АСТРОФИЗИКА

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APPROXIMATE SOLUTION TO THE FRACTIONAL SECOND-TYPE LANE-EMDEN EQUATION

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The spherical isothermal Lane-Emden equation is a second order non-linear differential equation that model many configurations in astrophysics. Using the fractal index technique and the power series expansion, the fractional Lane-Emden equation involving the modified Riemann-Liouville derivative is solved. The results indicate that the series converges over the range of radii $0 \le x < 2200$ for a wide spread of values for the fractional parameter α. Comparison with the numerical solution reveals good agreement with a maximum relative error of 0.05.

Key words: *isothermal gas sphere: fractal index: nonlinear fractional differential equation: modified Riemann-Liouville derivative*

1. *Introduction.* The isothermal Lane-Emden equation is often considered the asymptotic limit of the Lane-Emden equation where the polytropic index is taken to be very large. The self-gravitational isothermal gas sphere has been useful in many areas of astrophysics, such as stellar structure, star clusters, galaxies and galactic clusters [1-2].

Many numerical and analytical methods have been proposed in deriving solutions of the equations describing the isothermal gas sphere. These can be found in [3-8].

In the past two decades, there has been a surge of interest in studying fractional differential equations (FDEs) which appear in many branches of the sciences such as mathematics, chemistry, optics, plasmas, fluid dynamics, and engineering. Applications of fractional calculus and FDEs examples include: dielectric relaxation phenomena in polymeric materials [9], transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology [10], transport dynamics in systems governed by anomalous diffusion [11,12], and long-time memory in financial time series [13]. Hence it is very important to find efficient methods for solving FDEs. Finding analytical and approximate solutions of FDEs are two of the more useful approaches in understanding the physical mechanism of natural phenomenon and dynamical processes modeled by FDEs [14-17].

In the present paper, we introduce a new analytical solution of the equation governing the isothermal gas sphere. We derive a recurrence relation for the coefficients in a power series expansion of the solution of the fractional isothermal Lane-Emden equation. To the best of our knowledge, this is the first work dealing with the series solution of the fractional isothermal Lane-Emden equation.

The structure of the paper is as follows. In Section 2, some basic concepts of fractional calculus are introduced. The series solution to the fractional isothermal gas-sphere equation is described in Section 3. Section 4 is devoted to numerical results. Section 5 summarizes the conclusions reached.

2. *Basics of Fractional Calculus*. Fractional calculus generalizes notions of ordinary calculus. Depending on the definition and properties of the fractional derivative, there are many kinds of fractional calculus, such as Riemann-Liouville, Caputo, Kolwankar-Gangal, Oldham-Spanier, Miller-Ross, Cresson, Grunwald-Letnikov, and modified Riemann-Liouville [18-20].

We start by recalling the Jumarie modification of the Riemann-Liouville derivative, [21-25]. Assume that $f : R \to R, x \to f(x)$ denotes a continuous function, and let h denote a constant discretization span; the limit form of the modified Riemann-Liouville derivative is defined as

$$f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha}, \quad 0 < \alpha < 1,$$

where

$$\Delta^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} f[x + (\alpha - k)h].$$

This is analogous to the standard derivative (calculus for beginners), and gives the α -order derivative of a constant as zero. The integral form of the modified Riemann-Liouville derivative is

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)] d\xi, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, & 0 < \alpha < 1 \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_0^x (x-\xi)^{n-\alpha-1} [f(\xi) - f(0)] d\xi, & n \le \alpha < n+1, \ n \ge 1. \end{cases} \tag{1}$$

Other useful Jumarie modified formulae are

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0, \tag{2}$$

$$D_x^\alpha (c f(x)) = c D_x^\alpha f(x), \tag{3}$$

$$D_x^\alpha [f(x) g(x)] = g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x), \tag{4}$$

$$D_x^\alpha f[g(x)] = f_g[g(x)]D_x^\alpha g(x), \tag{5}$$

$$D_x^\alpha f[g(x)] = D_g^\alpha f[g(x)](g_x)^\alpha , \tag{6}$$

where c is a constant. Eqs. (4)-(6) are direct results from

$$D_x^\alpha f(x) \equiv \Gamma(\alpha+1)D_x f(x). \tag{7}$$

[26] modified the chain rule, Eq. (5) to

$$D_x^\alpha f[g(x)] = \sigma_x\, f_g[g(x)]D_x^\alpha g(x), \tag{8}$$

where σ_x is called the fractal index, which is usually determined in terms of gamma functions [26-28]. Therefore, Eqs. (4) and (6) are modified to give

