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FERMIONIC CONDENSATE IN de SITTER SPACETIME

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Fermionic condensate is investigated in (D+1)-dimensional de Sitter spacetime by using the cutoff function regularization. In order to fix the renormalization ambiguity for massive fields an additional condition is imposed, requiring the condensate to vanish in the infinite mass limit. For large values of the field mass the condensate decays exponentially in odd dimensional spacetimes and follows a power law decay in even dimensional spacetimes. For a massless field the fermionic condensate vanishes for odd values of the spatial dimension D and is nonzero for even D. Depending on the spatial dimension the fermionic condensate can be either positive or negative. The change in the sign of the condensate may lead to instabilities in interacting field theories.

Keywords: fermionic condensate: de Sitter spacetime: Bunch-Davies vacuum

1. Introduction. De Sitter (dS) spacetime is among the frequently used background geometries for the investigation of the influence of gravitational field on quantum matter. In the early stages of studies this interest was motivated by high symmetry of the corresponding geometry. The dS spacetime is the maximally symmetric solution of Einstein's equation with a positive cosmological constant as the only source of gravitational field and because of that a relatively large number of physical problems can be exactly solved on that background. This helps to shed light on the effects of gravity on quantum fields in more complicated geometries. The further increase of the interest to the investigations of quantum effects on dS bulk was related to the appearance of the inflationary scenario for the expansion of the early Universe (for reviews see [1,2]). In most inflationary models the expansion is described by an approximately dS geometry sourced by the potential energy of a scalar field (inflaton). A short period of the corresponding quasiexponential expansion provides a natural solution to a number of fine tuning problems of the standard Big Bang model (horizon and flatness problems, the problem of topological defects, etc.). In addition, the inflationary scenario leads to an interesting mechanism for the generation of small inhomogeneities in the energy distribution at the beginning of the radiation dominated cosmological expansion that seed the large scale structure of the Universe at late stages. This mechanism is based on the classicalization of quantum fluctuations of scalar fields by an inflationary expansion. Its predictions are in good agreement with the

observational data about the temperature anisotropies of the cosmic microwave background. Those data, in combination with observations of high redshift supernovae and galaxy clusters indicate that the expansion of the Universe at the present epoch is well approximated by a model where the dominant part of the energy content is described by the equation of state close to the one for a positive cosmological constant. The cosmological expansion with this type of gravitational source will lead to an asymptotically dS universe as the future attractor. This shows that the investigation of physical effects in dS spacetime is also important for the future of the Universe.

The expectation values of bilinear combinations of quantum fields with different spins (field squared, energy-momentum tensor) for the Bunch-Davies vacuum in dS spacetime have been investigated in a large number of papers (see [3-6] and references therein). In particular, the Green function and the effective Lagrangian for a spinor field have been discussed in [7]. The expression for the renormalized vacuum expectation value (VEV) of the energy-momentum tensor for a spinor field in 4-dimensional dS spacetime is derived in [8] by using the *n*-wave regularization method. The same result is obtained in [9] by using the regularization based on a cutoff function. In [9] the fermionic condensate is investigated as well. The fermionic condensate and the VEV of the energymomentum tensor for a spinor field in (D+1)-dimensional dS spacetime for even values of D have been investigated in [10] by using the point-splitting regularization technique. The shifts in the VEVs for spinor fields induced by the toroidal compactification of a part of spatial dimensions in dS spacetime were studied in [10-12]. Another class of topological effects caused by the presence of a cosmic string in dS bulk have been discussed in [9].

In the present paper we investigate the renormalized fermionic condensate in (D+1)-dimensional dS spacetime for general value of the spatial dimension D. The regularization procedure will be based on the introduction of a cutoff function in the corresponding integral representation. In addition to the VEV of the energy-momentum tensor, the fermionic condensate is an important local characteristic of the fermionic vacuum. Though the corresponding operator is local, because of the global nature of the notion of vacuum, it contains information about global properties of the background geometry. The fermionic condensate is an important characteristic in quantum chromodynamics, in the physics of superconductivity and phase transitions, in models of dynamical mass generation and symmetry breaking. It has been investigated in various types of physical models, including the ones for curved backgrounds (see, for example, [13-20]).

