АСТРОФИЗИКА

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EXACT STATIONARY SOLUTIONS TO A GENERAL FORM OF KOMPANEETS EQUATION

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In the present work, we are proposing a method to build solutions to a general form of the stationary Kompaneets equation. In the non-relativistic regime, a special attention is given to cases where the solutions are expressed in terms of Heun functions. A comparison with the results obtained within a numerical analysis is also briefly discussed.

Keywords: *Kompaneets equation: Comptonization processes: Heun functions*

1. *Introduction.* Soon after Kompaneets derived, in 1957, the equation bearing his name [1], for non-relativistic plasmas, where the Compton scattering is the dominant process responsible for energy transport, this has become a very useful tool for modeling fundamental phenomena in modern cosmology and astrophysics.

In this respect, the frequency distribution of photons provides important information on various astrophysical environments, as for example intracluster gas, the coronae of accretion disks around black holes in binary stellar systems and active galactic nuclei (AGNs), and plasma streams out owing from neutron stars.

Besides detecting and characterizing galaxy clusters, the Sunyaev-Zeldovich effect [2,3], meaning the distortion of the cosmic microwave background radiation through inverse Compton scattering by high energy electrons, is of major interest for constraining cosmological parameters [4-6].

The existence of high-temperature galaxy clusters, with hot electrons of about $kT \sim 20\,\mathrm{kEV}$, has revealed the need to take into account relativistic corrections [7,8] and generalized forms of the Kompaneets equation have been proposed [9,10].

After Duncan and Thompson introduced the term of magnetar for a type of neutron star with massively boosted magnetic fields [11], the resonant inverse Compton scattering (RCS), which occurs when the photon frequency equals the cyclotron one, has been seen as a main candidate for quiescent nonthermal gamma-ray emissions. This type of neutron stars have long periods and spin-down rates and they show a non-atomic Planckian spectrum. The corresponding Kompaneets equation is similar to the original one, with the difference that the frequency-depending RCS cross section must replace the Thomson's one [12].

Even though solutions to the Kompaneets-type equations are found mainly using numerical simulations, there are stationary cases which can be analytically solved in terms of Heun functions [13,14].

Let us remind the reader that these functions, introduced more than 100 years ago, by Karl Heun [15], have been brought to the scientific community attention at the Centennial Workshop on Heun Equations [16]. In brief, these are unique local Frobenius solutions to a second-order linear ordinary differential equation of the Fuchsian type with 4 regular singular points. Once the singularities coalesce, one gets the confluent Heun functions (with two regular and one irregular singularities) or the double confluent Heun functions (with two irregular singularities) [17]. Even though, in the last decade, there is a raising number of articles on the Heun functions and their applications in theoretical and applied science [18-21], there are serious gaps in understanding different properties of these functions and how MAPLE (the only program available) is dealing with their singularities and is computing the derivatives for specific ranges of the model's parameters. That is why, most of the people are preferring to rely on numerical methods.

2. *A method for generating solutions to stationary Kompaneets-type equations*. Let us start by considering the general equation

$$g\frac{dn}{dx}+h[n(n+1)]=f\,, \tag{1}$$

where the functions $g(x)$, $h(x)$ and $f(x)$ are defined on $\mathbf{R}_+$. By dividing the above relation by n^2 and introducing $z=1/n$, one comes to the relations

$$\frac{dz}{dx}=\frac{h}{g}(z+1)-\frac{f}{g}z^2$$

and

$$\frac{dw}{dx}=\frac{h}{g}w-\frac{f}{g}(w-1)^2\,, \tag{2}$$

with $w=z+1$. Using the following solution of the homogeneous equation

$$w(x)=K(x)\exp\left[\int^x\frac{h}{g}ds\right],$$

the relation (2) leads to the following Master Equation for the function $K(x)$ depending on the explicit form of the functions g, h and f,

$$\frac{dK}{dx}=-\frac{f}{g}\exp\left[-\int^x\frac{h}{g}ds\right]\left\{K\exp\left[\int^x\frac{h}{g}ds\right]-1\right\}^2\,. \tag{3}$$

In the above relations, s is a real-valued integration variable which runs from any inferior physical value up to the current real value x. Thence, as in the mathematical textbooks, the integration variable s is termed as *a mute variable*, so that it runs up to the superior value x, which is the true argument of the resulting function $w(x)$.