$$D_x^\alpha[f(x)g(x)] = \sigma_x\{g(x)D_x^\alpha f(x) + f(x)D_x^\alpha g(x)\}, \tag{9}$$

$$D_x^\alpha f[g(x)] = \sigma_x\, D_g^\alpha f[g(x)](g_x)^\alpha \tag{10}$$

Throughout this manuscript, we use Eq. (8) to solve the fractional isothermal Lane-Emden equation.

3. *Computational Developments.*

3.1. *Isothermal Lane-Emden Equation*. The Lane-Emden equation for an isothermal gas sphere [1], can be written as

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} = e^{-u}, \tag{11}$$

with the initial conditions

$$u(0) = 0, \quad \left.\frac{du}{dx}\right|_{x=0} = 0. \tag{12}$$

The power series solution of Eq. (11) has the form [6]

$$u(x) = \frac{x^2}{6} - \frac{x^4}{120} + \frac{x^6}{1890} - \frac{61x^8}{1632960} + -\ldots \tag{13}$$

The fractional isothermal Lane-Emden equation, which is the generalization of the isothermal Lane-Emden equation (11) can be written as:

$$x^{2\alpha}D_x^\alpha D_x^\alpha u + 2x^\alpha D_x^\alpha u + x^{2\alpha}e^{-u} = 0, \tag{14}$$

with the initial conditions

$$u(0) = 0, \quad D_x^\alpha u(0) = 0, \tag{15}$$

where $u = u(x)$ is the unknown function, D_x^α is the modified Riemann-Liouville derivative and $e^{-u} = \sum_{k=0}^{\infty}\frac{(-1)^k u^k}{k!}$.

3.2. *Series Solution of Eq. (14)*. With the transform $X = x^\alpha$, we assume the solution could be expressed as a series,

$$u(X)=\sum_{m=0}^{\infty}A_mX^m=A_0+A_1X+A_2X^2+A_3X^3+A_4X^4+A_5X^5+\ldots= \\ =A_0+A_1X^{\alpha}+A_2X^{2\alpha}+A_3X^{3\alpha}+A_4X^{4\alpha}+A_5X^{5\alpha}+\ldots \tag{16}$$

The first initial condition of Eq. (15) gives $u(0)=A_0$, or $A_0=0$. Applying Eqs. (2) and (4) to Eq. (16), we have

$$D_x^{\alpha}u=D_x^{\alpha}A_0+D_x^{\alpha}\left(A_1X^{\alpha}\right)+D_x^{\alpha}\left(A_2X^{2\alpha}\right)+D_x^{\alpha}\left(A_3X^{3\alpha}\right)+D_x^{\alpha}\left(A_4X^{4\alpha}\right)+ \\ +D_x^{\alpha}\left(A_5X^{5\alpha}\right)+\ldots=0+\frac{A_1\Gamma(\alpha+1)x^{\alpha-\alpha}}{\Gamma(\alpha+1-\alpha)}+\frac{A_1\Gamma(2\alpha+1)x^{2\alpha-\alpha}}{\Gamma(2\alpha+1-\alpha)}+ \\ +\frac{A_1\Gamma(3\alpha+1)x^{3\alpha-\alpha}}{\Gamma(3\alpha+1-\alpha)}+\ldots=A_1\Gamma(\alpha+1)+\frac{A_1\Gamma(2\alpha+1)x^{\alpha}}{\Gamma(\alpha+1)}+\frac{A_1\Gamma(3\alpha+1)x^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots \tag{17}$$

Applying the second initial condition of Eq. (15) gives

$$D_x^{\alpha}u(0)=A_1\Gamma(\alpha+1),\quad \text{or}\quad A_1=0. \tag{18}$$

Now suppose that

$$G(X)=\sum_{m=0}^{\infty}Q_mX^m=Q_0+Q_1X+Q_2X^2+Q_3X^3+Q_4X^4+Q_5X^5+\ldots \tag{19}$$

