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NOTE ON THE DERIVATION OF THE EQUATION OF MOTION OF A CHARGED POINT-PARTICLE FROM HAMILTON'S PRINCIPLE

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An alternative derivation of the equation of motion of a charged point particle from Hamilton's principle is presented. The variational principle is restated as a Bolza problem of optimal control, the control variable u^i, $i=0, ..., 3$, being the 4-velocity. The trajectory $\bar{x}^i(s)$ and 4-velocity $\bar{u}^i(s)$ of the particle is an optimal pair, i.e. it furnishes an extremum to the action integral. The pair $(\bar{x}, \bar{u})$ satisfies a set of necessary conditions known as the maximum principle. Because of the path dependence of proper time s, we are concerned with a control problem with a free end point in the space of coordinates $(s, x^0, ..., x^3)$. To obtain the equation of motion the transversality condition must be satisfied at the free end point.

Key words: *maximum principle: transversality condition*

1. *Introduction*. The equation accepted as describing the motion of a charged point-particle of proper mass m, charge e and 4-velocity v^i, moving in a given field of gravitation g_{ij} and electromagnetism F_{ij} may be written in the form

$$mcDv^i = \frac{e}{c} F^{ij} v_j \quad \left(D = \frac{\delta}{\delta s};\ i, j = 0, ..., 3\right), \tag{1}$$

where the 4-velocity $v^i = dx^i/ds = \dot{x}^i$ satisfies the equation

$$g_{ij}\dot{x}^i\dot{x}^j = 1. \tag{2}$$

The equation of motion (1), which is in fact a generalization to curved space-time of the Heaviside-Lorentz law of ponderomotive force, may also be derived from a variational principle (omitting the minus sign which is a matter of convention)

$$J = \int_{s_1}^{s_2} \left(mcds + kA_i dx^i\right), \rightarrow extr. \quad (k = e/c), \tag{3}$$

where A_i is a 4-potential of the field $F_{ij} = \partial_i A_j - \partial_j A_i$. The extremum is to be sought in the class Γ of admissible curves consisting of those smooth, future oriented timelike world lines joining the fixed end events E_1, E_2, with respective coordinates x_1^i and x_2^i.

Remark 1: Eq. (2) is not only equivalent to a specification of the parameter used on the world line, i.e. natural parameter or proper time s, but it also expresses the timelike nature of the trajectory.

For purposes of reference we reproduce the essentials of the derivation. To prove that the Euler necessary condition for the variational problem (3) assumes the form of the equation of motion (1) it is customary, in the physical literature (e.g., [1-3]), to introduce for the set Γ of admissible timelike world lines, a monotonic parameter w, increasing from $w=w_1$ at E_1 to $w=w_2$ at E_2 as s increases from $s=s_1$ to $s=s_2$ (the simplest strictly increasing function $w(s)$: $[s_1,s_2]\to[w_1,w_2]$ is given by $w(s)=w_1+(w_2-w_1)(s_2-s_1)^{-1}(s-s_1)$). By adopting this procedure the original problem (3) with the equality constraint (2) is written in the form of a parametric Lagrange problem with fixed end events E_1, E_2 in the x^i-space and with no side-conditions

$$J=\int_{w_1}^{w_2} F dw \to extr, \quad F=mc\left(g_{ij}x'^i x'^j\right)^{1/2}+kA_i x'^i. \tag{4}$$

The Lagrangian variables being $x^i(w)$ and the tangent vector $x'^i=dx^i/dw$ vanishing nowhere on $[w_1, w_2]$ (regular parametrization).The extremals satisfy the Euler-Lagrange equations

$$\frac{d}{dw}\frac{\partial F}{\partial x'^i}-\frac{\partial F}{\partial x^i}=0. \tag{5}$$

Calculating these expressions and setting $w=s$ on the extremal one obtains the equation of motion (1). The object of this note is to present an alternative treatment of the variational principle (3). Using the parametrization of admissible timelike world lines in terms of proper time s, Eq. (2) defining the parameter s is incorporated into the calculations by the Lagrange multiplier technique. We restate the problem of extremizing the functional J under the constraint (2) as a Bolza problem of optimal control, the 4-velocity $\dot{x}^i$ being our control variable u^i. In order to comply with the relativistic demand of the path dependence of proper time we have to consider a control problem with the first end point P fixed in the (s,x^i)-space but the second end point Q variable since s_2 is undetermined. To obtain the equation of motion the transversality condition must be satisfied at Q.