The paper is organized as follows. In the next section, we describe the background geometry and present the complete set of fermionic normal modes. The expression for the fermionic condensate, regularized with the help of cutoff

function, is provided. The extraction of divergences and the renormalization of the corresponding VEV differ for dS spacetimes with even and odd numbers of spatial dimensions and we describe the respective procedures in sections 3 and 4, respectively. Closed analytic expressions are derived for the renormalized fermionic condensate in both these cases. In section 5 we consider a model with interacting scalar and fermionic fields where the fermion condensate determines the effective mass of the scalar field. The main results are summarized in section 6.

2. Regularized fermionic condensate in dS spacetime. We consider a quantum fermionic field ψ on background of (D+1)-dimensional de Sitter spacetime described by the line element

$$ds^{2} = dt^{2} - e^{2t/\alpha} \sum_{i=1}^{D} (dz^{i})^{2}, \qquad (1)$$

in planar coordinates $(t, z^1, ..., z^D)$. The parameter α determines the Hubble constant and is related to the corresponding positive cosmological constant Λ by the formula $\alpha^2 = D(D-1)/2\Lambda$. In addition to comoving time coordinate t, we will use the conformal time τ defined by the relation $\tau = -\alpha e^{-t/\alpha}$, $-\infty < \tau < 0$. In terms of this coordinate, the line element (1) takes a conformally flat form with the conformal factor $(\alpha/\tau)^2$. The dynamics of the field in a curved spacetime is governed by the Dirac equation

$$i\gamma^{\mu}(\partial_{\mu}+\Gamma_{\mu})\psi-m\psi=0, \qquad (2)$$

where $\gamma^{\mu} = e_{(a)}^{\mu} \gamma^{(a)}$ are the curved spacetime Dirac matrices and Γ_{μ} is the spin connection. The vielbein fields obey the relation $e_{(a)}^{\mu} e_{(b)}^{\nu} \eta^{ab} = g^{\mu\nu}$, with η^{ab} being the Minkowski spacetime metric tensor and $g_{\mu\nu} = \text{diag}(1, -e^{2t/\alpha}, ..., -e^{2t/\alpha})$. The flat-space Dirac matrices $\gamma^{(a)}$ are $N \times N$ matrices with $N = 2^{[(D+1)/2]}$, where the square brackets mean the integer part of the enclosed expression. In the discussion below these matrices will be taken in the Dirac representation:

$$\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a^+ & 0 \end{pmatrix}, \tag{3}$$

with a=1, 2, ..., D and $(N/2) \times (N/2)$ matrices σ_a . By using the anticommutation relations for $\gamma^{(a)}$ one gets $\sigma_a \sigma_b^+ + \sigma_b \sigma_a^+ = 2\delta_{ab}$. For the geometry under consideration we can take the vielbein fields in the form $e_{\mu}^{(0)} = \delta_{\mu}^0$, $e_{\mu}^{(a)} = e^{t/a} \delta_{\mu}^a$, a=1, 2, ..., D. The components of the spin connection are expressed as $\Gamma_0 = 0$, $\Gamma_l = (e^{t/a}/2\alpha)\gamma^{(0)}\gamma^{(l)}$, l=1, 2, ..., D.

The fermionic condensate in the vacuum state $|0\rangle$ is defined as the VEV $\langle 0|\overline{\psi}\psi|0\rangle = \langle \overline{\psi}\psi\rangle$, where the Dirac adjoint is expressed as $\overline{\psi} = \psi^{\dagger}\gamma^{(0)}$. In the discussion below we will assume that the state $|0\rangle$ corresponds to the maximally symmetric Bunch-Davies vacuum. Note that the maximal symmetry does not uniquely define the vacuum state. As it has been discussed in [21], in dS spacetime

there is a one-complex-parameter family of maximally symmetric states. Among those states the Bunch-Davies vacuum is singled out as the only state having the Hadamard structure of singularities.

Given the complete set of solutions to the equation (2), denoted here as $\left\{\psi_{B}^{(+)},\psi_{B}^{(-)}\right\},$ the fermion condensate is written as the mode-sum

$$\left\langle \overline{\psi}\psi\right\rangle = \frac{1}{2} \sum_{\beta} \left(\overline{\psi}_{\beta}^{(-)}\psi_{\beta}^{(-)} - \overline{\psi}_{\beta}^{(+)}\psi_{\beta}^{(+)} \right). \tag{4}$$