By putting

$$e^{-u}=G(X), \tag{20}$$

we have

$$e^{-u(0)}=G(0)=1,\quad \text{or}\quad Q_0=1.$$

First, we focus on determining the fractional derivative of $e^{-u}=\sum_{k=0}^{\infty}\frac{(-1)^k u^k}{k!}$. Taking the fractional derivative on both sides, the fractional derivative of u^2 is considered as u times u; similarly u^3 will be considered as u times u^2, etc. Therefore,

$$D_x^{\alpha}e^{-u}=D_x^{\alpha}\left(\sum_{k=0}^{\infty}\frac{(-1)^k u^k}{k!}\right)=\sum_{k=1}^{\infty}\frac{(-1)^k D_x^{\alpha}u^k}{k!}=\sum_{k=0}^{\infty}\frac{(-1)^k ku^{k-1}D_x^{\alpha}u}{k!}= \\ =\sum_{k=1}^{\infty}\frac{(-1)^k u^{k-1}D_x^{\alpha}u}{(k-1)!}=-D_x^{\alpha}u\sum_{k=1}^{\infty}\frac{(-1)^{k-1}u^{k-1}}{(k-1)!}=-D_x^{\alpha}u\sum_{s=0}^{\infty}\frac{(-1)^s u^s}{s!}=-e^{-u}D_x^{\alpha}u,$$

which can be written in the form

$$GD_x^{\alpha}u=-D_x^{\alpha}G. \tag{21}$$

Differentiating both sides of Eq. (21) k times with the α -derivative, we have

$$\underbrace{D_x^{\alpha}\ldots D_x^{\alpha}}_{k\text{ times}}\left[GD_x^{\alpha}u\right]=-\underbrace{D_x^{\alpha}\ldots D_x^{\alpha}}_{k\text{ times}}\left(D_x^{\alpha}G\right),\quad \text{or}\quad \sum_{j=0}^{k}\binom{k}{j}u^{\alpha(j+1)}G^{\alpha(k-j)}=-G^{\alpha(k+1)}.$$

At $x=0$,

$$\sum_{j=0}^{k}\binom{k}{j}u^{\alpha(j+1)}(0)G^{\alpha(k-j)}(0)=-G^{\alpha(k+1)}(0). \tag{22}$$

As

$$u^{\alpha(j+1)}(0)=A_{j+1}\Gamma((j+1)\alpha+1),\quad G^{\alpha(k-j)}(0)=Q_{k-j}\Gamma((k-j)\alpha+1),$$
$$G^{\alpha(k+1)}(0)=Q_{k+1}\Gamma((k+1)\alpha+1),$$

we have

$$\sum_{j=0}^{k}\frac{k!A_{j+1}\Gamma((j+1)\alpha+1)Q_{k-j}\Gamma((k-j)\alpha+1)}{j!(k-j)!}=-Q_{k+1}\Gamma((k+1)\alpha+1)$$

That is,

$$Q_{k+1}=-\sum_{j=0}^{k}\frac{k!A_{j+1}\Gamma((j+1)\alpha+1)Q_{k-j}\Gamma((k-j)\alpha+1)}{j!(k-j)!\Gamma((k+1)\alpha+1)},$$

and setting $l=k+1$, then

$$Q_l=-\frac{(l-1)!}{\Gamma(l\alpha+1)}\sum_{j=0}^{l-1}\frac{A_{j+1}\Gamma((j+1)\alpha+1)Q_{l-1-j}\Gamma((l-1-j)\alpha+1)}{j!(l-1-j)!}$$

If $i=j+1$

$$Q_l=-\frac{(l-1)!}{\Gamma(l\alpha+1)}\sum_{i=1}^{l}\frac{A_i\Gamma(i\alpha+1)Q_{l-i}\Gamma((l-i)\alpha+1)}{(i-1)!(l-i)!} \tag{23}$$

