psi_n(y))\mathrm{d}y,\\
&=\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)[\mu(y,\phi(y))-\mu(y,\psi_n(y))]\mathrm{d}y\int_{-1}^{1}p(x,y)\eta(y,\phi(y))\mathrm{d}y\\
&+\int_{-1}^{1}\xi\left(\left|\frac{y-x}{\lambda}\right|\right)\mu(y,\psi_n(y))\mathrm{d}y\int_{-1}^{1}p(x,y)[\eta(y,\phi(y))-\eta(y,\psi_n(y))]\mathrm{d}y.
\end{aligned}$$

Using assumptions (i)–(iii), we get

$$\begin{aligned}
|\phi(x)-\psi_n(x)| &\le QkN_2\int_{-1}^{1}M_1(y)|\phi(y)-\psi_n(y)|\mathrm{d}y\int_{-1}^{1}\mathrm{d}y\\
&+QN_1k\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}M_2(y)|\phi(y)-\psi_n(y)|\mathrm{d}y,\\
&\le Ql_1kN_2\int_{-1}^{1}|\phi(y)-\psi_n(y)|\mathrm{d}y\int_{-1}^{1}\mathrm{d}y\\
&+QN_1kl_2\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}|\phi(y)-\psi_n(y)|\mathrm{d}y.
\end{aligned}$$

But

$$|\phi(x)-g(x)|\le QN_1kN_2\int_{-1}^{1}\mathrm{d}y\int_{-1}^{1}\mathrm{d}y=4QN_1kN_2, \tag{2.4}$$

and using the inequality (2.4) gives

$$|\phi(x)-\psi_n(x)|\le\left(4Qk[l_1N_2+l_2N_1]\right)^n.$$

Hence

$$\lim_{n\to\infty}\psi_n(x)=\phi(x)\Rightarrow\psi(x)=\phi(x),$$

which completes the proof. □

3. The kernel of position

Theorem 3.1. *The bad kernel of equation (1.2) takes the logarithmic form.*

Proof. For $w \in (0, \infty)$, the function $L(w)$ is positive and continuous. Then it can satisfy the asymptotic equalities:

$$L(w) = q - (q-1)w + O(w^2), \quad w \to 0, \tag{3.1}$$

$$L(w) = 1 + \frac{q-1}{w} + O(w^{-2}), \quad w \to \infty, \quad q \geq 1. \tag{3.2}$$

Most of the previous authors have been solved the Fredholm integral equations of the first and second kind in the problems of continuum mechanics, when $w \to \infty,\ q = 1$. i.e. $L(w) = 1$.

Here, we consider the case when $w \to 0$, then for the first and second approximate of $L(w)$ after using the two famous relations

$$\begin{aligned} &\int_0^\infty \frac{\cos(\frac{y-x}{\lambda}w)}{w}\mathrm{d}w = -\ln|y-x| + d; \quad d = \ln\frac{4\lambda}{\pi}, \\ &\int_0^\infty \cos(\frac{y-x}{\lambda}w)\mathrm{d}w = \delta(y-x); \quad \delta(y-x) \text{ is the Dirac function.} \end{aligned} \tag{3.3}$$

We can arrive

$$\xi\left(\left|\frac{y-x}{\lambda}\right|\right) = -\ln|y-x| + d. \tag{3.4}$$

4. The solution algorithm of the (SQIE)

Let $T_n(x) = \cos(n\cos^{-1}x)$; $x \in [-1,1]$; $n \geq 0$ denotes the Chebyshev polynomials of the first kind, while $U_n(x) = \sin[(n+1)\cos^{-1}x]/\sin(\cos^{-1}x)$, $n \geq 0$ denotes the Chebyshev polynomials of the second kind. It is well known that $T_n(x)$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{-1/2}$, while $U_n(x)$ form an orthogonal sequence of functions with respect to the weight function $(1-x^2)^{1/2}$.