Here, $\psi_{\beta}^{(+)}$ and $\psi_{\beta}^{(-)}$ are the analogs of the positive and negative energy mode functions in the Minkowski bulk and the collective index β presents the set of quantum numbers. In (4), the symbol \sum_{β} is understood as a summation over the discrete quantum numbers and an integration over the continuous ones. In the problem under consideration the mode functions are specified by the momentum $\mathbf{k} = (k^1, ..., k^D)$ and by the quantum number σ taking the values $\sigma = 1, ..., N/2$ (hence, $\beta = (\mathbf{k}, \sigma)$). They are given by the expressions

$$\psi_{\beta}^{(+)} = C(k)\eta^{(D+1)/2} e^{i\mathbf{k}\mathbf{r}} \begin{pmatrix} H_{1/2-i\alpha m}^{(1)}(k\eta)w_{\sigma}^{(+)} \\ -i(\mathbf{n}\sigma)H_{-1/2-i\alpha m}^{(1)}(k\eta)w_{\sigma}^{(+)} \end{pmatrix},
\psi_{\beta}^{(-)} = C(k)\eta^{(D+1)/2} e^{i\mathbf{k}\mathbf{r}} \begin{pmatrix} -i(\mathbf{n}\sigma)H_{-1/2+i\alpha m}^{(2)}(k\eta)w_{\sigma}^{(-)} \\ H_{1/2+i\alpha m}^{(2)}(k\eta)w_{\sigma}^{(-)} \end{pmatrix},$$
(5)

where $\eta = -\tau$, $k = |\mathbf{k}|$, $\mathbf{n} = \mathbf{k}/k$, $\mathbf{kr} = \sum_{i=1}^{D} k^{i} z^{i}$, $H_{\nu}^{(1,2)}(z)$ are the Hankel functions, and $\boldsymbol{\sigma} = (\sigma_{1}, \sigma_{2}, ..., \sigma_{D})$. In (5), the one-column matrices $w_{\sigma}^{(\pm)}$ have N/2 rows and the elements $w_{\sigma l}^{(+)} = \delta_{\sigma l}$, $w_{\sigma l}^{(-)} = i \delta_{\sigma l}$. The normalization coefficient C(k) is expressed as

$$C(k) = \frac{\sqrt{k} e^{\pi \alpha m/2}}{2^{D/2+1} \pi^{(D-1)/2} \alpha^{D/2}}.$$
 (6)

Similar mode functions in locally dS spacetime with a toroidally compactified subspace are presented in [10]. The mode functions for Dirac fermions in 4-dimensional dS spacetime have also been considered in [22]. For a massless field, by taking into account that $H_{1/2}^{(1)}(x) = -i\sqrt{2/\pi x} e^{ix}$, we get the conformal relation $\psi_{\beta}^{(\pm)} = (\eta/\alpha)^{D/2} \psi_{(M)\beta}^{(\pm)}$ with the corresponding modes in Minkowski spacetime.

Substituting the normal modes (5) in (4), for the fermionic condensate we find

$$\langle \overline{\psi}\psi \rangle = \frac{\eta^{D+1} e^{\pi \alpha m} N}{2^{D+2} \pi^{D/2-1} \Gamma(D/2) \alpha^{D}} \int_{0}^{\infty} dk k^{D} \times \left[H^{(1)}_{-1/2-i\alpha m}(k \eta) H^{(2)}_{-1/2+i\alpha m}(k \eta) - H^{(1)}_{1/2-i\alpha m}(k \eta) H^{(2)}_{1/2+i\alpha m}(k \eta) \right].$$
(7)

The expression on the right-hand side is divergent and some renormalization procedure is necessary. Introducing the Macdonald function instead of the Hankel function, the formula (7) is rewritten as

$$\left\langle \overline{\Psi}\Psi \right\rangle = \frac{2^{-D} \alpha^{-D} \eta^{D+1} N}{i \pi^{D/2+1} \Gamma(D/2)} \left(\partial_{\eta} + \frac{1-2 im \alpha}{\eta} \right)$$
$$\times \int_{0}^{\infty} dk k^{D-1} K_{1/2-im\alpha}(ik \eta) K_{1/2-im\alpha}(-ik \eta).$$
(8)

In deriving this representation we have used the relation

$$K_{\nu}(y)K_{\nu-1}(-y) - K_{\nu}(-y)K_{\nu-1}(y) = \left(\partial_{y} + \frac{2\nu}{y}\right)K_{\nu}(y)K_{\nu}(-y),$$
(9)

with y = ix and $v = 1/2 - im\alpha$. This relation directly follows from the recurrence relations for the Macdonald function.