Hence,

$$A_0=0,\quad A_1=1,\quad Q_0=1,\quad Q_1=0. \tag{24}$$

Alternatively, $D_x^\alpha u$ can be written as

$$\begin{aligned}D_x^\alpha u&=\sum_{m=1}^{\infty}A_m\sigma_x mX^{m-1}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)}x^{\alpha-\alpha}=\\&=\sum_{m=1}^{\infty}A_m mX^{m-1}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)}\frac{\Gamma(m\alpha+1)}{m\Gamma(\alpha+1)\Gamma(m\alpha+1-\alpha)}=\\&=\sum_{m=1}^{\infty}A_mX^{m-1}\frac{\Gamma(m\alpha+1)}{\Gamma(m\alpha+1-\alpha)},\end{aligned} \tag{25}$$

where the fractal index σ_x (see example 6 in [28]) is given by

$$\sigma_x=\frac{\Gamma(m\alpha+1)}{m\Gamma(\alpha+1)\Gamma(m\alpha+1-\alpha)}$$

Also, we have

$$D_x^{\alpha} D_x^{\alpha} u = \sum_{m=2}^{\infty} A_m \sigma_x (m-1) X^{m-2} \frac{\Gamma(m\alpha+1)}{\Gamma(m\alpha+1-\alpha)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)} x^{\alpha-\alpha} =$$

$$= \sum_{m=2}^{\infty} X^{m-2} \frac{A_m (m-1)\Gamma(m\alpha+1)}{\Gamma(m\alpha+1-\alpha)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\alpha)} \frac{\Gamma((m-1)\alpha+1)}{(m-1)\Gamma(\alpha+1)\Gamma((m-1)\alpha+1-\alpha)} = \tag{26}$$

$$= \sum_{m=2}^{\infty} X^{m-2} \frac{A_m \Gamma(m\alpha+1)}{\Gamma(m\alpha+1-2\alpha)}$$

with the fractal index σ_x (see example 6 in [28]) is given by

$$\sigma_x = \frac{\Gamma((m-1)\alpha+1)}{(m-1)\Gamma(\alpha+1)\Gamma((m-1)\alpha+1-\alpha)}.$$

Substituting Eqs. (19), (25), and (26) into Eq. (14) yields

$$x^{2\alpha} \sum_{m=2}^{\infty} X^{m-2} \frac{A_m \Gamma(m\alpha+1)}{\Gamma(m\alpha+1-2\alpha)} + 2x^{\alpha} \sum_{m=2}^{\infty} X^{m-1} \frac{A_m \Gamma(m\alpha+1)}{\Gamma(m\alpha+1-\alpha)} - x^{2\alpha}\left[1 + \sum_{m=1}^{\infty} Q_m X^m\right] = 0,$$

$$\sum_{m=2}^{\infty} X^m \frac{A_m \Gamma(m\alpha+1)}{\Gamma(m\alpha+1-2\alpha)} + \sum_{m=2}^{\infty} X^m \frac{2A_m \Gamma(m\alpha+1)}{\Gamma(m\alpha+1-\alpha)} - \left[X^2 + \sum_{m=1}^{\infty} Q_m X^{m+2}\right] = 0, \tag{27}$$

$$\sum_{m=0}^{\infty} X^{m+2} \frac{A_{m+2} \Gamma((m+2)\alpha+1)}{\Gamma(m\alpha+1)} + \sum_{m=0}^{\infty} X^{m+2} \frac{2A_{m+2} \Gamma((m+2)\alpha+1)}{\Gamma(m\alpha+1+\alpha)} - \left[X^2 + \sum_{m=1}^{\infty} Q_m X^{m+2}\right] = 0.$$

Equating the coefficients of X^2 and X^{m+2}, we get

$$A_2 = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)[\Gamma((\alpha+1)+2)]}, \tag{28}$$

and

$$A_{m+2} = \frac{\Gamma(m\alpha+1)\Gamma((m+1)\alpha+1)}{\Gamma((m+1)\alpha+1)\Gamma((m+2)\alpha+1) + 2\Gamma(m\alpha+1)\Gamma((m+2)\alpha+1)} Q_m. \tag{29}$$

Eqs. (23) and (29) are the recurrence relations of the power series expansion, Eq. (16).

4. *Results*. We elaborated a FORTRAN code to calculate the series coefficients for the range $0 \le x < 2200$, this range covers that used in [29]. We ran the code with a step $\Delta\alpha = 0.05$ to encompass a large range of values for the fractional parameters α and varied the number of series terms until we obtained the minimum value of the relative error.

Fig.1 plots the Emden function *u* versus *x*. Fig.2 shows the variation of the relative error with *x*; the maximum relative error is about 0.05, indicating good accuracy. The variation of α is plotted in Fig.3. Generally, as *x* increases, we see the α parameter decrease except for some intermediate values. This result shows the highly dependent nature of the convergence on α. The number of series terms required to obtain a solution with suitable relative error is lower than that of the accelerated series solution of [6] by more than 50%.