In this section, we use Chebyshev polynomial $T_n(x)$ of the first kind of order n to solve the following Quadratic integral equation with singular kernel $(-\ln|y-x|+d)$:

$$\psi(x) + \int_{-1}^{1}(\ln|y-x| - d)\psi(y)\mathrm{d}y \int_{-1}^{1} p(x,y)\psi(y)\mathrm{d}y = g(x), \tag{4.1}$$

here $\int_{-1}^{1}$ denotes integration in the sense of logarithmic principal value, the singular Quadratic integral equation with Carleman kernel can be obtained from (4.1) by using the famous relation, see [30]

$$\ln|y-x| = E(x,y)|y-x|^{-\alpha}; \qquad 0 \leq \alpha < 1, \tag{4.2}$$

where $E(x,y) = |y-x|^\alpha \ln|y-x| \in C[-1,1]$ for all $x \in [-1,1]$.

Assume the unknown function $\psi(x)$, in the light of weight function, takes the form

$$\psi(x) = R(x)H(x); \qquad R(x) = (1-x^2)^{-\frac{1}{2}}, \tag{4.3}$$

where $H(x)$ is unknown function and $R(x)$ represents the weight function of $T_n(x)$. Now, to obtain numerically a solution of Eq. (4.1), we express $H(x)$ as

$$H(x) = \sum_{n=0}^{\infty} C_n T_n(x). \tag{4.4}$$

Hence, we have

$$\psi(x) = \sum_{n=0}^{\infty} C_n \frac{T_n(x)}{\sqrt{1-x^2}}, \tag{4.5}$$

the formula (4.5) can be truncated to

$$\psi_N(x) = \sum_{n=0}^{N} C_n \frac{T_n(x)}{\sqrt{1-x^2}}, \tag{4.6}$$

where C_n are constants and $T_n(x)$ are the Chebyshev polynomials of the first kind that satisfy the orthogonal relation

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m = 0, \\ \frac{\pi}{2}, & n = m \neq 0. \end{cases} \tag{4.7}$$

$$T_m(x)T_n(x) = \frac{1}{2}[T_{m+n}(x) + T_{|m-n|}(x)], \qquad (m, n \geq 0), \tag{4.8}$$

$$\int_{-1}^{1} T_n(x) dx = \begin{cases} \frac{1}{1-n^2}, & n = 0, 2, 4, ... \\ 0, & n = 1, 3, 5, ... \end{cases} \tag{4.9}$$

Also, we say that, if $\psi(x) \in L_2([-1,1])$, then the polynomial series (4.6) belongs to $L_2([-1,1])$. In view of (4.6), the known term of (4.1) can be represented as

$$g_N(x) = \sum_{n=0}^{N} G_n \frac{T_n(x)}{\sqrt{1-x^2}}, \tag{4.10}$$

where the coefficients G_n; $n \geq 0$, are constants. If the known function $g(x) \in L_2([-1,1])$, it follows that the polynomial series (4.10) belongs to $L_2([-1,1])$. Since, any smooth function can be represented in the polynomial series form, therefore we assume the smooth function $p(x,y)$ of (4.1) in the form:

$$p(x,y) = \sum_{m=0}^{M} T_m(x)T_m(y); \qquad m = 0, 1, 2, ..., M. \tag{4.11}$$

Using (4.6), (4.10) and (4.11) in (4.1), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} C_n \frac{T_n(x)}{\sqrt{1-x^2}} + \sum_{n=0}^{\infty} C_n \int_{-1}^{1} (\ln|y-x| - d) \frac{T_n(y)}{\sqrt{1-y^2}} \mathrm{d}y \times \\ &\times \sum_{m=0}^{M} \sum_{n=0}^{\infty} C_n T_m(x) \int_{-1}^{1} \frac{T_m(y)T_n(y)}{\sqrt{1-y^2}} \mathrm{d}y = \sum_{n=0}^{\infty} G_n \frac{T_n(x)}{\sqrt{1-x^2}}, \end{aligned} \tag{4.12}$$

where

$$G_n = \begin{cases} \frac{2}{\pi}\int_{-1}^{1} g(x)T_n(x)dx, & n \neq 0, \\ \frac{1}{\pi}\int_{-1}^{1} g(x)T_n(x)dx, & n = 0. \end{cases} \tag{4.13}$$