2. *Mathematical preliminaries and derivation of the equation of motion.*

2.1. *Statement of the problem and necessary conditions for the optimum.* The variational problem (3) under the constraint (2) may be exhibited in the form

$$J(x,u)=mc(s_2-s_1)+k\int_{s_1}^{s_2} A_i u^i ds \to extr. \tag{6}$$

provided we adjoin the side conditions

$$\dot{x}^i - u^i = 0. \tag{7}$$

$$g_{ig} u^i u^j = 1. \tag{8}$$

The extremum is sought in the class P of all pairs (x, u) satisfying Eqs. (7) and (8). It remains to specify under what end conditions the functional $J(x, u)$ is to be extremized. Although the end events E_1, E_2 are fixed in the (x)-space, i.e in space-time, in the (s, x)-space the problem is one with the first end point fixed but the second end point variable. Indeed, if at the initial end point the values $s_1 = 0$ and $x^i(0) = x_1^i$ are all prescribed (the choice $s_1 = 0$ involves no loss of generality), at the final end point, because of the path dependence of proper time, we demand only $x^i(s_2) = x_2^i$, leaving s_2 variable. Considering a one-parameter family of admissible timelike world lines $x^i(s, b) = 0$, $(s_1 = 0 \le s \le s_2(b))$, the extremizing curve corresponding to the value of the parameter $b = 0$, the end conditions satisfy the equations

$$s_1 = 0, \quad x^i(0, b) = x_1^i, \quad x^i(s_2, b) = x_2^i. \tag{9}$$

Pairs (x, u) satisfying side conditions (7), (8) and end conditions (9) are called admissible. Formulated in the manner described above the original problem (3) becomes a special case of a Bolza problem of optimal control with a free end point [4,5], the velocity $\dot{x}^i$ being our control variable u^i. If the admissible pair $(\bar{x}, \bar{u})$ furnishes an extremum for the problem described above, the pair $(\bar{x}, \bar{u})$ is termed optimal (or subject to the maximum principle) if it has the following properties [4-6]. Our formulation of the necessary conditions which the optimal pair must satisfy is essentially but not precisely that of Hestenes [4].

(α) There exist multipiers $\lambda_0 = \text{const}$, $p_i(s)$, $\lambda(s)$ not vanishing simultaneously anywhere on (s_1, s_2)
and functions $\tilde{H}(x, u, p, \lambda)$ and $G(s_2)$ respectively defined by

$$\begin{aligned} \tilde{H} &= H(x, u, p) - \lambda\left(g_{ij} u^i u^j - 1\right), \\ H(x, u, p) &= p_i u^i - \lambda_0\, k A_i u^i, \\ G(s_2) &= \lambda_0\, m c s_2, \end{aligned} \tag{10}$$

such that:

(i) The class Γ of admissible curves consisting of smooth timelike world lines $x^i(s)$, the corresponding control variables $u^i(s)$ and the multipliers $p_i(s)$, $\lambda(s)$ are continuous functions of s for $s_1 \le s \le s_2$.

(ii) The demand that $\tilde{H}$ be invariant under general coordinate transformations implies that the functions $p_i(s)$ are the components of a covariant 4-vector and λ_0, $\lambda(s)$ invariants.

(β) At any given s, $s_1 \le s \le s_2$, the function $H(\bar{x}(s), u, p(s))$ of the variable u assumes for $u = \bar{u}(s)$ its maximum $\bar{H} = H(\bar{x}(s), \bar{u}(s), p(s))$. Since s does not appear explicitly, i.e. the control problem is autonomous, $\overline{H}$ is subject to the differential equation

$$\frac{d}{ds}\overline{H}=0 \tag{11}$$

for a non-autonomous problem, (11) generalizes to $\frac{d}{ds}\overline{H}=\frac{\partial}{\partial s}H(\overline{x},\overline{u},p,s)$.