In order to obtain an alternative integral representation of the fermionic condensate, for the product of the Macdonald functions (8) we use the formula [23]

$$K_{\nu}(ik\eta)K_{\nu}(-ik\eta) = \int_{0}^{\infty} dy \cosh(2\nu y) \int_{0}^{\infty} \frac{du}{u} \exp\left[-2(k\eta \sinh y)^{2} u - \frac{1}{2u}\right].$$
 (10)

Substituting this into (8), we first integrate over k. Then, instead of u we introduce a new integration variable $x = 1/(u\eta^2 \sinh^2 y)$. After changing the order of the integrations, the integral over y is expressed in terms of the Macdonald function and we find

$$\left\langle \overline{\psi}\psi\right\rangle = -\frac{i\,\alpha^{-D}\eta^{D+1}\,N}{2(2\pi)^{D/2+1}} \left(\partial_{\eta} + \frac{1-2\,im\,\alpha}{\eta}\right) \int_{0}^{\infty} dx x^{D/2-1} e^{x\eta^{2}} K_{1/2-im\alpha}\left(x\,\eta^{2}\right). \tag{11}$$

Using the relation

$$(\eta \partial_{\eta} + 2\nu) e^{x\eta^2} K_{\nu} (x\eta^2) = 2x\eta^2 e^{x\eta^2} [K_{\nu} (x\eta^2) - K_{\nu-1} (x\eta^2)],$$
 (12)

the condensate can also be presented in the form

$$\left\langle \overline{\Psi}\Psi \right\rangle = \frac{2\alpha^{-D}N}{(2\pi)^{D/2+1}} \int_{0}^{\infty} dx x^{D/2} e^{x} \operatorname{Im} K_{1/2-im\alpha}(x).$$
(13)

The integral in the right-hand side diverges in the upper limit.

For the further evaluation an explicit regularization scheme should be used. As such a scheme we will introduce an exponential cutoff function e^{-sx} , s > 0, in the integrand of (13) with the regularized expression

$$\langle \overline{\psi}\psi \rangle^{(s)} = \frac{2\alpha^{-D}N}{(2\pi)^{D/2+1}} \int_{0}^{\infty} dx x^{D/2} e^{(1-s)x} \operatorname{Im} K_{1/2-im\alpha}(x).$$
 (14)

The limit $s \rightarrow 0$ should be taken at the and of calculations.

In (14), the integral over x is explicitly evaluated in terms of the associated Legendre function (see [24]) and we find

$$\left\langle \overline{\psi}\psi\right\rangle^{(s)} = \frac{\alpha^{-D} N}{(2\pi)^{(D+1)/2}} \operatorname{Im}\left[\Gamma\left(\mu + im\,\alpha\right)\Gamma\left(\mu + 1 - im\,\alpha\right)\frac{P_{-im\alpha}^{-\mu}\left(-\gamma\right)}{\left(1 - \gamma^{2}\right)^{\mu/2}}\right],\tag{15}$$

with $\gamma = 1 - s$ and

$$\mu = \frac{D+1}{2}.\tag{16}$$

For the product of the gamma functions in this formula one has

$$\Gamma(\mu + im\alpha)\Gamma(\mu + 1 - im\alpha) = B_D(m\alpha)(\mu - im\alpha)\prod_{l=1}^{\lfloor D/2 \rfloor} [(\mu - l)^2 + m^2\alpha^2], \quad (17)$$

where [D/2] stands for the integer part of D/2, and the function

$$B_D(x) = \begin{cases} \pi x / \sinh(\pi x), & \text{for odd } D, \\ \pi / \cosh(\pi x), & \text{for even } D, \end{cases}$$
(18)

is introduced. Now we want to expand the regularized fermionic condensate in powers of s. The further discussion should be developed for even and odd values D separately.

3. Condensate in even dimensional spacetimes. First we consider odd values of the spatial dimension D. In this case μ is an integer and the corresponding Legendre function in (15) is expressed in terms of the hypergeometric function as follows:

$$P_{-im\alpha}^{-\mu}\left(-\gamma\right) = \frac{\Gamma\left(1 - im\alpha - \mu\right)}{\Gamma\left(1 - im\alpha + \mu\right)} \left(1 - \gamma^2\right)^{\mu/2} \partial_{\gamma}^{\mu} F\left(im\alpha, 1 - im\alpha; 1; \frac{1 + \gamma}{2}\right).$$
(19)