This procedure enables us to compute the number of scattered photons as

$$n(x)=\left[K(x)\exp\left[\int^{x}\frac{h}{g}ds\right]-1\right]^{-1}. \tag{4}$$

For example, let us deal with the famous original form of Kompaneets equation [1], for describing the evolution of the photon density, in the frequency space,

$$\frac{\partial n}{\partial t}=\frac{kT}{mc^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[Nc\sigma_T x^4\left(\frac{dn}{dx}+n+n^2\right)\right], \tag{5}$$

where x is the dimensionless frequency $x=h\nu/(kT)$, with $h\nu$ representing the photon energy and T being the electron temperature, n is the density of photons in the spectral interval dx, N is the electron number density (assuming homogeneous), σ_T is the Thomson cross section and c is the speed of light.

The above form, proposed by Kompaneets, is describing the so-called up-Comptonization and occurs in radio and infrared astronomy, where the condition $h\nu << kT << mc^2$ is satisfied.

In the stationary regime, the equation (5) leads to the following relation

$$\frac{dn}{dx}+n(n+1)=\frac{Q}{x^4}, \tag{6}$$

where the positive constant Q, associated to the photon flux by

$$j(x)=\frac{kT}{mc^2}Nc\sigma_T x^2\left[\frac{dn}{dx}+n+n^2\right]=\frac{Q}{x^2},$$

is given by the dimensionless quantity

$$Q=\frac{kT}{mc^2}Nc\tau\sigma_T,$$

where τ is the characteristic time (scattering optical depth). As expected, for $Q=0$, the current $j(x)$ is vanishing and one gets the familiar Bose-Einstein equilibrium distribution

$$n_{BE}=\left(e^x-1\right)^{-1}. \tag{7}$$

Beginning with the original work of Kompaneets [1], the steady-state equation (6) has been discussed in numerous papers.

The three terms in the left hand side of (6) are responsible for the following physical processes: the first term corresponds to the diffusion of photons due to the Doppler effect and the transfer of energy from electrons to the radiation (energy gain with consequent cooling of electrons), the second one stands for the Compton effect (downward photon flow along the frequency axis) and the third term accounts for cooling of photons due to induced Compton scatterings.

By comparing (6) with the general equation (1), one may identify the functions $h(x)=g(x)=1$ and $f(x)=Q/x^4$ so that the relations (3) and (4), with the standard Cauchy conditions for thermal equilibrium, turn into the simple forms

$$\frac{dK}{dx}=-\frac{Q}{x^4}e^{-x}\left[Ke^x-1\right]^2 \tag{8}$$

and

$$n(x)=\left[Ke^x-1\right]^{-1}, \tag{9}$$

with $K(0_+) = 1_+$, for the initial distribution (7).

The relations (3) and (4) for the particular situation presented above can be analytically solved and $K(x)$ turns out to be expressed in terms of Heun double confluent functions. Even though the solution of (6), expressed by the Heun double confluent function and its derivative have been discussed in [13,14], we underline the fact that it is dificult to operate with this type of functions since there are unsolved problems related to the normalization procedure and their derivatives are not converging for physical ranges of the variable and parameters.

In what it concerns the numerical procedure, since the stationary Kompaneets equation (1) has only one singularity point at $x=0$ and is a single-variable ODE, the explicit Heun method is sufficient to evaluate the solutions. However, one has to pay attention to: the initial condition $n(x_0) = n_0$, the initial point n_0 and the sign and value of Q. The term Q/x^4 is very important for fixing the initial condition. For example, a value $x_0 \approx 10^{-2}$ leads to the huge value $Q/x^4 \approx 10^8 Q$.

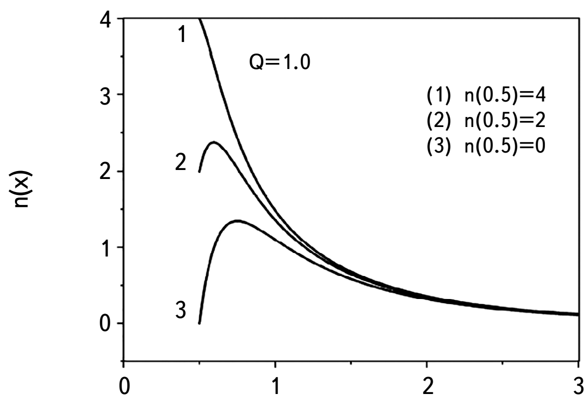

Fig.1. The effect of different initial conditions n_0.

