АСТРОФИЗИКА

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NOTE ON THE INTERPRETATION OF PROPER MASS AS A CONSTANT LAGRANGE MULTIPLIER

R.A.KRIKORIAN

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The timelike world line C_0 of a free particle is a maximizing curve for the integral $I=\int ds$ in the class Γ of neighbouring admissible timelike curves joining the events A, B, and satisfying the side-condition imposed on the 4-velocity $\varphi=g_{ij}\dot{x}^i\dot{x}^j=1$ ($\dot{x}^i=dx^i/ds$). Considering the problem of extremizing integral I as a time optimal problem, we show that the multiplier $\lambda(s)$ associated with the equation $\varphi=1$ is constant along C_0 and may be identified with the proper mass m of the free particle. The constancy of m can thus be regarded as a consequence of the path dependence of proper time.

Key words: *proper mass: Lagrange multiplier*

1. *Introduction*. According to the geodesic hypothesis, the world-line C_0 of a free mass point joining two events A, B is a timelike geodesic in Riemannian space-time, thus C_0 satisfies the variational equation

$$\delta I=\delta\int_A^B ds=0 \tag{1}$$

and this leads to the equations of the geodesic

$$D\dot{x}^i=0 \quad (D=\delta/\delta s) \tag{2}$$

where $\dot{x}^i=dx^i/ds$ is the unit tangent vector to the world line C_0 so that

$$\varphi(x,\dot{x})=g_{ij}\dot{x}^i\dot{x}^j=1. \tag{3}$$

From the assumed constancy of the invariant proper mass m, multiplication of (2) by m yields the equations for the parallel propagation of the 4-momentum $p^i=m\dot{x}^i$ along C_0, i.e.

$$Dm\dot{x}^i=0. \tag{4}$$

Put into words $\delta I(C_0)=0$ says that the world line C_0 furnishes an extremum (maximum) to the integral I in the class Γ of neighbouring admissible timelike curves. The class Γ is composed of the smooth, future-pointing timelike curves

$$x^i=x^i(s) \quad (s_1\le s\le s_2,\ i=0,1,2,3) \tag{5}$$

joining the fixed end-events

$$A: x^i(s_1) = x_1^i, \quad B: x^i(s_2) = x_2^i \tag{6}$$

and satisfying (3), which not only restricts the parameter s to be the naturel parameter (proper time) along the timelike curve (5) but it also specifies the timelike character of the tangent vector $\dot{x}(s)$.

Remark 1: Evaluation of the integral I along C_0 can be given a chronometric interpretation, it corresponds to the interval of proper time registered between the events A, B by a standard clock carried by the free particle. In the class Γ of admissible curves the geodesic C_0 is a maximizing curve for the integral $I = \int_A^B ds$ [1].

The purpose of this note is to show that, as a consequence of the multiplier rule imposed on C_0 in order that it be a maximizing curve for the integral I, the multiplier $\lambda(s)$ associated with the constraint (3), i.e. $\varphi = 1$, must be a positive constant, so that the Euler necessary condition for an extremum may be written $D\lambda\, \dot{x}^i = 0$; λ may thus be invested with physical meaning, it corresponds to the proper mass of the free particle.

Remark 2: Novozhilov and Yappa [2] choosing for a pointlike charge interacting with a given electromagnetic field the Lagrangian

$$L = \frac{1}{2} m\left(u^i u_i - c^2\right) - \frac{q}{c} A^j u_j \qquad \left(u^i = dx^i / d\tau\right),$$

interpret the factor $m/2$ as playing the role of the Lagrange multiplier, assumed constant, associated with the side condition (3), serving to specify the parameter s on C_0. As we shall see the proof that the multiplier, associated with C_0, must be constant is not so simple. For a free particle a Lagrangian of the form $L = (m/2)\left(u^i u_i - 1\right)$ has been proposed by Peres and Rosen [3].

2. *Mathematical preliminaries*. Formulated in the manner described in Section 1, the problem of extremizing integral I in the class Γ of admissible timelike curves, satisfying condition (3), can be regarded, in the space of points (s, x), as a nonparametric Lagrange problem [4]. Although the end-points are fixed in x-space (space-time), the probem in sx-space is one with variable right endpoint. Indeed, as a consequence of the path dependence of proper time, the interval $(s_1 \le s \le s_2)$ cannot be chosen to be the same for each admissible curve in Γ; without loss of generality, we can restrict the class Γ to timelike curves having the same end-value $s_1 = 0$ of the parameter s at the event A, but at the event B the end-value s_2 must vary when we pass from one admissible curve C_1 to a neighbouring curve C_2. As a consequence of the variability of s_2, besides the Euler necessary condition the extrimizing curve C_0 must satisfy a further necessary condition, known as the transversality condition. The combined results of the theorems stating these condtions are called by Bliss the multiplier rule [4]. In

the language of physics the problem under consideration is often designated as a time optimal problem. The results of this section follow from the non-parametric theory. The multiplier rule ensures that there exists a function [4,5] defined by

$$F(x,\dot{x},\lambda)=\lambda_0+\lambda(s)[\varphi(x,\dot{x})-1] \tag{7}$$

such that the Euler-Lagrange (EL) equations

$$\text{a)}\ \frac{d}{ds}F_{\dot{x}}-F_x=0,\quad \text{b)}\ \varphi-1=0 \tag{8}$$