The method of finding the approximate solution through (4.12) depends on comparing the coefficients associated with the same polynomial terms T_n, the approximate solution is obtained directly from (4.6), then the exact solution is obtained from (4.5). This method is said to be convergent of order r if and only if for N sufficiently large, there exists a constant $D > 0$ independent of N such that

$$\|\psi(x) - \psi_N(x)\| \leq DN^{-r}. \tag{4.14}$$

So, the transformation error E_N can be determined as

$$E_N = \|\psi(x) - \psi_N(x)\| \leq \left\| \sum_{n=N+1}^{\infty} C_n \right\|. \tag{4.15}$$

In the aid of (4.14), we write

$$E_N \leq DN^{-r}; \qquad \text{(D is a constant)}. \tag{4.16}$$

Using orthogonal the relation (4.7) and the following famous relation:

$$\int_{-1}^{1} \frac{(\ln|y-x| - d)T_n(y)}{\sqrt{1-y^2}} \mathrm{d}y = \begin{cases} \pi(\ln 2 - d), & n = 0, \\ \pi\frac{T_n(x)}{n}, & n \geq 1. \end{cases} \tag{4.17}$$

The solution of (4.12) can be obtained after discussing the following:

Case (i): For $n = 0,\ m = 0$, we have

$$C_0 \frac{T_0(x)}{\sqrt{1-x^2}} + C_0^2 \pi^2 (\ln 2 - d) T_0(x) = G_0 \frac{T_0(x)}{\sqrt{1-x^2}}, \tag{4.18}$$

multiplying both sides of (4.18) by the term $T_m(x)dx$, then integrating the result from -1 to 1, we get

$$C_0^2 + \frac{1}{\pi(\ln 2 - d)} C_0 - \frac{1}{\pi(\ln 2 - d)} G_0 = 0. \tag{4.19}$$

Case (ii): For $n = 0,\ m \neq 0$, after using the orthogonal relation and formula (4.17), (4.12) tends to

$$\begin{aligned} C_0 \frac{T_0(x)}{\sqrt{1-x^2}} &= G_0 \frac{T_0(x)}{\sqrt{1-x^2}}; \quad m \geq 1, \\ C_0 &= G_0, \end{aligned} \tag{4.20}$$

where

$$G_0 = \frac{1}{\pi} \int_{-1}^{1} g(x) dx.$$

Case (iii): For $n \neq 0,\ m = 0$, we deduce

$$\begin{aligned} C_n \frac{T_n(x)}{\sqrt{1-x^2}} &= G_n \frac{T_n(x)}{\sqrt{1-x^2}}; \quad n \geq 1, \\ C_n &= G_n; \qquad n \geq 1, \end{aligned} \tag{4.21}$$

where,

$$G_n = \frac{2}{\pi}\int_{-1}^{1} g(x)T_n(x)dx.$$

Case (iv): For $n = m \neq 0$, we can establish

$$C_n \frac{T_n(x)}{\sqrt{1-x^2}} = G_n \frac{T_n(x)}{\sqrt{1-x^2}} + \frac{1}{2n} C_n^2 \pi^2 T_n(x) T_m(x); \quad n,\, m \geq 1. \tag{4.22}$$

Multiplying both sides of (4.22) by the term $T_m(x)dx$, then integrating the result from -1 to 1, we get

$$C_n = G_n + \frac{1}{4n} C_n^2 \pi [H + 3H^*]; \quad n,\, m \geq 1, \tag{4.23}$$

where

$$H = \begin{cases} \frac{1}{1-(3m)^2}, & 3m = 0, 2, 4, ... \\ 0, & 3m = 1, 3, 5, ..., \end{cases} \tag{4.24}$$

and

$$H^* = \begin{cases} \frac{1}{1-m^2}, & m = 0, 2, 4, ... \\ 0, & m = 1, 3, 5, ... \end{cases} \tag{4.25}$$