Also, this complicates the use of normalized units in the numerical procedures, where Q has integer values, $Q \approx 10$, 100, 1000. Thus, the value of the term Q/x^4 will dictate the value of the first step of the Heun method:

$$\tilde{n}(x_0 + \Delta x) = n(x_0) + f(x_0, n(x_0))$$

and the process will continue, sending the solution to infinity. For this reason, it is recommended to make sure the denominator does not drift to higher negative powers.

In the Fig.1, one may notice the effect of different initial conditions n_0, for different curves, all starting at the same point x_0.

3. *Heun-type solutions for Kompaneets equation*. The equation (6) presented in the previous section is valid for $h\nu << kT << mc^2$ and it fails for describing the down-Comptonization of high energy photons, which is important in the hard X-ray or γ-ray astronomy. Therefore, in the last years, generalized forms of the original Kompaneets equation have been proposed.

Following Ross et al. [22], let us extend the equation (6) for the physically important case $kT << h\nu << mc^2$ by adding the contribution $ax^2 n'$, with

$$a = \frac{7}{10}\frac{kT}{mc^2}, \tag{10}$$

where the term

$$ax^2 = \frac{7}{10}\frac{h\nu}{kT}\frac{h\nu}{mc^2},$$

which has been neglected in the previous case, plays a significant role for highly energetic photons. Thus, one has to deal with the Riccati-type equation

$$\left(1 + ax^2\right)\frac{dn}{dx} + n(n+1) = \frac{Q}{x^4}, \tag{11}$$

which, compared to the general form (1), leads to the functions

$$g(x) = 1 + ax^2, \quad h(x) = 1, \quad f(x) = \frac{Q}{x^4}. \tag{12}$$

Consequently, the relations (3) and (4) turn into the following expressions

$$n(x) = \left[K(x)\exp\left(\frac{1}{\sqrt{a}}\arctan\left(\sqrt{a}\,x\right)\right) - 1\right]^{-1}, \tag{13}$$

with

$$\frac{dK}{dx} = -\frac{Q}{x^4\left(1 + ax^2\right)}\exp\left[-\frac{1}{\sqrt{a}}\arctan\left(\sqrt{a}\,x\right)\right] \times \left\{K\exp\left[\frac{1}{\sqrt{a}}\arctan\left(\sqrt{a}\,x\right)\right] - 1\right\}^2, \tag{14}$$

which can be solved by iterations.

In the particular case $Q=0$, the function K is a constant, $K=k=$const and the relation (13) is simply given by the expression

$$n=\left[k\exp\left(\frac{1}{\sqrt{a}}\arctan\left(\sqrt{a}\,x\right)\right)-1\right]^{-1}, \tag{15}$$

which, for $x\to\infty$, decreases to the constant nonzero value

$$n_0=\left(k\exp\left[\frac{\pi}{2\sqrt{a}}\right]-1\right)^{-1},$$

which depends on the electron's temperature.

In the Fig.2, there is a comparison between the numerical solutions of the equation (6) (termed as "case 1") and the equation (11) (the "case 2"), computed with the formalism described above. Once Q is increasing, the number of scattered photons, solutions of (11), is significantly increasing and the distributions get a prominent maximum at $x<1$. One may notice the effect of the ax^2 contribution on the high frequency photons.