Substituting this into the expression for the regularized fermionic condensate, we get

$$\left\langle \overline{\psi}\psi\right\rangle^{(s)} = -\frac{\pi N \alpha^{-D}}{(2\pi)^{\mu} \sinh(\pi m \alpha)} \operatorname{Re}\left[\partial_{s}^{\mu} F(im\alpha, 1 - im\alpha; 1; 1 - s/2)\right].$$
(20)

The expansion of the right-hand side of this expression is given by the formula [25]

$$F(im\alpha, 1 - im\alpha; 1; 1 - s/2) = \frac{i}{\pi} \sinh(\pi m\alpha) \sum_{n=0}^{\infty} a_n [b_n - \ln(s/2)] (s/2)^n , \qquad (21)$$

for the hypergeometric function. In this formula,

$$a_n = \frac{(im\alpha)_n (1 - im\alpha)_n}{(n!)^2}, \quad b_n = 2\Psi(n+1) - \Psi(n+im\alpha) - \Psi(n+1 - im\alpha), \tag{22}$$

where $(c)_n$ is Pochhammer's symbol, $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function (here we use the notation $\Psi(x)$ for the digamma function instead of the standard one $\psi(x)$ in order to avoid the confusion with the fermion field ψ). With the use of (21), we have the following expansion

$$\langle \overline{\psi}\psi\rangle^{(s)} = \frac{\alpha^{-D} N}{(4\pi)^{\mu}} \Biggl[\sum_{l=1}^{\mu} C_l^{\mu} (-1)^l (l-1)! \sum_{n=1}^{l} \frac{\mathrm{Im}(a_{\mu-n})}{(s/2)^n} (l-n+1)_{\mu-l} - \mu! \mathrm{Im}(a_{\mu}) \ln(s/2) + \sum_{l=1}^{\mu} C_l^{\mu} (-1)^l (l-1)! \mathrm{Im}(a_{\mu}) (l+1)_{\mu-l} + \mu! \mathrm{Im}(a_{\mu}b_{\mu}) + \ldots \Biggr],$$

$$(23)$$

where C_l^{μ} are the binomial coefficients and the dots stand for the terms which vanish in the limit $s \rightarrow 0$. As it is seen from (23), we have the power-law divergent terms, logarithmically divergent term, and the finite part. Note that the coefficients (22) can also be written in the form

$$a_{n} = \frac{im \alpha (n!)^{-2}}{n + im \alpha} \prod_{l=1}^{n} [l^{2} + m^{2} \alpha^{2}], \quad a_{0} = 1,$$

$$b_{n} = 2\Psi(n+1) - 2\operatorname{Re}[\Psi(n + im \alpha)] - \frac{1}{n - im \alpha}.$$
(24)

On the basis of the expansion (23), taking into account the finite renormalization terms, the renormalized fermionic condensate is written in the form

$$\left\langle \overline{\Psi}\Psi \right\rangle_{\text{ren}} = -\frac{\alpha^{-D} N}{(4\pi)^{\mu}} \frac{(m\alpha)^{2\mu-1}}{\Gamma(\mu)} \left\{ \sum_{l=1}^{\mu-1} \frac{f_l}{(m\alpha)^{2l}} + 2\left\{ \text{Re}[\Psi(im\alpha)] - \ln(m\alpha) \right\} \prod_{l=1}^{\mu-1} \left(1 + \frac{l^2}{m^2 \alpha^2} \right) \right\},$$

$$(25)$$

where we have used the relation $\operatorname{Re}[\Psi(1+im\alpha)] = \operatorname{Re}[\Psi(im\alpha)]$ which directly follows from the formula $\Psi(1+z) = \Psi(z) + 1/z$ for the digamma function (see [25]). In (25), the coefficients f_i should be fixed by an additional renormalization condition (for a discussion of ambiguities in the renormalization of the expectation value of the energy-momentum tensor in the Hadamard renormalization procedure for general number of spatial dimensions see [26]). As such a condition we require that $\langle \overline{\Psi}\Psi \rangle_{ren} \to 0$ in the limit $m \to \infty$. By using the expansion

$$\operatorname{Re}[\Psi(im\,\alpha)] = \ln(m\,\alpha) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n}}{2 n (m\,\alpha)^{2n}},$$
(26)

with B_{2n} being the Bernoulli coefficients, and requiring the cancellation of the terms in (25) with positive powers of the mass, we find

$$\sum_{n=1}^{\mu-1} \frac{(-1)^n B_{2n}}{nx^n} \prod_{l=1}^{\mu-1} \left(1 + \frac{l^2}{x}\right) = -\sum_{l=1}^{\mu-1} \frac{f_l}{x^l} + \dots,$$
(27)