5. Fredholm integral equation of the second kind

From the above discussion, we can establish a famous integral equation as the following: Assume in (4.1) the known smooth function $p(x, y) = 1$ and $\int_{-1}^{1} \psi(y)\mathrm{d}y = P$ where, P is constant, this result is called the pressure condition. Therefore, we have

$$\psi(x) + P\int_{-1}^{1} (\ln|y-x| - d)\psi(y)\mathrm{d}y = g(x). \tag{5.1}$$

The above formula is called Fredholm integral equation of the second kind with logarithmic kernel. In addition, using the famous relation (4.2), we have Fredholm integral equation of the second kind with Carleman kernel, see [30]

$$\psi(x) + P\int_{-1}^{1} E(x, y)|y-x|^{-\alpha}\psi(y)\mathrm{d}y = g(x). \tag{5.2}$$

The important of Carleman kernel came from the work of Aroytion, who proved that the first approximation of the nonlinear integral equation in the theory of plasticity, represent a Fredholm integral equation of the second kind with Carleman kernel, see [5].

Differentiating the integral equation (5.1) with respect to x, we have

$$\psi'(x) + P\int_{-1}^{1} \frac{\psi(y)\mathrm{d}y}{x-y} = g'(x). \tag{5.3}$$

The importance of the above equation came from the work of Frankel, see [16].

Many authors have discussed the solution of the above equations using different methods. For Cauchy methods, see [28], for potential theory method, see [17]. And for orthogonal method, see [31]. The importance of orthogonal polynomials came from its spectral relationships which have many applications in astrophysics and mathematical physics problems. For this, we us the following spectral relationships to discuss the solution of Fredholm integral equation of the second kind with logarithmic kernel. Moreover, we will establish many spectral relationships from the integral equation,

$$\int_{-1}^{1}\left(\ln\frac{1}{|y-x|}+d\right)\frac{T_{2n}(y)}{\sqrt{1-y^2}}\mathrm{d}y=\begin{cases}\pi(\ln 2+d); & n=0;\\ \frac{\pi}{2n}T_{2n}(x); & n\neq 0,\end{cases}\tag{5.4}$$

for even function, and

$$\int_{-1}^{1}\left(\ln\frac{1}{|y-x|}+d\right)\frac{T_{2n-1}(y)}{\sqrt{1-y^2}}\mathrm{d}y=\frac{\pi}{2n-1}T_{2n-1}(x);\tag{5.5}$$

for odd function, where $T_n(x)$ is the Chebyshev polynomial of the first kind.

Many different spectral relations, that have importance applications in mathematical physics, can be established.

(i) For even values, let

$$x=\frac{\sin\omega/2}{\sin u/2},\qquad y=\frac{\sin v/2}{\sin u/2},$$

and we get the integral relation operators of the first kind in the form:

$$\int_{-1}^{1}\left(\ln\frac{1}{2|\sin(\omega-v)/2|}+d\right)\frac{T_{2n}\left(\frac{\sin v/2}{\sin u/2}\right)}{\sqrt{2(\cos v-\cos u)}}dv=\tag{5.6}$$

$$=\begin{cases}\pi\left(\ln(\sin\frac{u}{2})+d\right); & n=0,\\ \frac{\pi}{2n}T_{2n}\left(\frac{\sin\omega/2}{\sin u/2}\right); & n=1,2,....\end{cases}$$

(ii) For odd values, let in (5.5),

$$x=\frac{\tan\omega/2}{\tan u/2},\qquad y=\frac{\tan v/2}{\tan u/2},$$

and we get

$$\int_{-1}^{1}\left(\ln\frac{1}{2|\sin(\omega-v)/2|}+d\right)\frac{T_{2n-1}\left(\frac{\tan v/2}{\tan u/2}\right)\sec(v/2)}{\sqrt{2(\cos v-\cos u)}}dv=\tag{5.7}$$

$$=\frac{\pi}{2n-1}T_{2n-1}\left(\frac{\tan\omega/2}{\tan u/2}\right);\quad n=1,2,....$$