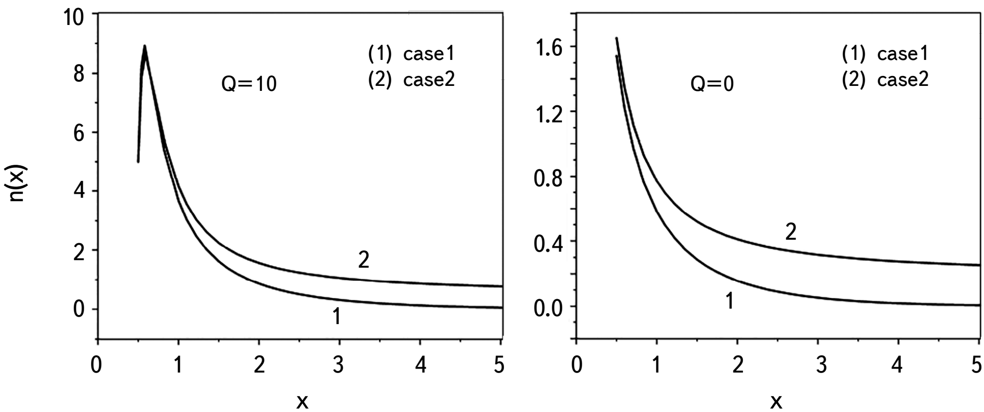

Fig.2. Comparison between cases 1 and 2 for different values of Q.

On the other hand, the substitution

$$n(x)=\left(1+ax^2\right)\frac{\psi'}{\psi}, \tag{16}$$

in the equation (11) leads to the following second order differential equation for the unknown function ψ,

$$\left(1+ax^2\right)^2\frac{d^2\psi}{dx^2}+\left(1+ax^2\right)\left(1+2ax\right)\frac{d\psi}{dx}-\frac{Q}{x^4}\psi=0, \tag{17}$$

which can be analytically solved. Indeed, its solution is expressed as

$$\psi(x)=F(x)H(x), \tag{18}$$

with

$$F(x) = \exp\left[-\frac{1}{2\sqrt{a}}\arctan\left(\sqrt{a}\,x\right)\right]\left(1+ax^2\right)^{\gamma/2} x^{-\gamma} \tag{19}$$

and

$$H(x) = C_1 HC^- + \frac{C_2}{x} HC^+ , \tag{20}$$

where C_1 and C_2 are two integration constants and $HC^{\mp}$ are the Heun Confluent functions [16,17]

$$HC^{\mp} = [\alpha, \beta, \gamma, \delta, \eta, \zeta] \tag{21}$$

of parameters

$$\alpha = 0, \quad \beta = \mp\frac{1}{2}, \quad \gamma = \sqrt{-Qa - \frac{1}{4a}}, \quad \delta = \frac{Qa}{4}, \quad \eta = \frac{4a-1}{16a} \tag{22}$$

and variable

$$\zeta = -\frac{1}{ax^2}. \tag{23}$$

One may easily check that, in terms of the variable (23), the functions (21) are solutions to the following second-order differential equation

$$\zeta(\zeta - 1)y'' + \left[\left(\gamma + \frac{3}{2}\right)\zeta - \frac{1}{2}\right]y' - \left[\frac{1}{16a} - \frac{\gamma}{4} - \frac{Qa}{4}\zeta\right]y = 0, \tag{24}$$

which is a Heun-type equation satisfied by the Heun Confluent functions of parameters (22) [16,17]. Obviously, these functions are real for γ a real quantity and this imposes $Q < 0$ and $|Q| > 1/\left(4a^2\right)$. A negative Q means that there is a constant photon supply at high frequencies ($x \to \infty$) and a sink at $x = 0$. One may notice from

$$\frac{dn}{dx} = -n(n+1) + \frac{Q}{x^4(1+bx)},$$

that, for $Q < 0$, the solution will always decrease from its initial condition, i.e.

$$n(x_0) > n(x_1) > n(x_2) > \ldots$$

Thus, depending on the initial condition n_0, there are cases where the solution will have negative values (numerically but not physically). However, the drop below zero slowly converges to zero, due to the $-n$ term which becomes positive and causes an increase of n, for $x \to \infty$.

In the case $Q = 0$, the particular form of the equation (24) is satisfied by the hypergeometric functions.

Putting everything together, the photon number density defined in (16) can be computed as

$$n = \left(1 + ax^2\right)\left[\frac{F'}{F} + \frac{H'}{H}\right], \tag{25}$$

where ' means the derivative with respect to x so that

$$HC' = \frac{d}{dx}HC = \frac{2}{ax^3}HeunCPrime.$$

The first term in the right hand side of (25) is the negative quantity

$$\frac{F'}{F} = -\frac{1}{2x}\frac{\left(x + 2\gamma\right)}{\left(ax^2 + 1\right)}$$

which is competing against the second contribution containing the Heun Confluent functions (21) and their derivatives. One may impose values for the constants C_1 and C_2 so that the photon density (25) gets positive (physical) values. For example, with the particular choice $C_1 = -C_2 > 0$, the function $n(x)$ given in (25) is represented in the Fig.3. As x goes to small values, one may notice in the left side of the Fig.3, that the function $n(x)$ is rapidly increasing, being limited on the left by an asymptote which separates the region where the plot goes to (nonphysical) negative values. For large x values (see the right plot in the Fig.3), $n(x)$ is approaching a constant non-zero value.