This relation defines the values of the coefficients f_1 in the expression (25) for the renormalized fermionic condensate. In particular, one has $f_1 = -1/6$ for D=3, $f_1 = -1/6$, $f_2 = -17/20$ for D=5, and $f_1 = -1/6$, $f_2 = -47/20$, $f_3 = -5297/630$ for D=7. In the cases D=3 and D=5 from (25) one finds

$$\left\langle \overline{\psi}\psi\right\rangle_{\rm ren} = \frac{m}{2\pi^2\alpha^2} \left\{ \left(\ln(m\alpha) - \operatorname{Re}[\Psi(im\alpha)]\right) \left(m^2\alpha^2 + 1\right) + \frac{1}{12} \right\},$$

$$\left\langle \overline{\psi}\psi\right\rangle_{\rm ren} = \frac{m}{8\pi^3\alpha^4} \left\{ \frac{m^2\alpha^2}{12} + \frac{17}{40} + \left(\ln(m\alpha) - \operatorname{Re}[\Psi(im\alpha)]\right) \left(m^2\alpha^2 + 1\right) \left(m^2\alpha^2 + 4\right) \right\}.$$
(28)

For D=3 the result (28) coincides with the corresponding expression obtained previously in [9]. For a massless field the renormalized fermionic condensate vanishes. For large masses, $m\alpha >>1$, the condensate behaves as $\alpha^{-D}/(m\alpha)$. In Fig.1 we have plotted the fermionic condensate as a function of $m\alpha$ for D=3and D=5. In these cases the fermionic condensate is negative for massive fields.



Fig.1. Fermionic condensate versus $m\alpha$ for the spatial dimensions D=3 and D=5.

4. *Fermionic condensate in odd dimensional spacetime*. In the renormalization procedure we need the expansion of the expression on the right-hand side of (15) near the point $\gamma = 1$. For even values of *D*, this expansion for the associated Legendre function directly follows from the formula

$$\frac{P_{-im\alpha}^{-\mu}(-\gamma)}{\left(1-\gamma^{2}\right)^{\mu/2}} = \frac{(2-s)^{-\mu}\sinh(\pi\,m\,\alpha)}{i\,\Gamma(1+\mu)\sin(\mu\pi)}F(im\alpha,1-im\alpha;1+\mu;s/2)$$

$$+\frac{1}{\pi}\sin[\pi(im\alpha+\mu)]\frac{\Gamma(1-im\alpha-\mu)\Gamma(\mu)}{\Gamma(1-im\alpha+\mu)s^{\mu}}F(im\alpha,1-im\alpha;1-\mu;s/2).$$
(29)

The standard definition of the hypergeometric function in terms of the series over s provides the required expansion. For the case under consideration μ is a half-

582

integer and, hence, the second term on the right-hand side of (29) does not contribute to the finite part, whereas the first term is finite in the limit $s \rightarrow 0$. Substituting expression (29) into formula (15) and using the expansion for the hypergeometric function in the second term on the right-hand side, we find the following expansion for the regularized fermionic condensate

$$\langle \overline{\psi}\psi \rangle^{(s)} = \frac{N\Gamma((1-D)/2)}{(4\pi)^{\mu}\alpha^{D}} \left\{ \frac{\Gamma(\mu)}{m\alpha} \sum_{n=1}^{[\mu]} \frac{n(s/2)^{n-\mu}}{\Gamma(1-\mu+n)n!} \prod_{l=0}^{n-1} (l^{2}+m^{2}\alpha^{2}) - \tanh(\pi m\alpha) \prod_{l=0}^{D/2-1} [(l+1/2)^{2}+m^{2}\alpha^{2}] + \ldots \right\},$$
(30)

where, as before, the dots stand for the terms which vanish in the limit $s \rightarrow 0$.