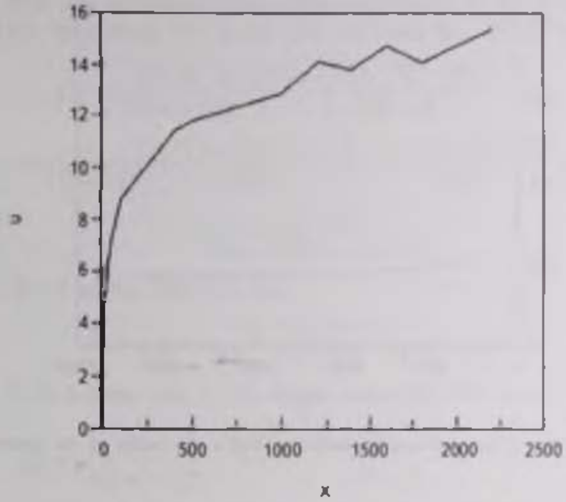

Fig 1. The Emden function u versus x the radius of the isothermal gas sphere

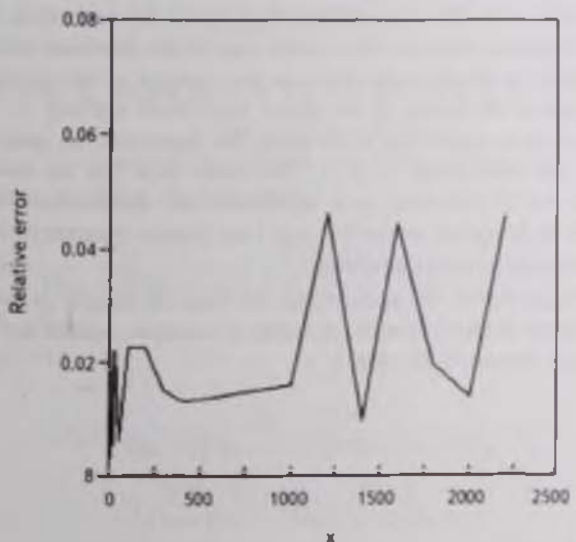

Fig.2. Absolute relative errors for the Emden function u

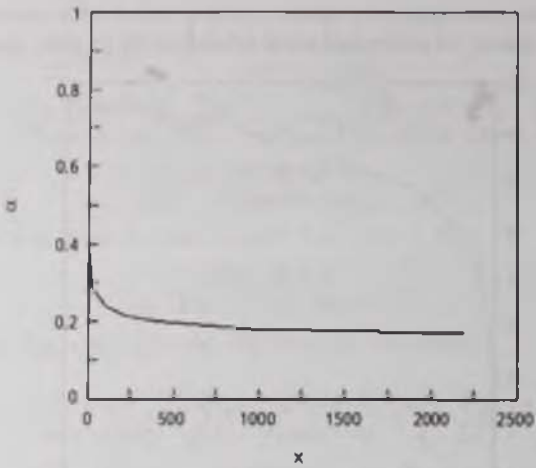

Fig.3. Variation of the fractional parameter α with x the radius of the isothermal gas sphere.

5. *Conclusion*. We derived a power series solution for the fractional isothermal Lane-Emden equation. Two recurrence relations are derived and solved simultaneously. The fractal index is deduced for each term in the expansion. By running code set with a small step for the fractional parameter α, we explored the effects of this factor on the accuracy of the calculations. The series reached the surface of the sphere faster when applying α, which may be viewed as an accelerator of the series. We found that the range of α spreads over the entire range $0 < \alpha < 1$. The results show that the maximum relative error is 0.05 indicating good agreement with numerical values. An application of the procedure on the first-type Lane-Emden equation (polytropic gas sphere) remains an open problem.

