АСТРОФИЗИКА

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WHITE DWARF STARS AS POLYTROPIC GAS SPHERES

M.I.NOUH[1,2], A.S.SAAD[2,3], W.H.ELSANHOURY[2,4], A.A.SHAKER[2], B.KORANY[2,5], T.M.KAMEL[2]

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We expect that relativistic effects have an important role in compact stars, because their electrons are highly degenerate. In this paper, we study properties of the condensed matter in white dwarfs using Newtonian and relativistic polytropic fluid spheres. We propose two polytropic indices ($n=3$ and $n=1.5$) to investigate the physical characteristics of the models. We numerically solve the Lane-Emden equations, and demonstrate that the relativistic effect is small in white dwarf stars.

Key words: *Compact stars: White dwarfs: Polytropes: Relativistic effects*

1. *Introduction*. The theoretical and observational study of compact stars remains one of the most exciting fields in modern physics. Predictions of the properties of white dwarfs serve to test our understanding on the matter at these high densities, while theories of high-density matter serve as a basis for interpreting observational results regarding these objects. Most excitingly, these objects bring together all four of the fundamental forces of nature and probe regimes not accessible in the terrestrial laboratory [1].

Matter in the interior of compact objects is highly degenerate. Because degenerate electrons are excellent conductors of heat, the interior is nearly isothermal and the core temperature approximately equals the temperature at the core envelope-boundary. Furthermore, since the pressure of the degenerate matter is nearly independent of the temperature, we may use polytropic models.

Polytropic models are vital to two classes of theoretical astrophysics: stellar structure and galactic dynamics. In stellar structures, the Lane-Emden equation governs the physical variables of the configurations [2,3].

Tooper conducted relativistic studies of the polytropic equation of state in 1964 [4], and derived two nonlinear differential equations that are analogues to the non-relativistic Lane-Emden equation. By numerically solving these two equations, they obtained the physical parameters of the polytrope. In [5] and [6], the two first order differential equations obtained by Tooperwere solved numerically to investigate the effect of increasing a specific relativistic parameter of the polytropes (indices $n=1.5$ and $n=3$). In [7], the Tolman-Oppenheimer-Volkoff TOV equation was solved analytically for different polytropic indices.

In this paper, we describe our study of the structure of white dwarfs using relativistic polytropic fluid spheres. The paper is organized as follows. In Section 2, we discuss polytropic and TOV equations. Section 3 contains our results and their interpretations. Section 4 contains our conclusions.

2. *The Polytropic Gas Sphere*. The polytropic equation of state has the form

$$p = K\rho^{\Gamma}, \quad \Gamma = 1 + \frac{1}{n},$$

where n is the polytropic index and K is the pressure constant. $\Gamma = 5/3$ for the non-relativistic case and $\Gamma = 4/3$ for relativistic case.

The equilibrium structure of a self-gravitating object is derived from the hydrostatic equilibrium equations. The simplest case is that of a spherical, non-rotating, static configuration, where, for a given equation of state, all macroscopic properties are parameterized by a single parameter, for example, the central density. Using algebraic manipulations, the structure equations can be combined to derive the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0, \tag{1}$$

where θ and ξ are dimensional and given by $\xi = rA$ and $\theta = \rho/\rho_c$. ρ_c is the central density and ρ is the density.

In the case of compact objects, the gravitational fields are sufficiently strong that the calculations must be performed in the context of general relativistic (rather than Newtonian) gravity. The fundamental equation of hydrostatic equilibrium in its general relativistic form has been derived [8,9], and is known as the "TOV" equation. It is written as

$$\xi^2\frac{d\theta}{d\xi}\frac{1-2\sigma(n+1)\nu/\xi}{1+\sigma\theta} + \nu + \sigma\xi\theta\frac{d\nu}{d\xi} = 0, \tag{2}$$

and

$$\frac{d\nu}{d\xi} = \xi^2\theta^n, \tag{3}$$

where

$$\nu = \frac{A^3 m(r)}{4\pi\rho_c}, \quad A = \left(\frac{4\pi G\rho_c}{\sigma(n+1)c^2}\right)^{1/2}, \quad \sigma = \frac{P_c}{c^2\rho_c}. \tag{4}$$