From formula (30) we find the following expression for the renormalized fermionic condensate:

$$\left\langle \overline{\psi}\psi\right\rangle_{\rm ren} = -\frac{Nm^D}{\left(4\pi\right)^{\mu}} \Gamma\left(\frac{1-D}{2}\right) \left\{ \sum_{l=0}^{D/2} \frac{c_l}{\left(m\,\alpha\right)^{2l}} + \tanh\left(\pi\,m\,\alpha\right) \prod_{l=1}^{D/2} \left[1 + \left(\frac{l-1/2}{m\,\alpha}\right)^2\right] \right\}.$$
 (31)

where the coefficients c_i are determined from the renormalization condition $\langle \overline{\psi} \psi \rangle_{ren} \rightarrow 0$ for $m \rightarrow \infty$. From this condition it follows that

$$\sum_{l=0}^{D/2} \frac{c_l}{(m\alpha)^{2l}} = -\prod_{l=1}^{D/2} \left[1 + \left(\frac{l-1/2}{m\alpha}\right)^2 \right].$$
 (32)

This leads to the following formula for the fermionic condensate

$$\left\langle \overline{\psi}\psi\right\rangle_{\rm ren} = \frac{(-1)^{D/2} (4\pi)^{(1-D)/2} N \alpha^{-D}}{2\Gamma((D+1)/2) (e^{2\pi m \alpha} + 1)} \prod_{l=1}^{D/2} \left[m^2 \alpha^2 + (l-1/2)^2\right].$$
(33)

This expression coincides with the result obtained in [10] by using the pointsplitting procedure and the adiabatic subtraction. Hence, we have shown that the different renormalization schemes give the same result for the renormalized fermionic condensate. The sign of the fermionic condensate (33) coincides with the sign of $(-1)^{D/2}$. For large values of the mass, $m\alpha \gg 1$, the fermionic condensate (33) is suppressed by the factor $m^D \alpha^D e^{-2\pi m\alpha}$. Unlike to the case of odd *D*, in even number of spatial dimensions the fermionic condensate for a massless field differs from zero (see also [10]):

$$\langle \overline{\psi}\psi \rangle_{\rm ren} = \frac{(-1)^{D/2} N \Gamma((D+1)/2)}{(4\pi)^{(D+1)/2} \alpha^D}.$$
 (34)

In Fig.2 the dependence of the fermionic condensate on $m\alpha$ is presented for several values of the spatial dimension.

583



Fig.2. Fermionic condensate as a function of $m\alpha$ for D=2, 4, 6.

5. *Interacting scalar and fermion fields*. Nonzero fermionic condensate can be of considerable importance in interacting field theories. As an example, here we consider a system of interacting fermionic and scalar fields described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{2} M^2 \varphi^2 + \frac{i}{2} \left[\overline{\psi} \gamma^{\mu} \nabla_{\mu} \psi - \left(\nabla_{\mu} \overline{\psi} \right) \gamma^{\mu} \psi \right] - m \overline{\psi} \psi - \lambda \varphi^2 \overline{\psi} \psi , \qquad (35)$$

with the coupling constant λ having the dimension (length)^{*D*-2}. The corresponding field equations read

$$\left(\Box + M^{2} + 2\lambda\overline{\psi}\psi\right)\phi = 0, \quad \left(i\gamma^{\mu}\nabla_{\mu} - m - \lambda\phi^{2}\right)\psi = 0, \quad (36)$$

where \Box stands for d'Alembert operator for scalar fields.

Assume that the field Ψ is quantized and the field Φ is a classical field. If $\langle \overline{\psi} \psi \rangle_{ren}$ is the renormalized fermion condensate, then the classical dynamics of the scalar field is described by the equation

$$\left(\Box + M^2 + 2\lambda \langle \overline{\psi}\psi \rangle_{\rm ren}\right)\varphi = 0.$$
(37)

As it is seen, the effect of the interaction of the scalar field with the fluctuations of the fermionic field is equivalent to the change of the mass term. For a general background the effective mass depends on the spacetime point. In the case of dS bulk the fermion condensate is constant and the interaction leads to a constant shift in the squared mass term for the scalar field. In general, this shift can be negative and under the condition $M^2 + 2\lambda \langle \overline{\psi}\psi \rangle_{ren} < 0$ the effective mass becomes

tachyonic. The tachyonic mass may lead to an instability of the corresponding field theory (for instabilities in interacting scalar field theories induced by background geometry, nontrivial topology and boundaries see [27]). Note that, in a similar way, the quantum fluctuations of the scalar field lead to the correction of the fermionic mass term in the form $\lambda \langle \phi^2 \rangle_{ren}$. In a more general case of a scalar field with the potential $V(\phi)$, the interaction with the vacuum fluctuations of a fermionic field leads to the correction with the effective potential $V_{eff}(\phi) = V(\phi) + \lambda \langle \overline{\psi}\psi \rangle_{ren} \phi^2$. In particular, this type of correction to the inflaton potential can have important consequences in the inflationary scenario.