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Appendix

We shall now determine some of the coefficients in the series expansion of Eq. (16). By putting $l=2$ in Eq. (23) and using Eq. (28) for A_2, we obtain

$$Q_2 = -\frac{(2-1)!}{\Gamma(2\alpha+1)}\sum_{i=1}^{2}\frac{A_i\Gamma(i\alpha+1)Q_{2-i}\Gamma((2-i)\alpha+1)}{(i-1)!(2-i)!} =$$
$$= -\frac{1}{\Gamma(2\alpha+1)}\left[\frac{A_1\Gamma(\alpha+1)Q_1\Gamma(\alpha+1)}{0!(1)!}+\frac{A_2\Gamma(2\alpha+1)Q_0\Gamma(1)}{1!(0)!}\right] = \tag{A1}$$
$$= -\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)[\Gamma((\alpha+1)+2)]}$$

Putting $m=1$ in Eq. (29), we have

$$A_3 = \frac{\Gamma(\alpha+1)\Gamma(2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)+2\Gamma(\alpha+1)\Gamma(3\alpha+1)}Q_1 \tag{A2}$$

as $Q_1=0$. It follows that $A_3=0$. Again, using Eq. (23) with $l=3$, we have

$$Q_3 = -\frac{2!}{\Gamma(3\alpha+1)}\sum_{j=0}^{2}\frac{A_{j+1}\Gamma((j+1)\alpha+1)Q_{2-j}\Gamma((2-j)\alpha+1)}{j!(2-j)!} =$$
$$= -\frac{2!}{\Gamma(3\alpha+1)}\left[\frac{A_1\Gamma(\alpha+1)Q_2\Gamma(2\alpha+1)}{2!}+\frac{A_2\Gamma(2\alpha+1)Q_1\Gamma(\alpha+1)}{1!1!}+\right. \tag{A3}$$
$$\left.+\frac{A_3\Gamma(3\alpha+1)Q_0}{2!}\right]=0.$$

To calculate A_4, we put $m=2$ in Eq. (29) and using Q_2 from Eq. (A1) gives

$$A_4 = -\frac{\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)\Gamma(2\alpha+1)[\Gamma(3\alpha+1)+2\Gamma(2\alpha+1)][\Gamma(\alpha+1)+2]}. \tag{A4}$$

Proceeding as above, we next obtain

$$Q_4 = \frac{\Gamma(\alpha+1)}{\Gamma(4\alpha+1)[\Gamma(\alpha+1)+2]}\left[\frac{3\Gamma(\alpha+1)}{[\Gamma(\alpha+1)+2]}+\frac{\Gamma(3\alpha+1)}{[\Gamma(3\alpha+1)+2\Gamma(2\alpha+1)]}\right] \tag{A5}$$

$$A_5 = 0,$$
$$Q_5 = 0$$

$$A_6 = \frac{\Gamma(\alpha+1)\Gamma(5\alpha+1)}{\Gamma(6\alpha+1)[\Gamma((\alpha+1)+2)][\Gamma(5\alpha+1)+2\Gamma(4\alpha+1)]}\times$$
$$\times\left[\frac{3\Gamma(\alpha+1)}{\Gamma(\alpha+1)+2}+\frac{\Gamma(3\alpha+1)}{\Gamma(3\alpha+1)+2\Gamma(2\alpha+1)}\right] \tag{A6}$$

Now putting $\alpha=1$ in Eqs. (A1)-(A6), we get the coefficient of the series expansion, Eq. (16), as

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = \frac{1}{6}, \quad A_3 = 0, \quad A_4 = -\frac{1}{120}, \quad A_5 = 0, \quad A_6 = \frac{1}{1890},$$

which is the same as the series solution of Eq. (12).

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ПРИБЛИЖЕННОЕ РЕШЕНИЕ ДРОБНОГО УРАВНЕНИЯ ЛАНЕ-ЕМДЕНА ВТОРОГО РОДА

Е.А-Б.АБДЕЛ-САЛАМ, М.И.НОУХ

Сферическое изотермическое уравнение Лане-Емдена представляет собой нелинейное дифференциальное уравнение второго порядка и моделирует многие астрофизические конфигурации. Пользуясь методом дробного индекса и разложением в ряд по степеням, решается уравнение Лане-Емдена, содержащее модифицированную производную Римана-Лиувилля. Результаты показывают, что ряд сходится в интервале значений радиусов $0 \le x < 2200$ для широкого набора значений параметра α. Сравнение с численным решением показывает хорошее согласие с максимальной относительной ошибкой, равной 0.05.

Ключевые слова: *изотермический газ: дробный индекс: нелинейное дробное дифференциальное уравнение: модифицированное производное Римана-Лиувилля*

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