Differentiating (5.4) and (5.5) with respect to x, we obtain the integral relations for the Chebyshev polynomials with Cauchy kernel in the form:

$$\int_{-1}^{1}\frac{T_n(y)}{y-x}\frac{\mathrm{d}y}{\sqrt{1-y^2}}=\pi U_{n-1}(x);\qquad n=1,2,...,\tag{5.8}$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind. Using the two cases (i) and (ii) in polar coordinates, relation (5.8) takes the form:

$$\int_{-1}^{1} \cot\frac{v-\omega}{2} T_n\left(\frac{\tan v/2}{\tan u/2}\right)\frac{\cos(v/2)}{\sqrt{2(\cos v-\cos u)}}dv$$
$$=\begin{cases} 0; & n=0,\\ \pi\csc(u/2)U_{2m-1}\left(\frac{\tan\omega/2}{\tan u/2}\right); & n=2m;\ m\geq 1,\\ \pi\csc(u/2)U_{2m-1}\left(\frac{\tan\omega/2}{\tan u/2}\right)+(-1)^m\frac{\sin u/2}{1+\cos u/2}(\tan u/4); & n=2m-1,\end{cases}$$

and

$$\int_{-1}^{1} \cot\frac{v-\omega}{2} T_n\left(\frac{\tan v/2}{\tan u/2}\right)\frac{\sec(v/2)}{\sqrt{2(\cos v-\cos u)}}dv$$
$$=\begin{cases} \pi\csc(u/2)\sec^2(\omega/2)U_{n-1}\left(\frac{\tan\omega/2}{\tan u/2}\right); & n=1,2,3,...,\\ \sec(u/2)\tan(\omega/2); & n=0;\ |\omega|<u.\end{cases}$$

6. Application and numerical results

In this section, we applied presented method in this paper for solving singular Quadratic integral equation (4.1).
To obtain the numerical solution of SQIE (4.1), we calculate the constant C_n of Eqs (4.19)–(4.21) and (4.23). Then, with the aid of the results of main relation, we can calculate the unknown function $\psi_N(x)$; $-1\leq x\leq 1$, when $N=50$, $M=8$, $\lambda=0.1$, $g(x)=x^2$. The tables and Figs are given for different cases.

In Table 1, we presented the absolute error $|\psi(x)-\psi_N(x)|$, $N=50$, using the introduced numerical method (Chebyshev polynomial) with $m=0$ in the interval $x\in[-1,1]$. Here in the following table, we have taken $m=0$ and this is present in two cases, the case (i) and from which we get C_0, case (iii) we get C_n, $n\geq 1$.

Table 1. Case (i, iii); represents the solution $\psi_N(x)$ and its error for different position in the simple case $m=0$.

x	$\psi_N(x)$	$\lvert\psi(x)-\psi_N(x)\rvert$
0.9	0.811029582	4.99495683×10^{-5}
0.7	0.489852149	3.45213542×10^{-6}
0.5	0.249993254	1.23514698×10^{-6}
0.3	0.089735841	1.02514368×10^{-6}
0.1	0.011009523	9.23514569×10^{-7}
-0.1	0.032154867	5.63251201×10^{-7}
-0.3	0.075216987	7.45238743×10^{-6}
-0.5	0.262514871	8.85221476×10^{-6}
-0.7	0.532648796	9.85416321×10^{-6}

In Figure 1, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method (Chebyshev polynomial) with different values of x.