Similar to the case of the system of interacting fermion and scalar fields, the nonzero fermionic condensate leads to the shift of the fermion effective mass in the Nambu-Jona-Lasinio type models. These models contain four fermion interaction term $g(\overline{\psi}\psi)^2$ in the Lagrangian density, with g being the four fermion coupling constant. They were applied to describe the dynamical symmetry breaking in electroweak theory and quantum chromodynamics (for symmetry breaking in the Nambu-Jona-Lasinio model in curved spacetime see, for example, [28,29]). The corresponding effective mass for a fermion field becomes $m-2g(\overline{\psi}\psi)_{ren}$. Again, we see that, depending on the fermionic condensate, the effective mass may become negative.

6. Conclusion. In the present paper we have investigated the fermionic condensate for a massive spinor field in dS spacetime in an arbitrary number of spatial dimensions. In Section 2, an expression for the corresponding regularized quantity is derived assuming that the field is prepared in the Bunch-Davies vacuum state. The renormalization procedure for even and odd dimensional spacetimes is considered separately. In even dimensional dS spacetime the renormalized fermionic condensate is given by expression (25), where the coefficients are obtained from the condition of vanishing the condensate in the limit $m \rightarrow \infty$. These coefficients are defined by the relation (27). For large values of the field mass, the condensate decays as $1/(m\alpha)$ and it vanishes for a massless field field. In odd dimensional dS spacetime, for the renormalized fermionic condensate we derived the formula (31), with the coefficients c_i defined from the relation (32). In this case, for large values of the mass the fermionic condensate decays exponentially. Unlike the case of even dimensions, for a massless field the condensate does not vanish.

Another vacuum state in dS spacetime is the hyperbolic vacuum [30-33]. It is naturally realized by the normal modes of quantum fields in the coordinate system foliating the spacetime by spatial sections with constant negative curvature. Unlike to the Bunch-Davies vacuum, the hyperbolic vacuum is not maximally symmetric and the corresponding fermionic condensate will depend on time. This feature has been demonstrated for the expectation values of the field squared and

energy-momentum tensor in the case of a scalar field (see [30,33]). For a massless fermionic field we expect that the difference in the fermionic condensates for the Bunch-Davies and hyperbolic vacua will decay at late stages of the expansion like $1/t^D$. This is in agreement with the general result in accordance of which the Bunch-Davies vacuum is a future attractor for relatively large class of states in dS spacetime. Note that the renormalization of the fermionic condensate for the hyperbolic vacuum is reduced to the renormalization for the Bunch-Davies vacuum and the difference in the corresponding VEVs is finite.

In interacting field-theoretical models (self-interacting fermionic field, fermionic fields interacting with scalar or vector fields) the formation of nonzero fermionic condensate may lead to phase transitions. We have considered two examples. The first one presents a system of scalar and fermionic fields with the interaction Lagrangian density proportional to $\varphi^2 \overline{\psi} \psi$ and the second one corresponds to the Nambu-Jona-Lasinio type model with the self interaction $(\overline{\psi}\psi)^2$. Depending on the value and sign of the condensate, the effective mass squared may become negative. Scalar-fermionic models with the interaction $\cosh(\overline{\psi}\psi)^2 + \cosh(\overline{\psi}\psi)^2$ have also been considered in the literature. In this type of models the nonzero condensate may lead to the change of the sign of the kinetic term for the scalar field (ghost field).

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FERMIONIC CONDENSATE IN de SITTER SPACETIME

ФЕРМИОННЫЙ КОНДЕНСАТ В ПРОСТРАНСТВЕ-ВРЕМЕНИ де СИТТЕРА

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Исследован фермионный конденсат в (D+1)-мерном пространстве-времени де Ситтера с помощью регуляризации обрезающей функцией. Для устранения неоднозначности перенормировки в случае массивных полей, накладывается дополнительное условие, требующее зануление конденсата в пределе бесконечной массы. Для больших значений массы поля конденсат экспоненциально затухает в пространстве-времени нечетной размерности и стремится к нулю по степенному закону в четно-мерном пространстве-времени. Для безмассового поля фермионный конденсат обращается в нуль при нечетных значениях пространственной размерности D и отличен от нуля при четном D. В зависимости от пространственной размерности фермионный конденсат может быть как положительным, так и отрицательным. Изменение знака конденсата может привести к неустойчивостям во взаимодействующих теориях поля.

Ключевые слова: фермионный конденсат: пространство-время де Ситтера: вакуум Банча-Девиса

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587

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