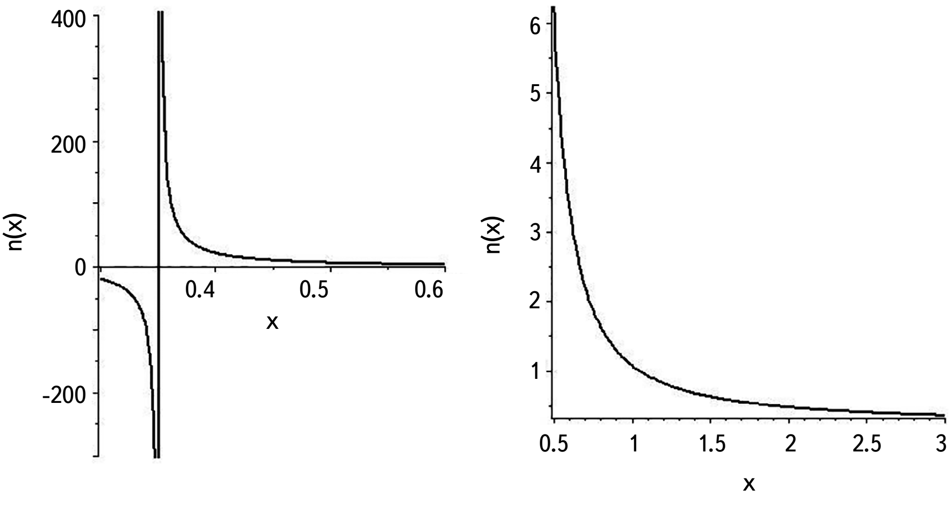

Fig.3. The photon density (25).

4. *General Compton scatterings in non-relativistic regime*. For describing more general Compton scattering processes in the nonrelativistic energy regime ($h\nu << mc^2$ and $kT << mc^2$) and with no comparison between $h\nu$ and kT, the Kompaneets equation has been generalized to new forms, as for example the one in [23], i.e.

$$(1+bx)\left[\frac{dn}{dx}+n^2+n\right]=\frac{Q}{x^4}, \tag{26}$$

where

$$b=\frac{14}{5}\frac{kT}{mc^2}. \tag{27}$$

One may easily check, with MAPLE, that the equation (26) does not lead to a differential equation satisfied by the Heun functions and therefore we have to use the general formalism described in the Section 2. Thus, by identifying the functions

$$g(x)=h(x)=1+bx, \quad f(x)=\frac{Q}{x^4},$$

we get the Bose-Einstein-type distribution

$$n(x)=\left[Ke^x-1\right]^{-1}, \tag{28}$$

with $K(x)$ numerically evaluated, using Maple or Mathematica, from the relation

$$\frac{dK}{dx}=-\frac{Q}{x^4(1+bx)}e^{-x}\left[Ke^x-1\right]^2. \tag{29}$$

The numerical solution is represented in the Fig.4, (the "case 3"). One can compare the distributions corresponding to the three cases analyzed in the present work, namely the numerical solutions of the equations (6), (11) and (26).