hold at each point of C_0. The multipliers $\lambda_0=\text{const}$ and $\lambda(s)$ do not vanish simultaneously at any point of C_0. Moreover because of the variability of s_2, C_0 must satisisfy the transversality condition [4-6]

$$\left[\left(F-\dot{x}^iF_{\dot{x}^i}\right)ds_\alpha+F_{\dot{x}^i}dx^i_\alpha\right]_{\alpha=2}^{\alpha=1}=0 \tag{9}$$

where $(s,x,\dot{x},\lambda)$ is taken at the first end-point of C_0 when $\alpha=1$ and at the second when $\alpha=2$. In sx-space, the end-conditions satisfy the equations

$$s_1=0,\quad A:x^i(s_1)=x^i_1,\quad B:x^i(s_2)=x^i_2. \tag{10}$$

Thus the transversality condition (9) takes the form

$$\left(F-\dot{x}^iF_{\dot{x}^i}\right)_{s_2}=0. \tag{11}$$

Remark 3: In the derivation of Lagrange's equations from the variational principle $\delta\int L ds=0$, the multiplier rule is often treated inadequately, e.g. [7]. The Lagragian L is simply modified by setting $\bar{L}=L+\lambda(s)(\varphi-1)$, i.e. the multiplier λ_0 is arbitrarily chosen as unity.

3. *Derivation of the equation of motion and physical interpretation of the multiplier* λ. We briefly sketch the computations for obtaining, from (8) and (9), (11) the equations of motion of a free partcle in the form (4). Upon multiplying (8a) by $\dot{x}^i$ and with the aid of the relation

$$\dot{x}^iF_{\dot{x}^i}=(2F-\lambda_0+\lambda), \tag{12}$$

which is obtained from (7) by making use of the homogeneity of φ in the variables $\dot{x}^i$, $\dot{x}^i\varphi_{\dot{x}^i}=2$, it follows that

$$\dot{x}^i\left(\frac{d}{ds}F_{\dot{x}^i}-F_{x^i}\right)=\frac{d}{ds}F+2\frac{d}{ds}\lambda=0.$$

Substituting for F, eq. (7), we obtain

$$\frac{d\lambda}{ds}=0. \tag{13}$$

Consequently, λ has a constant value along C_0. This is all the information obtainable from the EL equations. Further useful information concerning λ comes from the tranversality condition (11).

In view of (12), (11) reads

$$F - \dot{x}^i F_{\dot{x}^i} = \left[F - 2\left(F - \lambda_0 + \lambda\right)\right]_2 .$$

Substituting for F we find

$$\lambda_0 = 2\lambda(s_2) \tag{14}$$

On account of (13) we deduce that if C_0 is to maximize the integral I, the multiplier λ associated with equation (3) must be constant and equal to $\lambda_0/2 > 0$. Moreover we observe that if $\lambda_0 = 0$, by (13) and (14) $\lambda = 0$. The multiplier rule assures that if $\lambda_0 = 0$, λ does not vanish simultaneously anywhere on $[s_1, s_2]$. The maximizing curve C_0 is said to be normal. For a normal curve, multipliers $\lambda_0 = 1$, $\lambda(s)$ always exist and in this form they are unique. Furthermore, C_0 as a normal extremizing curve may be imbedded in a one-parameter family of admissible curves $\in \Gamma$, which satisfy the end conditions (10) [4]. As a consequence of the constancy of λ, the EL equations (8a) can be written in explicit form

$$\frac{d}{ds} F_{\dot{x}^i} - F_{x^i} = 2\lambda\left(g_{ij}\ddot{x}^j + \frac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right)\dot{x}^k \dot{x}^j\right) = 0 \tag{15}$$

on multiplying by g^{il} we get

$$2\lambda\left(\ddot{x}^l + \Gamma^l_{jk}\dot{x}^j \dot{x}^k\right) = 2\lambda D\dot{x}^l \qquad (D = \delta/\delta s) \tag{16}$$

from the constancy of λ we deduce from (16)

$$D\lambda\dot{x}^l = 0. \tag{17}$$

Comparison with the equation of motion (4) shows that λ may be identified with the proper mass m of the free particle. Adopting the point of view of variational analysis the constancy of m may thus be regarded as a consequence of the path dependence of proper time.

Collège de France - Sorbonne Université, CNRS,
Institut d'Astrophysique de Paris, France, e-mail: krikoria@iap.fr

ОБ ИНТЕРПРЕТАЦИИ СОБСТВЕННОЙ МАССЫ КАК ПОСТОЯННОГО МНОЖИТЕЛЯ ЛАГРАНЖА

Р.А.КРИКОРИАН

Временно-подобная мировая линия C_0 свободной частицы, это максимизирующая кривая для интеграла $I = \int ds$ в классе Γ соседних допустимых временно-подобных кривых, соединяющих события A, B и удовлетворяющих боковым условиям, наложенным на 4-скорость $\varphi = g_{ij}\dot{x}^i\dot{x}^j = 1$ ($\dot{x}^i = dx^i/ds$). Рассматривая максимизацию интеграла как проблему оптимального времени, показано, что множитель $\lambda(s)$, связанный с уравнением $\varphi = 1$, постоянная вдоль C_0 и ее можно отождествить с собственной массой m свободной частицы. Постоянство m можно рассматривать как следствие зависимости собственного времени от траектории.

Ключевые слова: *собственная масса: множитель Лагранжа*

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