Here, ξ is the dimensionless radius, ν is a dimensional finite stellar mass $m(r)$ at a radius r, A is a constant with a dimension of inverse length, σ is the relativistic parameter (which can be considered as related to the relativistic corrections), and P_c is the central density. In Eqs. (1)-(3), the Lane-Emden functions θ are the solutions that satisfy the condition $\theta = 1$ at $\xi = 0$ and $\theta = 0$ at $\xi = \xi_1$. We can determine the radius R and the mass M from

$$R = A^{-1}\xi_1, \quad M' \equiv \sigma^{(3-n)/2}\nu(\xi_1),$$

$$M = \frac{4\pi\rho_c}{A^3}\nu(\xi_1) = \left[\frac{(n+1)c^2}{4\pi G}\left(\frac{K}{c^2}\right)^n\right]^{1/2} M'. \tag{5}$$

3. ***Results.*** We numerically integrated Equations (2) and (3) using the Runge-Kutta method. A Mathematica routine was used to determine the zeroes of the TOV equation at different polytropic indices n and for different values of the relativistic parameter σ. The integral had initial values $\xi = 0$, $\theta = 1$ and $\nu = 0$ proceeded forward using step size $\Delta\xi$. The zero of θ is denoted by ξ_1, and was determined by integrating until we obtained a negative value of θ. Then, a small step size $\Delta\xi$ was used to give more accurate results.

In the TOV equations (Equations (2) and (3)), the functions $\theta(\xi)$, and $\nu(\xi)$ depend on two parameters, n and σ. When $\sigma \to 0$, these reduce to the non-relativistic Lane-Emden equation (Equation (1)).

Fig.1 contains a plot of the relativistic function $\nu(\xi_1)$ as a function of n and σ. The curve for $\sigma = 0$ reduces to the non-relativistic Lane-Emden function. The function $\nu(\xi_1)$ decreases with increasing n because the equation of state softens, and with increasing σ, because general relativity, becomes more important.

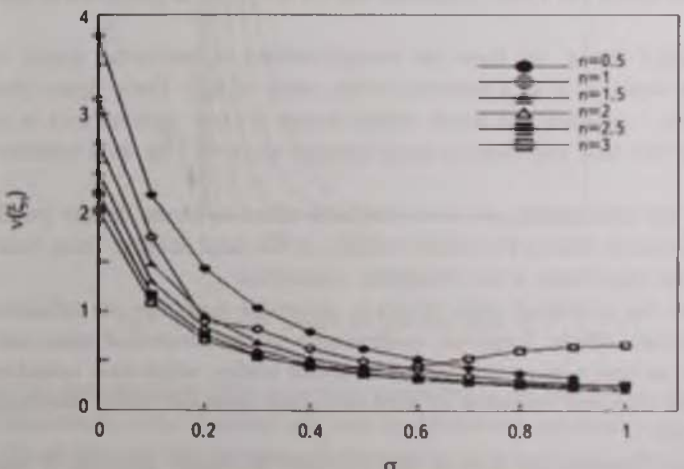

Fig.1. Variation of the relativistic function $\nu(\xi_1)$ with respect to polytropic index n and the relativistic parameter.

Fig.2 contains a plot of M', which determines the stellar mass. For $n < 3$, M' increases with σ up to a certain maximum value, which is the critical value σ_{CR} and represents the onset of instability. For $n = 3$, M' has two minima, $\sigma = 0.4$ $(M' = 0.4516)$ and $\sigma = 0.5$ $(M' = 0.4214)$, and two maxima at $\sigma \to 0$ and $\sigma_{CR} \approx 0.42$.