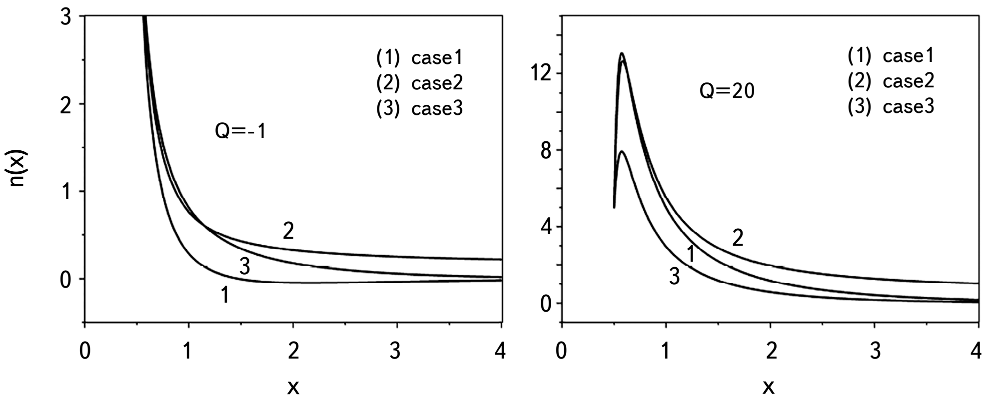

Fig.4. Comparison between the three cases, for different values of Q.

For general Compton scattering processes (case 3), when Q turns into a positive integer, the lower frequencies are impacted by having a lower photon number, while for higher frequencies, the distribution in the case 3 is decreasing to zero, once n approaches a Bose-Einstein-type distribution.

5. *Conclusions*. The aim of the present work is to discuss different forms

of the Kompaneets equation, valid in the non-relativistic limit where the typical photon energy and the plasma temperature are negligibly small compared to the electron rest energy, mc^2.

Even though Kompaneets-type equations have been solved numerically, there are a few situations where their stationary forms can be tackled analytically. For example, the original Kompaneets equation (5), in the stationary regime, turns into (6), its solution being given by the Heun Double Confluent functions [13,14]. However, there are difficulties in dealing with these functions for physically important ranges of model's parameters [21].

In the non-relativistic regime, for describing the down-Comptonization of high energy photons with $h\nu >> kT$, one has to use the equation (11), where the correction ax^2n' is important for situations which take place, for example, during a supernova explosion [24]. It turns out that the photon density is expressed in terms of Heun Confluent functions and their derivatives. Because the argument of the Heun function has a singularity at $x=0$, one can expect that, at small values of x, the solution (25) is more complicated than the conventional Bose-Einstein spectrum.

Both the up- and down-Comptonization processes (in the non-relativistic regime) can be described by the equation (26), which doesn't assume any relation between $h\nu$ and kT [23]. It is valid for various cases, $h\nu << kT$, $h\nu \sim kT$ and $h\nu >> kT$ and leads to the equation (6) in the low frequency limit. Moreover, unlike the equation (11), the equation (26) contains the factor $n' + n(n+1)$ which ensures the invariance of the total number of photons and the thermal equilibrium distribution [25]. Indeed, if one imposes the current of the general form

$$j(x) = f_1(x)\left[n' + f_2(n, x)\right]$$

to vanish for steady states, i.e.

$$n_0 = \left[ke^x - 1\right]^{-1},$$

one gets the function f_2 of the form

$$f_2(n, x) = n(n+1).$$

Since the equation (26) does not lead to a Heun-type equation, we have used the general approach described in the Section 2. As it can be noticed in the Fig.4, for $Q<0$ and low frequencies, the three graphs are rapidly increasing, being limited on the left by an asymptote (see the Fig.2). Depending on the initial condition n_0, one may have negative (nonphysical) solutions. Once we turn to positive Q values, the numerical solutions are represented in the right graph of the Fig.4. For high frequencies, the distributions in the cases 1 and 3 are approaching the zero value, lying below the curve corresponding to the equation (11) (case 2).

The analysis developed in the present paper can be extended to integrate the more general time-depending Kompaneets equation, employing the variables separation original method proposed by Dubinov and Kitayev [26].

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ТОЧНЫЕ СТАЦИОНАРНЫЕ РЕШЕНИЯ ОБЩЕГО ВИДА УРАВНЕНИЯ КОМПАНЕЙЦА

М.А.ДАРИЕСКУ, Ч.ДАРИЕСКУ, Г.АМАНОЛОАИ

В настоящей работе мы предлагаем метод построения решений общего вида стационарного уравнения Компанейца. В нерелятивистском режиме особое внимание уделяется случаям, когда решения выражаются через функции Гойна. Кратко обсуждается также сравнение с результатами, полученными в рамках численного анализа.

Ключевые слова: *уравнение Компанейца: процессы комптонизации: функции Гойна*

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