(γ) The functions $\overline{x}(s)$, $\overline{u}(s)$, $p_i(s)$, $\lambda(s)$ satisfy the Euler-Lagrange equations

$$\text{(a)}\ \frac{d}{ds}x^i=\frac{\partial}{\partial p_i}\overline{H}=u^i,\quad \text{(b)}\ \frac{d}{ds}p_i=-\frac{\partial}{\partial x^i}\overline{H}=\lambda g_{lm,i}u^l u^m+\lambda_0 kA_{l,i}u^l,$$
$$\text{(c)}\ \frac{\partial}{\partial u^i}\overline{H}=0=p_i-\lambda_0 kA^i-2\lambda g_{ij}u^j. \tag{12}$$

(δ) The problem in this form is one with the first end point fixed in the (s, x)-space but the second end point variable since s_2 is undetermined (viz. 9). Accordingly, for fixed x-coordinates at the second end point, the transversality condition reduces to $(mc\lambda_0+[-\overline{H}]_2)ds_2=0$, where ds_2 is arbitrary. We then have the relation

$$mc\lambda_0=\left[p_i\overline{u}^i-\lambda_0 kA_i\overline{u}^i\right]_2 \tag{13}$$

which must hold at the second end point. We have set $[\overline{H}]_2=\overline{H}(\overline{x}(s_2),\overline{u}^i(s_2),p_i(s_2))$ and $[p_i]_2$ for the components of p_i evaluated at $s=s_2$.

2.2. *Derivation of the equation of motion.* Forming the scalar product of (12c) with $\overline{u}$ we obtain

$$p_i\overline{u}^i-\lambda_0 kA_i\overline{u}^i=2\lambda(s),\quad \text{i.e}\quad \overline{H}=H(\overline{x},\overline{u},p)=2\lambda(s). \tag{14}$$

By virtue of (11) $d\overline{H}/ds$ vanishes hence

$$\frac{d\lambda(s)}{ds}=0\quad\rightarrow\lambda(s)=\text{const}. \tag{15}$$

Further information to determine λ is obtained from the transversality condition (13), which yields, taking into account (14) and (15),

$$mc\lambda_0=[-\overline{H}]_2=2\lambda(s)\quad\rightarrow\lambda(s)=\text{const}=\frac{1}{2}mc\lambda_0. \tag{16}$$

We observe that if $\lambda_0=0$, then by (16) and (12c) $\lambda(s)=p_i(s)=0$. But the necessary condition (α) assures that these multipliers exist and there are not all zero, hence $\lambda_0\neq0$. The optimal trajectory $\overline{x}(s)$ is called normal and we can set $\lambda_0=1$. Differentiation of (12c) with respect to s then gives

$$\frac{dp_i}{ds}=\lambda_0 kA_{i,l}\overline{u}^l+mc\lambda_0 g_{ij}\frac{d\overline{u}^j}{ds}+mc\lambda_0 g_{ij,l}\overline{u}^l\overline{u}^j. \tag{17}$$

Setting $\lambda=(1/2)mc\lambda_0$ in (12b) and equating to (17) we get the equation of motion of a charged point particle in curved space-time

$$\frac{d}{ds}\overline{u}^r + \Gamma^r_{lm}\,\overline{u}^l\overline{u}^m = \frac{e}{mc^2}F^{rl}\overline{u}_l\,. \tag{18}$$

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К ВЫВОДУ УРАВНЕНИЯ ДВИЖЕНИЯ ЗАРЯЖЕННОЙ ТОЧЕЧНОЙ ЧАСТИЦЫ ИЗ ПРИНЦИПА ГАМИЛЬТОНА

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Представлен альтернативный вывод уравнения движения заряженной точечной частицы из принципа Гамильтона. Вариационный принцип переформулирован как проблема Больца оптимального контроля, где параметры контроля есть 4-векторы u^i, $i=0, ..., 3$. Траектория $\overline{x}^i(s)$ и 4-вектор скорости $\overline{u}^i(s)$ частицы являются оптимальной парой, т.е. приводят к экстремуму интеграла действия. Пара $(\overline{x},\overline{u})$ удовлетворяет набору необходимых условий, известных как принцип максимума. Из-за зависимости собственного времени s от пути мы имеем дело с проблемой контроля с свободной конечной точкой в пространстве координат $(s, x^0, ..., x^3)$. Для нахождения уравнения движения условие трансверсальности должно выполняться в свободной конечной точке.

Ключевые слова: ***принцип максимума: условие трансверсальности***

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