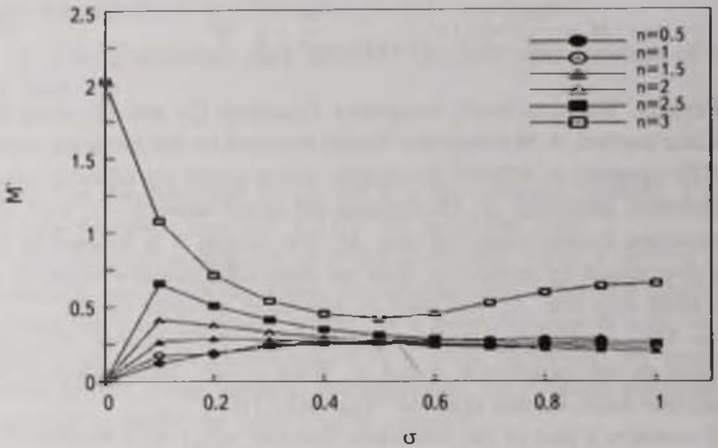

Fig.2. Effect of σ on the stellar mass function M'.

The matter in compact stars is highly degenerate, so we expect that the relativistic effect has a very important role on the physical properties of these stars.

In Fig.3 and 4, we show the density profiles of the stellar matter for different values of σ as a function of the radius $R(R_{\odot})$. These figures show that when increases, the stellar matter density is more concentrated in the center of the star. The effect is much stronger for $n = 3$ (the ultra-relativistic case).

For the mass profile, we found the same effect as in the density profiles because the star mass is the volume integral of the mass density. These results reflect the importance of the relativistic corrections.

Next, we considered white dwarfs to determine how they are influenced by relativistic effects. Empirical confirmation of the theoretical mass-radius relation has been a prime objective of numerous studies, which have considered individual stars and ensembles of stars with good mass and radius determinations [10].

Recent observational projects have provided the masses and radii of many white dwarf stars, so we can establish the inverse problem for white dwarfs. That is, if in the relativistic case, M, R and n are considered given quantities,

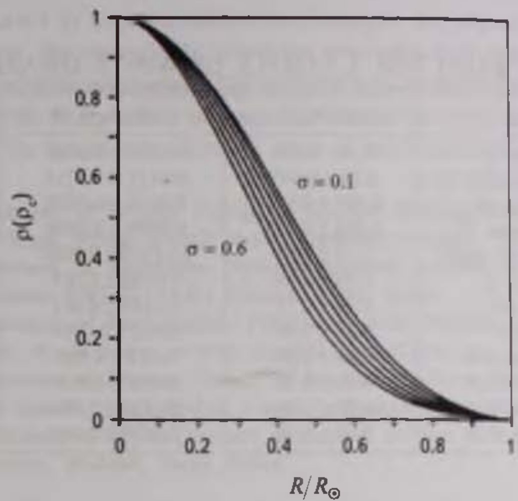

Fig.3. Star density profiles for different values of σ at $n = 1.5$.

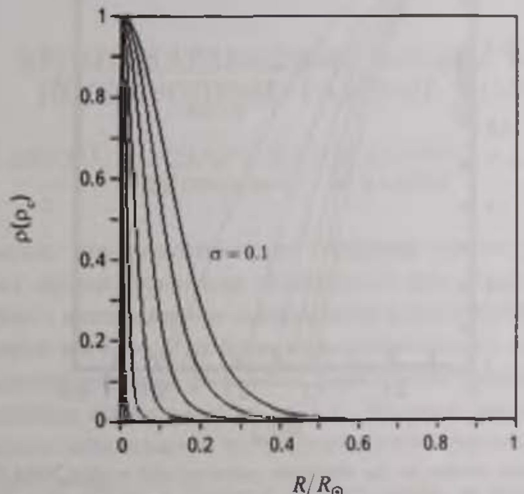

Fig.4. Star density profiles for different values of σ at $n = 3$.

then the determination of the relativistic parameter σ (or range of σ) becomes a characteristic value problem and can be determined graphically [4].

To do this, we use the observed mass-radius relation in [11], based on the parallax of 10 white dwarfs observed by HIPPARCOS. The masses and radii are listed in Table 1.