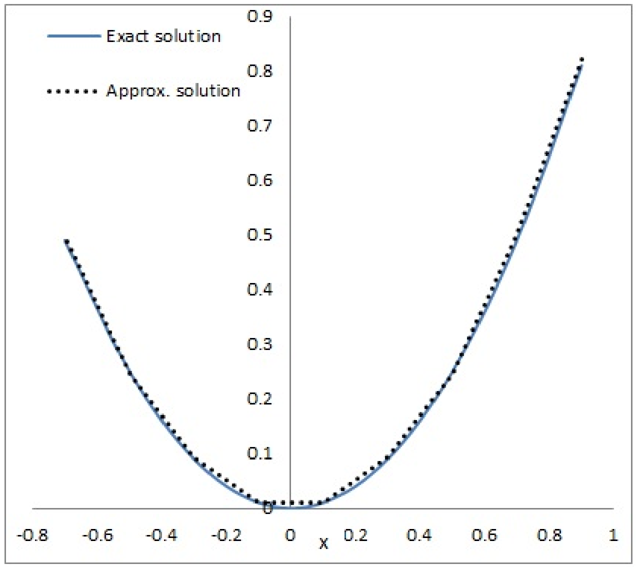

Figure 1.
Exact and approximate solution of Chebyshev polynomial method for $N = 50$.

In Table 2, we have presented approximated solution and the absolute error of approximate solution in some arbitrary points. We take here $m \geq 1$ this is achieved in both cases (ii, iv), where we get C_n, $n \geq 0$.

Table 2. Case (ii, iv); represents the solution $\psi_N(x)$ and its error for different x in $m \neq 0$.

x	$\psi_N(x)$	$\lvert\psi(x) - \psi_N(x)\rvert$
0.9	0.815456797	6.45572136×10^{-5}
0.7	0.489231464	8.21365475×10^{-6}
0.5	0.244625825	6.32014578×10^{-6}
0.3	0.086321456	5.36985212×10^{-6}
0.1	0.012698721	4.14785412×10^{-7}
-0.1	0.039574215	3.32145698×10^{-7}
-0.3	0.0765654566	6.25814736×10^{-6}
-0.5	0.221466574	5.23541587×10^{-6}
-0.7	0.538412301	9.25413698×10^{-6}

In Figure 2, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method (Chebyshev polynomial) with different values of x.

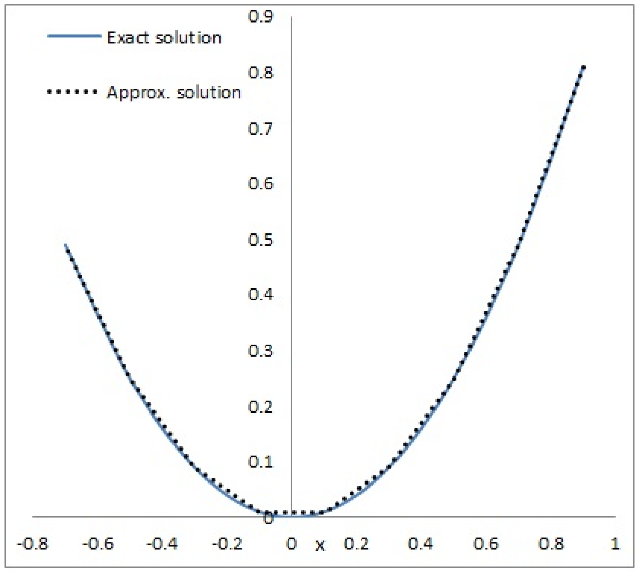

Figure 2.
Exact and approximate solution of Chebyshev polynomial method for $N = 50$.

7. Conclusion and Remarks

In this paper, from the above results and discussion, the following may be concluded, the equation (1.1) has a unique solution $\psi(x)$ in the space $L_2([-1,1])$, under some conditions. Singular Quadratic integral equation is usually difficult to solve analytically, in many cases, it is required to obtain the approximate solutions. From the Tables 1, 2, we note that the error takes maximum value at the ends when $x = 1$ and $x = -1$, while it is minimum at the middle when $x = 0$.

The smooth function $p(x, y)$ has an effect for the potential function $\psi(x)$, that is the error becomes smaller for bigger powers of x and y in $p(x, y)$. The singular Quadratic integral equation with Carleman kernel can be established from this work by using Eq. 4.2. The Fredholm integral equations of the second kind with logarithmic and Carleman kernels are considered, now, as special cases of this work. Many spectral relations are established from the problem these relations have many important applications in mathematical physics problems.

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