Fig.5 illustrates the positions of the white dwarfs selected from Table 1 with

Table 1

MASS AND RADII FOR A SAMPLE OF WHITE DWARFS [11]

Name	$M(M_\odot)$	$R(R_\odot)$
Sirius B	1.0 ± 0.016	0.0084 ± 0.0002
Stein 2051 B	0.48 ± 0.045	0.0111 ± 0.0015
40 Eri B	0.501 ± 0.011	0.0136 ± 0.0002
Procyon B	0.604 ± 0.018	0.0096 ± 0.0004
CD-38 10980	0.74 ± 0.04	0.01245 ± 0.0004
W485 A	0.59 ± 0.04	0.0150 ± 0.001
L268-92	0.70 ± 0.12	0.0149 ± 0.001
L481-60	0.53 ± 0.05	0.012 ± 0.0004
G154-B5B	0.46 ± 0.08	0.011 ± 0.001
G181-B5B	0.50 ± 0.05	0.011 ± 0.001
G156-64	0.59 ± 0.001	0.0110 ± 0.001
G154-B5B	0.46 ± 0.08	0.0130 ± 0.002

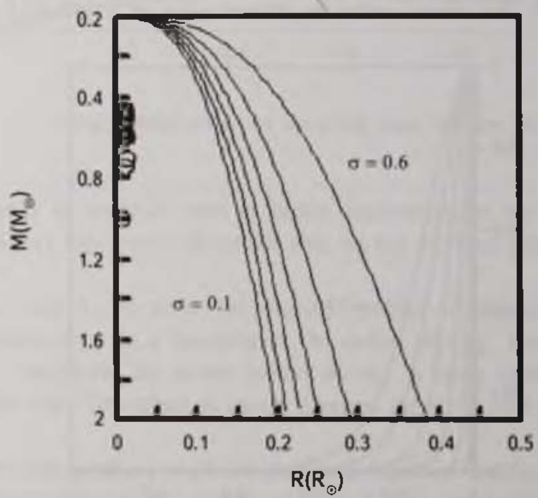

Fig.5. Mass radius relation for the relativistic polytrope with $n = 1.5$. Solid lines represent the mass radius relation for different relativistic parameters σ and the open circles represent the mass and radius from [11].

the polytropic mass-radius relations calculated for polytropic index $n = 1.5$ and for different values of the relativistic parameter σ. Most of the objects tend to have small σ, except for two that had σ values between 0.1 and 0.3.

4. *Conclusions*. In this paper, we studied properties of the condensed matter in white dwarfs using a polytropic fluid sphere. Two polytropic indices

($n = 3$ and $n = 1.5$) were considered to investigate the physical characteristics of the models. We numerically solved the relativistic fluid sphere equations for different relativistic parameters. The deduced mass-radius relation at $n = 1.5$ was compared with observations of a selected sample of white dwarfs. The result shows that the relativistic effect on most of the selected sample was small.

[1] Department of Physics, College of Science, Northern Border University, Arar, Saudi Arabia, e-mail: abdo_nouh@hotmail.com
[2] Department of Astronomy, National Research Institute of Astronomy and Geophysics NRIAG, 11421 Helwan, Cairo, Egypt
[3] Department of Mathematics, Preparatory Year, Qassim University, Qassim, Saudi Arabia, e-mail: saad6511@gmail.com
[4] Department of Physics, Faculty of Science, Northern Border University, Rafha Branch, Saudi Arabia, e-mail: welsanhoury@gmal.com
[5] Department of Physics, Faculty of Applied Science, Umm AL-Qura University, Makaha, Saudi Arabia

БЕЛЫЕ КАРЛИКОВЫЕ ЗВЕЗДЫ КАК ПОЛИТРОПНЫЕ ГАЗОВЫЕ ШАРЫ

М.И.НОУХ[1,2], А.С.СААД[2,3], В.Г.ЭЛСАНОРИ[2,4], А.А.ШАКЕР[2], Б.КОРАНИ[2,5], Т.М.КАМЕЛ[2]

Ожидается, что релятивистские эффекты играют важную роль в компактных звездах, поскольку электроны в них сильно вырождены. В данной работе с использованием ньютоновских и релятивистских политропных жидких шаров изучаются свойства конденсированного вещества в белых карликах. Для исследования физических характеристик моделей мы предлагаем два политропных индекса ($n = 3$ и $n = 1.5$). Численно решаются уравнения Лане-Эмдена и показывается, что релятивистские эффекты в белых карликовых звездах малы.

Ключевые слова: *компактные звезды: белые карлики: политропы: релятивистские эффекты*

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